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A Properties of Mandelbrot Set of Quartic Rational Maps

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Abstract:

We present some properties the Mandelbrot set of Quartic rational maps. Every Quartic rational functions is conjugate toz^4+c or λ(z^3+1/z+b). We study the Mandelbrot set M_λ, the set of parramrters b for which the Julia set of λ(z^3+1/z+b) is connected.

Keywords: Mandelbrot set, Julia set, Quartic rational maps.

Introduction:

Early in the 20th century, Julia Set and Fatou started researching complicated dynamics [1]. They investigated the Julia sets of rational functions on the complex sphere and discovered the majority of the fundamental properties of complex sphere dynamics.In 1982, Sullivan [2], found a solution to an ancient problem involving the characteristics of members of the Fatou set, and since then, complexity has advanced significantly. For instance, Douady and Hubbard [3], attained crucial findings on the Mandelbrot set's structure of $z^2 + c$. The intricate and stunning computer graphics of the Mandelbrot set and Julia sets of $z^2 + c$ are another factor that contributed to the complex dynamics' enormous popularity. Each map of the Quartic rationale is conjugate to $z^4 + c$ or $\lambda(z^3 + \frac{1}{z} + b)$. For the function $\lambda(z^3 + \frac{1}{z} + b)$ with $\lambda > 1$, obtained a criterion of its Julia set to be cantor set. In this study, some properties of the dynamics of Quartic rational functions for $\lambda = 1$ are presented. To see more [4],[5],[6],[7],[8]

1 Definitions and Prerequisites:

In this section, we introduce some basic theorems and not at ions of complex dynamics.

Suppose $\bar{\mathbb{C}}$ be the extended complex plan. Assume $R : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ be rational function with degree $d \geq 1$. For $n \in N$ the n-th iteration of R is written R^n. The set

$$F(R) = \{z : \{R^n\} \text{ is a normal family in some neighborhood of } z\}$$

is named the Fatou set of R.

The complemetary set $J(R) = \bar{\mathbb{C}} - F(R)$ is named the Julia set.

Definition 2.1. [9] Suppose $R : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ be rational function with degree $d \geq 2$, then z is named a critical point of R if $R'(z) = 0$. The number of critical points of a subset E of $\bar{\mathbb{C}}$ is named a total deficiency $\sigma_R(E)$ of R over E. We define the connectivity $c(E)$ of subset E as the number of members of ∂E.

Theorem 2.1. [10] Suppose ρ_0 and ρ_1 be the members of the Fatou set of a rational function R and assume that $R: \ \rho_0 \ \rightarrow \ \rho_1$. Then, for some $s \in Z$, R is an $s-$fold : $\rho_0 \ \rightarrow \ \rho_1$ and

$$2 - c(\rho_0) + \sigma_R(\rho_0) = s(2 - c(\rho_1)).$$

A subset E of $\overline{\mathbb{C}}$ is forward invariant under R if $R(E) = E$, backward invariant under R if $R^{-1}(E) = E$ and completely invariant under R if E is forward and backward invariant under R.

Theorem 2.2. [10] $\partial\rho_0 = J(R)$. If ρ_0 is a completely invariant member of $F(R)$. Suppose $\{\xi_1 \ \cdots \xi_h\}$ be a cycle of a rational function R. Then the deritvative of R^h at every point ξ_i is

$$\left(R^h\right)'(\xi_i) = \prod_{j=1}^{h}(R)'\left(R^j(\xi_i)\right) = \prod_{j=1}^{h}(R)'(\xi_i).$$

The number is named the multiplier of the cycle.

Definition 2.2. [10] Suppose $\{\xi_1 \ \cdots \xi_h\}$ be a cycle of a rational function R. Then for each $i \ \in \{1,2,\cdots h\}$ this cycle is named by

1. attracting if $|\left(R^h\right)'(\xi_i)| < 1$;
2. rationally indifferent if $\left(R^h\right)'(\xi_i)$is a root of unity;
3. irrationally indifferent if$|\left(R^h\right)'(\xi_i)| = 1$, but $\left(R^h\right)'(\xi_i)$ is not a root of unity;
4. repelling if $|\left(R^h\right)'(\xi_i)| > 1$;
5. nonrepelling if $|\left(R^h\right)'(\xi_i)| \leq 1$;

Theorem 2.3. [10] A rational function R of degree d has at most $2d - 2$ nonrepelling cycles.

Therefore every quartic rational function has at most six nonrepelling cycles.

Theorem 2.4. [10] Suppose $\{\xi_1 \ \cdots \xi_h\}$ be a cycle of a rational function R. If the cycle is attracting, then any ξ_i lies in a member, say ρ_j, of the fatou set F(R), and $R^{nh} \rightarrow \xi_j$ locally uniformly on ρ_j as $n \rightarrow \infty$. If the cycle is rationally indifferent or repelling, then the cycle lies in the Julia set J(R).

Theorem 2.5. [10] Suppose $\{\xi_1 \ \cdots \xi_h\}$ be an irrational indifferent cycle which lies in F(R) and ξ_i lies in a member ρ_j of F(R). Then ρ_j is simply connected, and $R^h: \rho_j \rightarrow \rho_j$ is conjugate a rational of infinite order of the unite disc D.

Each member of the Fatou set of this kind is named a Siegel disk after C.L. Siegle established its existence in 1941. Every analytic function containing a rationally indifferent fixed point at ξ can be seen to be conjugate to

$$g(z) = z - z^{q-1} + O(z^{q-2}) \quad (1)$$

for some $q \in Z$, in some neighborhood of the origin. For the dynamics of an analytic function near a rationally indifferent fixed point, we have the following theorems.

Theorem 2.6. [10] Assume that g is an analytic function satisfying (1) and let $\upsilon_1 \cdots \upsilon_h$ be the h−th roots of unity and $\zeta_1 \cdots \zeta_h$ be the h−th roots of −1. Then for sufficiently small nonnegative number v_0 and θ_0, we find that $|g(z)| < |z|$ on any strip

$$S_i = \left\{ z \middle| 0 < \left|\frac{z}{\upsilon_i}\right| < \upsilon_0, \left|\arg\left(\frac{z}{\upsilon_i}\right)\right| < \theta_0 \right\}$$

and $|g(z)| > |z|$ on every strip

$$\Sigma_i = \left\{ z \middle| 0 < \left|\frac{z}{\zeta_i}\right| < v_0, \left|\arg\left(\frac{z}{\zeta_i}\right)\right| < \theta_0 \right\}$$

. For any nonnegative number r, every nonnegative number q, and all $k = 0,1,\cdots q-1$ we define the set $\Pi_k(r)$ as following

$$\Pi_k(r) = \left\{ ve^{i\theta} : v^q < r\big(1 + \cos(q\theta)\big) \text{and } \left|\frac{2k\pi}{q} - \theta\right| < \frac{\pi}{q} \right\}$$

These sets are named petals.

Not that the petals are pairwise disjoint, and that every petal subtends an angle $\frac{2\pi}{q}$at the origin, so that the total angle subtends at the origin by each the petals is 2π. We call the line of symmetry of $\Pi_k(r)$ (the ray$\theta = \frac{2\pi}{q}$) by the axis of petal $\Pi_k(r)$.

Theorem 2.7. [10] Assume that an analytic function g has a Taylor expansion

$$g(z) = z - z^{q-1} + O(z^{q-2})$$

at the origin. Then for each sufficiently small s and for $k = 1,\cdots q$

- g functions every petal $\Pi_k(s)$.into itself;
- $g^n \to 0$ uniformly on every petal as $n \to \infty$;
- $\arg g^n(z) \to \frac{2k\pi}{q}$ locally uniformly on $\Pi_k(s)$.as $n \to \infty$;
- $|g(z)| < |z|$ on a neighborhood of the axis of every petal.
  In particular if g is a rational function, then the Fatou set has membersρ_k every of which contains Π_k respectively, such that
- $g^n \to 0$ uniformly on every members ρ_k as $n \to \infty$;
- $\arg g^n(z) \to \frac{2k\pi}{q}$locally uniformly on ρ_k as $n \to \infty$.

Theorem 2.8. [2] Suppose that R be a rational function and ρ_0 be a periodic member of the Fatou set of R with period m and assume $S = R^m$. Then ρ_0 must be one of the following kinds

- $\rho_0$be an attracting member if it contains a periodic q s.t $0 \le |S'(q)| < 1$.

- ρ_0 be a parabolic member if there exists a periodic point q on $\partial\rho_0$ whos period divides m and $S^k(z) \to q$ as $k \to \infty$ $\forall z \in \rho_0$.
- ρ_0 be a Siegel disc if ρ_0 is simply connected and $S\backslash\rho_0$ is conjugate to a rotation.
- ρ_0 is a Herman ring if ρ_0 is conformally equivalent to an annulus $A = \{z \in C : s_1 < |z| < s_2\}$ (where $s_1, s_2 \in R, s_1 \geq 0, s_2 > 0$ and the function $S\backslash\rho_0$ is conjugate to a rigid rotation of the annulus.

The immediate basin of the cycle is the union of those members of F(R) whose closure contains an attractive $\xi,\cdots,\xi_p$. The immediate basin of an attracting fixed point ξ is denoted by $A(\xi)$, which is made up of only one attractive member of the Fatou set.

Theorem 2.9. [10] A critical point of R can be found in the immediate basin of an attractive cycle or parabolic cycle.

We refer the set of critical points of R by $C(R)$ and we utilize $C^+(R)$ to refer the union of forward $C(R)$ in another meaning

$$C^+(R) = \bigcup_{m=0}^{\infty} R^m(C)$$

Theorem 2.10. [10] Suppose that $\varrho_1,\cdots\ \varrho_p$ be a cycle of Siegel discs or Herman rings of rational function R. Then the closure of $C^+(R)$ contains $\cup\, \partial\varrho_j$.

Theorem 2.11. [10] The derived set of $C^+(R)$ contains any irrationally indifferent cycle of R in J.

2 Mandelbrot Set of Quartic Rational Function

In this part, we'll look at some of the Mandelbrot set's features for Quartic rational functions. Theorem (2.10), in particular, characterizes the Mandelbrot set of the Quartic rational function $\lambda(z^3 + \frac{1}{z} + b)$., with $\lambda = 1$, implying that the Julia set is related to the set of parameter b.

Lemma 3.1. Every Quartic rational function is conjugate by a Mobius transformation:

$z^4 + c$.

Proof. Let $f(z) = \alpha z^4 + \beta z^3 + \gamma z^2 + \delta z + \rho$ represent a general quartic. Let $g_c(z) = z^4 + c$ and define $h(z) = Az + B, A, B \neq 0$. From the definition of topological conjugacy, we want to find A and B such that $h \circ f = g \circ h$. The commutativity of this expression depends of the equality of the two equations

$$(h \circ f)(z) = A\alpha z^4 + A\beta z^3 + A\gamma z^2 + A\delta z + A\rho + B$$

$$(gc \circ h)(z) = A^4 z^4 + 4A^3 B z^3 + 6A^2 B^2 z^2 + 4AB^3 z + B^4 + c$$

If we set each of the corresponding coefficients of both equations equal to one another, it yields the following equations

$$\alpha A = A^4 \Longrightarrow A = \sqrt[3]{\alpha}$$
$$A\beta = 4A^3B \Longrightarrow B = \frac{4\alpha^{\frac{2}{3}}}{\beta}$$
$$A\gamma = 6A^2B^2 \Longrightarrow 24\frac{\alpha}{\beta}$$
$$A\delta = 4AB^3 \Longrightarrow \delta = \frac{256\alpha\alpha^2}{\beta^3}$$
$$A\rho + B = B^4 + c \Longrightarrow c = A\rho + B - B^4$$
$$\Longrightarrow c = \alpha^{\frac{1}{3}}\rho + \frac{4\alpha^{\frac{2}{3}}}{\beta} + \frac{256\alpha^{\frac{8}{3}}}{\beta^4}$$

whence we obtain that f is topologically conjugate to the family g_c, denoted by $f \sim g_c$, via the affine transformation $h(z) = \alpha^{\frac{1}{3}}z + \frac{4\alpha^{\frac{2}{3}}}{\beta}$ and on the condition that the third equation be satisfied, i.e.

$$c = \alpha^{\frac{1}{3}}\rho + \frac{4\alpha^{\frac{2}{3}}}{\beta} + \frac{256\alpha^{\frac{8}{3}}}{\beta^4}$$

. □

In the following statements, we give some characteristics of the Fatou set and Julia set of Quartic rational functions.

Proposition 3.2. The member ρ_0 of F(R) is member invariant if it is an attractive or parabolic forward invariant member.

Proof. By the conjugation , we may suppose that R is either $z^4 + c$ or $\lambda(z^3 + \frac{1}{z} + b)$. First we assume $R = z^4 + c$ Then since $R(z) = R(-z)$,

$$z \in J(R) \text{ iff } -z \in J(R).$$

Suppose $\rho_1 = \{-z : z \in \rho_0\}$. Then ρ_1 is also a member of $F(R)$. Since the forward invariant member ρ_0 contains a critical point 0 or ∞, the member ρ_1 also contains the same critical point whatever ρ_0 has. Therefore, ρ_0 and ρ_1 must be the same member or F(R), and furthermore, if $z \in \rho_0$ then $-z \in \rho_1 = \rho_0$. To see that ρ_o is completely invariant, we suppose $\xi \in \rho_0$. If $z_0 \in R^{-1}(\xi)$, then $-z_0 \in R^{-1}(\xi)$. Since ρ_0 is forward invariant. One of z_0 and $-z_0$ must be in ρ_0 and hence both z_0 and $-z_0$ belong to ρ_0. Therefore, ρ_0 is completely invariant. In the same way is the proof of the second part. □

Proposition 3.3. There exists an attracting forward invariant member containing four critical point if and only if four critical points iterate to an attracting fixed point.

Proof. By theorem 2.8, if a member is attracted forward invariant, every points in the member iterate to an attracting fixed point. If any critical points iterate to an attracting fixed point ξ, then an attracting forward invariant member ξ is prop-invariant 3.2. As a result, it has two crucial points. □

Proposition 3.4. There exists a parabolic forward invariant member of R containing all critical points if and only if any critical points in F(R) iterate to a parabolic fixed point ξ with $R'(\xi) = 1, R''(\xi) \neq 0, R'''(\xi) = 0$ and $R^{(4)}(\xi) = 0$.

Proof. If a parabolic forward invariant member exists, then a cording theorem 2.8 states that each point in the member iterates to ξ and $R'(\xi) = 1$. Assume that $R''(\xi)$ equals 0. Then R must have at least four petals in close proximity to the Theorem 2.7. Each petal is contained in its own forward invariant component, which has at least one critical point. Because each member has four critical points, the union of all members has five critical points, this is contradiction. Conversely, suppose every critical points in $F(R)$ iterate to a parabolic fixed point ξ with $R'(\xi) = 1, R''(\xi) \neq 0, R'''(\xi) = 0$ and $R^{(4)}(\xi) = 0$. Because R is analytic we gate

$$\begin{aligned} R(z) &= \xi + R'(\xi)(z-\xi) + \frac{R''(\xi)}{2}(z-\xi)^2 + \frac{R'''(\xi)}{3!}(z-\xi)^3 + \frac{R^{(4)}(\xi)}{4!}(z-\xi)^4 + \cdots \\ &= z + \frac{R''(\xi)}{2}(z-\xi)^2 + \cdots . \end{aligned}$$

because $R''(\xi) \neq 0$, R has only petal by theorem 2.7 and a member containing the petal is forward invariant. A cording theorem 2.6, we know that every critical points iterating to ξ eventually lie in the parabolic member. □

Proposition 3.5. If a forward invariant member of $F(R)$ containing two critical points then it is an attracting component or parabolic forward invariant member.

Suppose ρ_0 be a forward invariant member of $F(R)$ which contains at least two critical points. By theorem 2.1, we get

$$2 - c(\rho_0) + \sigma R(\rho_0) = m\big(2 - c(\rho_0)\big).$$

Because $\sigma_R(\rho_0) = 4$ and $m = 1$ or 2 or 3 or 4, $c(\rho_0) = \infty$ or 0 then ρ_0 equals to the complex sphere, which contradicts to the fact that $J(R) \neq \emptyset$. Hance $c(\rho_0)$ must be infinity. Therefore a cording to theorem 2.8, ρ_0 is an attracting or parabolic forward invariant member.

Proposition 3. 6. The Julia set of a quartic function R is a Cantor set iff each critical points iterate to

- an attracting fixed point, or
- a parabolic fixed point ξ with $R'(\xi) = 1, R''(\xi) \neq 0, R'''(\xi) = 0$ and $R^{(4)}(\xi) = 0$.

Proof. A cording to theorem 2.11 $J(R)$ is a Cantor set iff there exists a forward invariant member containing every critical points. (see prop(3.3).), this is comparable to the existence of any critical points in an attractive or parabolic forward invariant member. Hence the assertion is obtained by (prop(3.3). and (3.4)) . □

Proposition 3.7. If there exist two non-repelling cycle of R, then $J(R)$ is connected.

Proof. Assume that $J(R)$ is a cantor set. A cording to prep. 3.5, one of the two cycle is an attracting or a parabolic fixed point, denoted ξ, to which any critical points in $F(R)$ iterate . Suppose $\{\xi_1, \cdots, \xi p\}$ be the another cycle of R for some integer p. Then is one of the following Kinds:

1. attracting cycle,
2. parabolic cycle,
3. irrationally indifferent cycle in $F(R)$;
4. irrationally indifferent cycle in $J(R)$.

In either case, the first or the second by theorem (2.9), there exists a critical point and the derived set of the orbit is $\cup \{\xi_i\}$. This contradicts to the fact that any critical points iterate to ξ. In the third case ξ are the center of Siegel discs, which is impossible since J(R) is totally disconnected. In the fourth case, ξ_i lie in the derived set of $C^+(R)$ by theorem (2.11), which is a contradiction because the derived set of $C^+(R)$ is $\{\xi\}$. □

Proposition 3.8. If there exist an indifferent cycle $\upsilon_1, \cdots, \upsilon_p$ of R such that υ_i is parabolic fixed point withe $R'(\upsilon_1) = 1$, $R''(\upsilon_1) \neq 0$, $R'''(\upsilon_1) = 0$ and $R^{(4)}(\upsilon_1) = 0$., then J(R) is connected.

Proof. Assume that J(R) is a cantor set. By prep.(3.1), there exists an attracting or a parabolic fixed point ξ, which cannot be υ_i. From prep.(3.7) that J(R) is connected. □

Suppose $R_{\lambda,b}$ be quartic rational function $\lambda(z^3 + \frac{1}{z} + b)$. for λ is real number and b is complex number. Define the Mandelbrot set M_λ, for any λ, of the quartic rational functions $R_{\lambda,b}$ as a set of parameter b for which the Julia set $J(R_{\lambda,b})$ is connected. By prep.(3.6) and (3.8), one can easily properties the Mandelbrot set M_λ, for any $|\lambda| = 1$.

Theorem 3.9. If $\lambda = 1, i$ and $\lambda \neq 1, i$, then the Mandelbrot set M_λ whole complex plan. If $\lambda = 1$, then

$$M_1 = \mathbb{C} - \left\{b \neq 0 \lim_{n\to\infty} R^n{}_{1,b}(\pm 1, \pm i) = \infty \text{ and } R^n{}_{1,b}(\pm 1, \pm i) \neq 0\ \forall n\right\}.$$

Proof. Suppose $R_{\lambda,b}(z)$ conjugate to the function r(z) utilize Moebius transformation $g(z) = \frac{1}{z}$; we get:
$r(z) = \frac{z}{\lambda(z^4+1)+bz}$ and $R'_{b,\lambda}(\infty) = (g.R.g^{-1})(0) = r'(0) = \frac{1}{\lambda}$ when $|\lambda| = 1$ and λ 6= 1,∞ is a fixed point of $R_{\lambda,b}(z)$ with $R'_{b,\lambda}(\infty) \neq 1$.
Since $J(R_{\lambda,b})$ is connected for every b . Then we obtain:

$M_\lambda = C$ for $\lambda = 1$ and λ 6= 1

If $\lambda = 1$,∞ is aparabolic fixed point of $R_{1,b}(\infty) = 1$.
If b = 0 then $R''_{1,b}(\infty) = r''(0) = 0$, the julia set $J(R_{1,0})$ is connected by dependind on prop(2.8).
Hance $b = 0 \in M_1$, if b 6= 0, then $R''_{1,b}(\infty) = r''(0) = -4b$. By depend on 3.6 $b \notin M_1$ if and only if every critical points in $F(R_{1,b})$ iterate to ∞. Therefore $0 \in M_1$, and if $b \neq 0$ then b /∈ M_1 iff:

$$\left\{\lim_{n\to\infty} R^n{}_{1,b}(\pm 1, \pm i) = \infty \text{ and } R^n{}_{1,b}(\pm 1, \pm i) \neq 0\ \forall n\right\}.$$

Similarly If $\lambda = i$ Explain $\lambda \neq i$.

4-Conclusion

If $\lambda = 1, i$ and $\lambda \neq 1, i$, then the Mandelbrot set M_λ whole complex plan. If $\lambda = 1$, then

$$M_1 = \mathbb{C} - \left\{b \neq 0 \lim_{n\to\infty} R^n{}_{1,b}(\pm 1, \pm i) = \infty \text{ and } R^n{}_{1,b}(\pm 1, \pm i) \neq 0\ \forall n\right\}$$ □

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