Approximate Methods For Solving Fractional Differential Equations

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Abstract:

In this paper, exact and approximate solutions of the nonlinear Burgers equation, heat-like equation and coupled nonlinear fractional Burger’s equations with Caputo-Fabrizio fractional operator by using Daftardar-Jafari method (DJM) and Sumudu decomposition method (SDM) are presented and discussed. The solutions of our equations are calculated in the form of rabidly convergent series with easily computable components. Three illustrative applications are given to demonstrate the effectiveness and the leverage of the present methods. Graphical results are utilized and discussed quantitatively to illustrate the solution. The results reveal that the methods are very effective and simple in determination of solution of the fractional partial differential equations.

Keywords: Daftardar-Jafari method; nonlinear Burger equation; heat-like equation, Caputo-Fabrizio fractional operator.

1 INTRODUCTION:

Fractional calculus has become increasingly popular in the scientific world in recent decades as a result of its relevance in solving a variety of physical, engineering, thermodynamics, and other scientific challenges. Nonlinear differential equations are used to describe the majority of natural occurrences. As a result, scientists from diverse fields of study attempted to tackle the problem. However, because of their non-linear nature of them, Finding a precise solution to a collection of equations is difficult. [1,3,4-7,10-14].

In the past decade, Caputo and Fabrizio introduced a new fractional differential operator and many researchers studied this operator and researchers are still interested in this operator because of its importance, as some studies have applied methods of approximate solutions to equations that include this fractional operator[2,9,15,16].

In this research, we use the DJM and SDM to solve nonlinear fractional partial differential equations with the Caputo-Fabrizio fractional operator. The following is how the paper is structured: Section 2 contains the fundamental concepts of fractional calculus, Section 3 and Section 4 provide an examination of the techniques utilized, Section 5 has various test problems that demonstrate the usefulness of the suggested method, and Section 6 contains the conclusion.
2 BASIC DEFINITIONS:

Definition 1. [2,9]. The Caputo-Fabrizio operator's fractional derivative for $0 < \alpha \leq 1$ is defined as:

$$
\mathcal{D}_{t}^{\alpha} u(t) = \frac{\varepsilon(\alpha)}{1 - \alpha} \int_{0}^{t} \exp \left[ -\frac{\alpha(t - s)}{1 - \alpha} \right] u'(s) \, ds, \quad t \geq 0,
$$

where $u'(s)$ is the derivative of $u$, and $\varepsilon(\alpha)$ is a normalization function such that $\varepsilon(0) = \varepsilon(1) = 1$.

The operator's fundamental attributes $\mathcal{D}_{t}^{\alpha}$ are as follows [11-17].

1. $\mathcal{D}_{t}^{\alpha} u(t) = u(t)$, where $\alpha = 0$.

2. $\mathcal{D}_{t}^{\alpha} [u(t) + v(t)] = \mathcal{D}_{t}^{\alpha} u(t) + \mathcal{D}_{t}^{\alpha} v(t)$

3. $\mathcal{D}_{t}^{\alpha} (c) = 0$, when $c$ is a constant.

Definition 2. [15,16]. The fractional integral operator of order $0 < \alpha \leq 1$ of Caputo-Fabrizio and $t > 0$ is given by:

$$
\mathcal{I}_{t}^{\alpha} u(t) = \frac{1 - \alpha}{\varepsilon(\alpha)} u(t) + \frac{\alpha}{\varepsilon(\alpha)} \int_{0}^{t} u(s) \, ds
$$

The operator's fundamental attributes $\mathcal{I}_{t}^{\alpha}$ are as follows:

1. $\mathcal{I}_{t}^{\alpha} [u(t)] = u(t)$, where $\alpha = 0$.

2. $\mathcal{I}_{t}^{\alpha} [u(t) + v(t)] = \mathcal{I}_{t}^{\alpha} u(t) + \mathcal{I}_{t}^{\alpha} v(t)$

3. $\mathcal{I}_{t}^{\alpha} \left( \mathcal{D}_{t}^{\alpha} u(t) \right) = u(t) - u(0)$.

Definition 3. [8] Over a set of functions $\mathcal{A}$, the Sumudu transform is defined,

$$
\mathcal{A} = \left\{ u(t) \left| \exists M, \tau_1, \tau_2 > 0, |u(t)| < Me^{\frac{|t|}{\tau_2}}, \text{if } t \in (-1)^j \times [0, \infty) \right. \right\},
$$

by the following formula

$$
\mathcal{S}\{u(t)\} = \int_{0}^{\infty} u(\omega t) e^{-t} \, dt, \quad \omega \in (\tau_1, \tau_2).
$$

3 ANALYSIS OF DJM:

In the Caputo-Fabrizio operator sense, consider the following nonlinear partial differential equation:
\[ CF_0 D_t^\alpha u(x, t) + R[u(x, t)] + N[u(x, t)] = g(x, t), \]  
(4)  

with initial condition \( \frac{\partial u(x, 0)}{\partial t} = u_0(x), \)

where \( CF_0 D_t^\alpha u(x, t) \) is Caputo-Fabrizio operator of \( u(x, t) \), a linear operator is \( R \), a nonlinear operator is \( N \) and a source term is \( g \).

When we apply the Caput-Fabrizio integral to both sides of Eq.(4), we get

\[ CF_t^\alpha \left[ CF_0 D_t^\alpha u(x, t) \right] + CF_t^\alpha R[u(x, t)] + CF_t^\alpha N[u(x, t)] = CF_t^\alpha [g(x, t)], \]  
(5)  

By integral properties of Caputo-Fabrizio, we get

\[ u(x, t) = u_0(x) + CF_t^\alpha [g(x, t)] - CF_t^\alpha R[u(x, t)] - CF_t^\alpha N[u(x, t)], \]  
(6)  

We're seeking for \( u(x,t) \) solution to Eq.(6) with the following series form:

\[ u(x, t) = \sum_{n=0}^{\infty} u_n(x, t) \]  
(7)  

It is possible to decompose the nonlinear operator \( N \) as follows:

\[ N \left[ \sum_{n=0}^{\infty} u_n(x, t) \right] = N[u_0] + \sum_{n=1}^{\infty} \left( N \left[ \sum_{i=0}^{n} u_i \right] - N \left[ \sum_{i=0}^{n-1} u_i \right] \right) \]  
(8)  

In view of Eq.(6) and Eq.(7), Eq.(8) is equivalent to

\[ \sum_{n=0}^{\infty} u_n(x, t) = u_0(x) + CF_t^\alpha [g(x, t)] - CF_t^\alpha R \left[ \sum_{n=0}^{\infty} u_n \right] - CF_t^\alpha N[u_0] \]

\[ - CF_t^\alpha \left( \sum_{n=1}^{\infty} \left( N \left[ \sum_{i=0}^{n} u_i \right] - N \left[ \sum_{i=0}^{n-1} u_i \right] \right) \right). \]  
(9)  

Furthermore, the relationship is defined by recurrence, thus

\[ u_0(x, t) = u_0(x) + CF_t^\alpha [g(x, t)] \]

\[ u_1(x, t) = - CF_t^\alpha R[u_0] - CF_t^\alpha N[u_0] \]

\[ u_n(x, t) = - CF_t^\alpha R[u_n] - CF_t^\alpha \left( N \left[ \sum_{i=0}^{n} u_i \right] - N \left[ \sum_{i=0}^{n-1} u_i \right] \right), \quad n = 1, 2, ... \]  
(10)  

The following is the \( n \)-term approximate solution of Eq.(4):

\[ u(x, t) = u_0(x, t) + u_1(x, t) + u_2(x, t) + \cdots \]  
(11)  

4 **ANALYSIS OF SDM:**

When you apply the Sumudu transform to Eq.(4),

\[ S \left[ CF_0 D_t^\alpha u(x, t) \right] + S \{ R[u(x, t)] \} + S \{ N [u(x, t)] \} = S \{ g(x, t) \}. \]  
(12)  

If the Sumudu transform differentiation property is used, then
\[ S\{u(x,t)\} = u_0(x) + (1 - a + aw)S\{g(x,t)\} \\
- (1 - a + aw)S\{R[u(x,t)] + N[u(x,t)]\} \quad (13) \]

Taking the inverse of Sumudu transform for both sides of Eq.(13), we have

\[ u(x,t) = S^{-1}\left(u_0(x)\right) + S^{-1}\left((1 - a + aw)S\{g(x,t)\}\right) \\
- S^{-1}\left((1 - a + aw)S\{R[u(x,t)]\}\right) - S^{-1}\left((1 - a + aw)S\{N[u(x,t)]\}\right). \quad (14) \]

Assume that \( u(x,t) \) is a solution to Eq.(14) which can be rewritten as

\[ u(x,t) = \sum_{n=0}^{\infty} u_n(x,t), \quad (15) \]

It is possible to decompose the nonlinear term as follows:

\[ N\{u(x,t)\} = \sum_{n=0}^{\infty} A_n, \quad (16) \]

where

\[ A_n = \frac{1}{n!} \frac{\partial^n}{\partial \zeta^n} \left[ N\left( \sum_{i=0}^{\infty} \zeta^i u_i(x,t) \right) \right]_{\zeta=0}, \quad n = 0,1,2, \ldots \quad (17) \]

We get the following result by substituting Eq.(15) and Eq.(16) into Eq.(14):

\[ u(x,t) = S^{-1}(u_0(x)) + S^{-1}\left((1 - a + aw)S\{g(x,t)\}\right) \\
- S^{-1}\left((1 - a + aw)S\left( R \left\{ \sum_{n=0}^{\infty} u_n(x,t) \right\} \right) \right) \\
- S^{-1}\left( N(1 - a + aw)S\left( \sum_{n=0}^{\infty} A_n \right) \right). \quad (18) \]

By comparing the left and right sides of Eq.(18), we get

\[ u_0(x,t) = S^{-1}(u_0(x)) + S^{-1}(1 - a + aw)S\{g(x,t)\}) \]
\[ u_1(x,t) = -S^{-1}(1 - a + aw)S\{R[u_0(x,t)]\}) - S^{-1}(N(1 - a + aw)S\{A_0\}) \]
\[ u_2(x,t) = -S^{-1}(1 - a + aw)S\{R[u_1(x,t)]\}) - S^{-1}(N(1 - a + aw)S\{A_1\}) \]
\[ \vdots \]

The general form of recursive relation is

\[ u_0(x,t) = S^{-1}(u_0(x)) + S^{-1}(1 - a + aw)S\{g(x,t)\}) \]
\[ u_1(x,t) = -S^{-1}(1 - a + aw)S\{R[u_0(x,t)]\}) - S^{-1}(N(1 - a + aw)S\{A_0\}) \]
\[ u_n(x,t) = -S^{-1}(1 - a + aw)S\{R[u_n(x,t)]\}) - S^{-1}(N(1 - a + aw)S\{A_n\}) \quad (19) \]
The approximate solution is given by

\[ u(x, t) = u_0(x, t) + u_1(x, t) + u_3(x, t) + \cdots \]  \hspace{1cm} (20)

5 Illustrative Examples:

Example 1. Consider the following, Burger equation which is nonlinear in the Caputo-Fabrizio sense

\[ c_F^0 D_t^a u + u \frac{\partial u}{\partial x} = \frac{\partial^2 u}{\partial x^2}, \quad 0 < a \leq 1 \]  \hspace{1cm} (21)

subject to the initial condition \( u(x, 0) = x \)

Below we present the DJM.

Taking the integral of Caputo-Fabrizio to both sides of Eq.(21), we obtain

\[ u(x, t) = u(x, 0) + c_F^0 J_t^a \left[ \frac{\partial^2 u}{\partial x^2} \right] - c_F^0 J_t^a \left[ u \frac{\partial u}{\partial x} \right] \]  \hspace{1cm} (22)

As a result of Eq.(10), we may get the following approximate solution:

\[ u_0(x, t) = u(x, 0) \]

\[ u_1(x, t) = c_F^0 J_t^a \left[ \frac{\partial^2 u_0}{\partial x^2} \right] - c_F^0 J_t^a \left[ u_0 \frac{\partial u_0}{\partial x} \right] \]

\[ u_2(x, t) = c_F^0 J_t^a \left[ \frac{\partial^2 u_1}{\partial x^2} \right] - c_F^0 J_t^a \left[ (u_0 + u_1) \frac{\partial(u_0 + u_1)}{\partial x} - u_0 \frac{\partial u_0}{\partial x} \right] \]  \hspace{1cm} (23)

\[ \vdots \]

By the above algorithms, we obtain:

\[ u_0(x, t) = x \]

\[ u_1(x, t) = -x(1 - a + at) \]

\[ u_2(x, t) = x(2a^2 - 4a + a^2t^2 - 4a^2t + 4at + 2) \]  \hspace{1cm} (24)

\[ \vdots \]

and so on.

Therefore, the series solution \( u(x, t) \) of Eq.(21) is given by

\[ u(x, t) = x - x(1 - a + at) + x(2a^2 - 4a + a^2t^2 - 4a^2t + 4at + 2) - \cdots \]  \hspace{1cm} (25)

The SADM is then used as a further step:

We get Eq.(21), by applying the Sumudu transform on both sides.

\[ S \left\{ c_F^0 D_t^a u \right\} = S \left\{ \frac{\partial^2 u}{\partial x^2} \right\} - S \left\{ u \frac{\partial u}{\partial x} \right\} \]  \hspace{1cm} (26)

By using the inverse Elzaki transform to both sides of Eq.(19), we obtain
\[ u = S^{-1}\left\{ (1 - \alpha + \alpha \omega) \left[ \mathcal{S}\left\{ \sum_{n=0}^{\infty} A_n \right\} - \mathcal{S}\left\{ \sum_{n=0}^{\infty} u_n \right\} \right] \right\} \] 

As a result, the approximate solution may be derived using Eq.(15)

\[ u_0(x, t) = u(x, 0) \]
\[ u_1(x, t) = S^{-1}\left\{ (1 - \alpha + \alpha \omega) \left[ \mathcal{S}\{A_0\} - \mathcal{S}\left\{ \sum_{n=0}^{\infty} A_n \right\} \right] \right\} \]
\[ u_2(x, t) = S^{-1}\left\{ (1 - \alpha + \alpha \omega) \left[ \mathcal{S}\{A_1\} - \mathcal{S}\left\{ \sum_{n=0}^{\infty} u_n \right\} \right] \right\} \]

By the above algorithms, we obtain:

\[ u_0(x, t) = x \]
\[ u_1(x, t) = -x(1 - \alpha + \alpha t) \]
\[ u_2(x, t) = x \left( 2\alpha^2 - 4\alpha + \alpha^2 t^2 - 4\alpha^2 t + 4\alpha t + 2 \right) \] (28)

and so on.

Therefore, the series solution \( u(x, t) \) of Eq.(17) is given by

\[ u(x, t) = x - x(1 - \alpha + \alpha t) + x \left( 2\alpha^2 - 4\alpha + \alpha^2 t^2 - 4\alpha^2 t + 4\alpha t + 2 \right) - \cdots \] (29)

If we put \( \alpha \to 1 \) in Eq.(22), we get the exact solution

\[ u(x, t) = x - xt + xt^2 - \cdots = x \sum_{k=0}^{\infty} (-t)^k = \frac{x}{1 + t} \] (30)

From Eq.(25) and Eq.(30), The outcome of utilizing DJM to approximate the solution of the given issue Eq.(21), is the same as that achieved using SDM.

If we put \( \alpha \to 1 \) in Eq.(25) and Eq.(30), we get the exact solution

\[ u(x, t) = x - xt + xt^2 - \cdots = x \sum_{k=0}^{\infty} (-t)^k = \frac{x}{1 + t} \] (31)

**Example 2.** Consider the following heat-like equation in the Caputo-Fabrizio sense

\[ c_F^{\alpha} u(x, y, t) = \frac{\partial^2 u(x, y, t)}{\partial x^2} + \frac{\partial^2 u(x, y, t)}{\partial y^2}, \quad 0 < \alpha \leq 1 \] (32)

where \( 0 \leq x, y \leq 2\pi \), \( t > 0 \), with the initial condition \( u(x, y, 0) = \sin(x)\sin(y) \).

First step by using DJM:

Taking the integral of Caputo-Fabrizio to both sides of Eq.(32), we obtain
\[ u(x, y, t) = u(x, y, 0) + c_F \int_0^t \left[ \frac{\partial^2 u(x, y, t)}{\partial x^2} + \frac{\partial^2 u(x, y, t)}{\partial y^2} \right] \] \tag{33}

As a result of Eq.(10), the following approximate solution may be obtained:

\[ u_1(x, y, t) = u(x, y, 0) \]
\[ u_1(x, y, t) = c_F \int_0^t \left[ \frac{\partial^2 u_0(x, y, t)}{\partial x^2} + \frac{\partial^2 u_0(x, y, t)}{\partial y^2} \right] \]
\[ u_2(x, y, t) = c_F \int_0^t \left[ \frac{\partial^2 u_1(x, y, t)}{\partial x^2} + \frac{\partial^2 u_1(x, y, t)}{\partial y^2} \right] \]

By the above algorithms, we obtain:

\[ u_0(x, y, t) = \sin(x) \sin(y) \]
\[ u_1(x, y, t) = -2(1 - a + at) \sin(x) \sin(y) \]
\[ u_2(x, y, t) = 4\sin(x) \sin(y) \left[ (1 - 2a + a^2) + (2a - 2a^2)t + \frac{1}{2} a^2 t^2 \right] \tag{34} \]

and so on.

Therefore, the series solution \( u(x, y, t) \) of Eq.(32) is given by

\[ u(x, y, t) = \sin(x) \sin(y) - 2(1 - a + at) \sin(x) \sin(y) + 4\sin(x) \sin(y) \left[ (1 - 2a + a^2) + (2a - 2a^2)t + \frac{1}{2} a^2 t^2 \right] \tag{35} \]

Second step by using SDM:

We've achieved this by using the Sumudu transform differentiation feature.

\[ S\left\{ \frac{c_F D_t^a u(x, y, t)}{(1 - a + aw)} \right\} = (1 - a + aw) \left( S \left\{ \frac{\partial^2 u(x, y, t)}{\partial x^2} \right\} + S \left\{ \frac{\partial^2 u(x, y, t)}{\partial y^2} \right\} \right), \tag{36} \]

by using the inverse Sumudu transform to both sides of (36), we obtain

\[ u(x, y, t) = u(x, y, 0) + S^{-1} \left\{ (1 - a + aw) \left( S \left\{ \frac{\partial^2 u_0(x, y, t)}{\partial x^2} \right\} + S \left\{ \frac{\partial^2 u_0(x, y, t)}{\partial y^2} \right\} \right) \right\}, \tag{37} \]

Hence, from Eq.(25) and Eq.(28), we give the components as follows:

\[ u_0(x, y, t) = \sin(x) \sin(y) \]
\[ u_1(x, y, t) = S^{-1} \left\{ (1 - a + aw) \left( S \left\{ \frac{\partial^2 u_0(x, y, t)}{\partial x^2} \right\} + S \left\{ \frac{\partial^2 u_0(x, y, t)}{\partial y^2} \right\} \right) \right\} \]
\[ = -2 \sin(x) \sin(y) (1 - a + at) \]
\[ u_2(x, y, t) = S^{-1} \left\{ (1 - a + aw) \left( S \left\{ \frac{\partial^2 u_0(x, y, t)}{\partial x^2} \right\} + S \left\{ \frac{\partial^2 u_0(x, y, t)}{\partial y^2} \right\} \right) \right\} \]
\[ u(x, y, t) = \sin(x) \sin(y) (1 - 2a + a^2) + (2a - 2a^2)t + \frac{1}{2}a^2 t^2 \]

and so on.

Therefore, the approximate solution \( u(x, t) \) of Eq.(29) is given by

\[ u(x, y, t) = \sin(x) \sin(y) - 2 \sin(x) \sin(y) (1 - a + at) + 4 \sin(x) \sin(y) \left[(1 - 2a + a^2) + (2a - 2a^2)t + \frac{1}{2}a^2 t^2\right] + \cdots \quad (38) \]

From Eq.(35) and Eq.(38), The outcome of utilizing DJM to approximate the solution of the given issue Eq.(21), is the same as that achieved using SDM.

If we put \( a \rightarrow 1 \) in Eq.(35) and Eq.(38), we get the exact solution

\[ u(x, y, t) = \sin(x) \sin(y) (1 - 2t + \frac{(2t)^2}{2!} - \cdots = \sin(x) \sin(y) e^{-2t} \quad (39) \]

6 Conclusion:

The DJM and SDM have been used to solve the nonlinear Burgers equation, heat-like equation, and linked nonlinear fractional Burger's equation with Caputo-Fabrizio fractional operator analytically. The examples indicate that DJM's findings are quite similar to those of SDM, and that all of the computations may be done with simple manipulations. In applied research, these strategies may be used to tackle a variety of linear and nonlinear fractional issues.

References:


