DOI: http://doi.org/10.32792/utq.jceps.12.02.04

Approximate Methods For Solving Fractional Differential Equations

Mohammed A. Hussein*

Educational Directorate of Thi-Qar , Nasiriyah, Iraq

Educational Directorate of Thi-Qar , Alayen University, Nasiriyah, Iraq.

Received 17/4/2022 Accepted 20/6/2022 Published 01/12/2022

Abstract:

In this paper, exact and approximate solutions of the nonlinear Burgers equation, heat-like equation and coupled nonlinear fractional Burger's equations with Caputo-Fabrizio fractional operator by using Daftardar-Jafari method (DJM) and Sumudu decomposition method (SDM) are presented and discussed. The solutions of our equations are calculated in the form of rabidly convergent series with easily computable components. Three illustrative applications are given to demonstrate the effectiveness and the leverage of the present methods. Graphical results are utilized and discussed quantitatively to illustrate the solution. The results reveal that the methods are very effective and simple in determination of solution of the fractional partial differential equations.

Keywords: Daftardar-Jafari method; nonlinear Burger equation; heat-like equation, Caputo-Fabrizio fractional operator.

1 INTRODUCTION:

Fractional calculus has become increasingly popular in the scientific world in recent decades as a result of its relevance in solving a variety of physical, engineering, thermodynamics, and other scientific challenges. Nonlinear differential equations are used to describe the majority of natural occurrences. As a result, scientists from diverse fields of study attempted to tackle the problem. However, because of their non-linear nature of them, Finding a precise solution to a collection of equations is difficult. [1,3,4-7,10-14].

In the past decade, Caputo and Fabrizio introduced a new fractional differential operator and many researchers studied this operator and researchers are still interested in this operator because of its importance, as some studies have applied methods of approximate solutions to equations that include this fractional operator[2,9,15,16].

In this research, we use the DJM and SDM to solve nonlinear fractional partial differential equations with the Caputo-Fabrizio fractional operator. The following is how the paper is structured: Section 2 contains the fundamental concepts of fractional calculus, Section 3 and Section 4 provide an examination of the techniques utilized, Section 5 has various test problems that demonstrate the usefulness of the suggested method, and Section 6 contains the conclusion.

2 BASIC DEFINITIONS:

Definition 1. [2,9]. The Caputo-Fabrizio operator's fractional derivative for $0 < a \leq 1$ is defined as:

$$ {}^{\mathcal{CF}}\mathcal{D}_t^a u(t) = \frac{\varepsilon(a)}{1-a} \int^{t} \exp\left[-\frac{a(t-s)}{1-a}\right] u'(s)\mathrm{d}s \,, t \geq 0, \quad (1)$$

where $u'(s)$ is the derivative of u , and $\varepsilon(a)$ is a normalization function such that ε(0)= ε(1)=1.

The operator's fundamental attributes ${}^{\mathcal{CF}}\mathcal{D}_t^a$ are as follows [11-17].

1. ${}^{\mathcal{CF}}\mathcal{D}_t^a u(t) = u(t)$, where $a = 0$.
2. ${}^{\mathcal{CF}}\mathcal{D}_t^a [u(t) + v(t)] = {}^{\mathcal{CF}}\mathcal{D}_t^a u(t) + {}^{\mathcal{CF}}\mathcal{D}_t^a v(t)$
3. ${}^{\mathcal{CF}}\mathcal{D}_t^a (c) = 0$, when c is a constant.

Definition 2. [15,16]. The fractional integral operator of order $0 < a \leq 1$ of Caputo-Fabrizio and $t > 0$ is given by:

$$ {}^{\mathcal{CF}}\mathcal{J}_t^a u(t) = \frac{1-a}{\mathcal{E}(a)} u(t) + \frac{a}{\mathcal{E}(a)} \int_0^t u(s)ds \quad (2)$$

The operator's fundamental attributes ${}^{\mathcal{CF}}\mathcal{J}_t^a$ are as follows:

1. ${}^{\mathcal{CF}}\mathcal{J}_t^a [u(t)] = u(t)$, where $a = 0$.
2. ${}^{\mathcal{CF}}\mathcal{J}_t^a [u(t) + v(t)] = {}^{\mathcal{CF}}\mathcal{J}_t^a u(t) + {}^{\mathcal{CF}}\mathcal{J}_t^a v(t)$
3. ${}^{\mathcal{CF}}\mathcal{J}_t^a \left({}^{\mathcal{CF}}\mathcal{D}_t^a u(t)\right) = u(t) - u(0)$,

Definition 3. [8] Over a set of functions $\mathcal{A}$, the Sumudu transform is defined,

$$\mathcal{A} = \left\{ u(t) \middle| \exists \mathcal{M}, \tau_1, \tau_2 > 0, |u(t)| < \mathcal{M} e^{\frac{|t|}{\tau_j}}, if\ t \in (-1)^j \times [0,\infty) \right\},$$

by the following formula

$$\mathcal{S}\{u(t)\} = \int_0^\infty u(wt)e^{-t}dt\,,\ \ w \in (\tau_1, \tau_2). \quad (3)$$

3 ANALYSIS OF DJM:

In the Caputo-Fabrizio operator sense, consider the following nonlinear partial differential equation:

$$ {}_{0}^{CF}\mathcal{D}_t^a\, u(x,t) + \mathcal{R}[u(x,t)] + \mathcal{N}\,[u(x,t)] = g(x,t), \quad (4)$$

with initial condition $\frac{\partial u(x,0)}{\partial t} = u_0(x)$,

where ${}_{0}^{CF}\mathcal{D}_t^a\, u(x,t)$ is Caputo-Fabrizio operator of $u(x,t)$, a linear operator is $\mathcal{R}$, a nonlinear operator is $\mathcal{N}$ and a source term is g.

When we apply the Caput-Fabrizio integral to both sides of Eq.(4), we get

$$ {}^{CF}\mathcal{J}_t^a\left[{}_{0}^{CF}\mathcal{D}_t^a\, u(x,t)\right] + {}^{CF}\mathcal{J}_t^a\mathcal{R}[u(x,t)] + {}^{CF}\mathcal{J}_t^a\mathcal{N}[u(x,t)] = {}^{CF}\mathcal{J}_t^a[g(x,t)], \quad (5)$$

By integral properties of Caputo-Fabrizio , we get

$$ u(x,t) = u_0(x) + {}^{CF}\mathcal{J}_t^a[g(x,t)] - {}^{CF}\mathcal{J}_t^a\mathcal{R}[u(x,t)] - {}^{CF}\mathcal{J}_t^a\mathcal{N}[u(x,t)], \quad (6)$$

We're seeking for u(x,t) solution to Eq.(6) with the following series form:

$$ u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) \quad (7)$$

It is possible to decompose the nonlinear operator N as follows:

$$ \mathcal{N}\left[\sum_{n=0}^{\infty} u_n(x,t)\right] = \mathcal{N}[u_0] + \sum_{n=1}^{\infty}\left(\mathcal{N}\left[\sum_{i=0}^{n} u_i\right] - \mathcal{N}\left[\sum_{i=0}^{n-1} u_i\right]\right) \quad (8)$$

In view of Eq.(6) and Eq.(7), Eq.(8) is equivalent to

$$ \sum_{n=0}^{\infty} u_n(x,t) = u_0(x) + {}^{CF}\mathcal{J}_t^a[g(x,t)] - {}^{CF}\mathcal{J}_t^a\mathcal{R}\left[\sum_{n=0}^{\infty} u_i\right] - {}^{CF}\mathcal{J}_t^a\mathcal{N}[u_0]$$
$$ - {}^{CF}\mathcal{J}_t^a\left(\sum_{n=1}^{\infty}\left(\mathcal{N}\left[\sum_{i=0}^{n} u_i\right] - \mathcal{N}\left[\sum_{i=0}^{n-1} u_i\right]\right)\right). \quad (9)$$

Furthermore, the relationship is defined by recurrence, thus

$$ u_0(x,t) = u_0(x) + {}^{CF}\mathcal{J}_t^a[g(x,t)]$$
$$ u_1(x,t) = -{}^{CF}\mathcal{J}_t^a\mathcal{R}[u_0] - {}^{CF}\mathcal{J}_t^a\mathcal{N}[u_0]$$
$$ u_n(x,t) = -{}^{CF}\mathcal{J}_t^a\mathcal{R}[u_n] - {}^{CF}\mathcal{J}_t^a\left(\mathcal{N}\left[\sum_{i=0}^{n} u_i\right] - \mathcal{N}\left[\sum_{i=0}^{n-1} u_i\right]\right).\ , n = 1,2,\ldots \quad (10)$$

The following is the n-term approximate solution of Eq.(4):

$$ u(x,t) = u_0(x,t) + u_1(x,t) + u_3(x,t) + \cdots \quad (11)$$

4 ANALYSIS OF SDM:

When you apply the Sumudu transform to Eq.(4),

$$ \mathcal{S}\{{}_{0}^{CF}\mathcal{D}_t^a\, u(x,t)\} + \mathcal{S}\{\mathcal{R}[u(x,t)]\} + \mathcal{S}\{\mathcal{N}\,[u(x,t)]\} = \mathcal{S}\{g(x,t)\}. \quad (12)$$

If the Sumudu transform differentiation property is used, then

$$\mathcal{S}\{u(x,t)\} = u_0(x) + (1-a+aw)\mathcal{S}\{g(x,t)\} - (1-a+aw)\mathcal{S}\{\mathcal{R}[u(x,t)] + \mathcal{N}[u(x,t)]\} \quad (13)$$

Taking the inverse of Sumudu transform for both sides of Eq.(13), we have

$$u(x,t) = \mathcal{S}^{-1}\big(u_0(x)\big) + \mathcal{S}^{-1}\big((1-a+aw)\mathcal{S}\{g(x,t)\}\big) - \mathcal{S}^{-1}\big((1-a+aw)\mathcal{S}\{\mathcal{R}[u(x,t)]\}\big) - \mathcal{S}^{-1}\big((1-a+aw)\mathcal{S}\{\mathcal{N}[u(x,t)]\}\big). \quad (14)$$

Assume that u(x,t) is a solution to Eq.(14) which can be rewritten as

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t), \quad (15)$$

It is possible to decompose the nonlinear term as follows:

$$\mathcal{N}\,[u(x,t)] = \sum_{n=0}^{\infty} \mathcal{A}_n, \quad (16)$$

where

$$\mathcal{A}_n = \frac{1}{n!}\frac{\partial^n}{\partial \zeta^n}\left[\mathcal{N}\left(\sum_{i=0}^{\infty} \zeta^i u_i(x,t)\right)\right]_{\zeta=0} \quad , n = 0,1,2,\dots \quad (17)$$

We get the following result by substituting Eq.(15) and Eq.(16) into Eq.(14):

$$u(x,t) = \mathcal{S}^{-1}\big(u_0(x)\big) + \mathcal{S}^{-1}\big((1-a+aw)\mathcal{S}\{g(x,t)\}\big) - \mathcal{S}^{-1}\left((1-a+aw)\mathcal{S}\left\{\mathcal{R}\left[\sum_{n=0}^{\infty} u_n(x,t)\right]\right\}\right) - \mathcal{S}^{-1}\left(\mathcal{N}(1-a+aw)\mathcal{S}\left\{\sum_{n=0}^{\infty} \mathcal{A}_n\right\}\right). \quad (18)$$

By comparing the left and right sides of Eq.(18), we get

$$u_0(x,t) = \mathcal{S}^{-1}\big(u_0(x)\big) + \mathcal{S}^{-1}\big((1-a+aw)\mathcal{S}\{g(x,t)\}\big)$$
$$u_1(x,t) = -\mathcal{S}^{-1}\big((1-a+aw)\mathcal{S}\{\mathcal{R}[u_0(x,t)]\}\big) - \mathcal{S}^{-1}\big(\mathcal{N}(1-a+aw)\mathcal{S}\{\mathcal{A}_0\}\big)$$
$$u_2(x,t) = -\mathcal{S}^{-1}\big((1-a+aw)\mathcal{S}\{\mathcal{R}[u_1(x,t)]\}\big) - \mathcal{S}^{-1}\big(\mathcal{N}(1-a+aw)\mathcal{S}\{\mathcal{A}_1\}\big)$$
$$\vdots$$

The general form of recursive relation is

$$u_0(x,t) = \mathcal{S}^{-1}\big(u_0(x)\big) + \mathcal{S}^{-1}\big((1-a+aw)\mathcal{S}\{g(x,t)\}\big)$$

$$u_{n+1}(x,t) = -\mathcal{S}^{-1}\big((1-a+aw)\mathcal{S}\{\mathcal{R}[u_n(x,t)]\}\big) - \mathcal{S}^{-1}\big(\mathcal{N}(1-a+aw)\mathcal{S}\{\mathcal{A}_n\}\big) \quad (19)$$

The approximate solution is given by

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_3(x,t) + \cdots \tag{20}$$

5 Illustrative examples:

Example 1. Consider the following, Burger equation which is nonlinear in the Caputo-Fabrizio sense

$${}^{CF}_{0}\mathcal{D}_t^a\, u + u\frac{\partial u}{\partial x} = \frac{\partial^2 u}{\partial x^2} \quad , \quad 0 < a \leq 1 \tag{21}$$

subject to the initial condition $u(x,0) = x$

Below we present the DJM.

Taking the integral of Caputo-Fabrizio to both sides of Eq.(21), we obtain

$$u(x,t) = u(x,0) + {}^{CF}\mathcal{J}_t^a\left[\frac{\partial^2 u}{\partial x^2}\right] - {}^{CF}\mathcal{J}_t^a\left[u\frac{\partial u}{\partial x}\right] \tag{22}$$

As a result of Eq.(10), we may get the following approximate solution:

$$u_0(x,t) = u(x,0)$$

$$u_1(x,t) = {}^{CF}\mathcal{J}_t^a\left[\frac{\partial^2 u_0}{\partial x^2}\right] - {}^{CF}\mathcal{J}_t^a\left[u_0\frac{\partial u_0}{\partial x}\right]$$

$$u_2(x,t) = {}^{CF}\mathcal{J}_t^a\left[\frac{\partial^2 u_1}{\partial x^2}\right] - {}^{CF}\mathcal{J}_t^a\left[(u_0+u_1)\frac{\partial(u_0+u_1)}{\partial x} - u_0\frac{\partial u_0}{\partial x}\right] \tag{23}$$

$$\vdots$$

By the above algorithms, we obtain:

$$u_0(x,t) = x$$

$$u_1(x,t) = -x(1 - a + at)$$

$$u_2(x,t) = x(2a^2 - 4a + a^2t^2 - 4a^2t + 4at + 2) \tag{24}$$

$$\vdots$$

and so on.

Thererfore, the series solution $u(x,t)$ of Eq.(21) is given by

$$u(x,t) = x - x(1 - a + at) + x(2a^2 - 4a + a^2t^2 - 4a^2t + 4at + 2) - \cdots \tag{25}$$

The SADM is then used as a further step:

We get Eq.(21), by applying the Sumudu transform on both sides.

$$\mathcal{S}\{{}^{CF}_{0}\mathcal{D}_t^a\, u\} = \mathcal{S}\left\{\frac{\partial^2 u}{\partial x^2}\right\} - \mathcal{S}\left\{u\frac{\partial u}{\partial x}\right\} \tag{26}$$

By using the inverse Elzaki transform to both sides of Eq.(19), we obtain

$$u = \mathcal{S}^{-1}\left\{(1-a+aw)\left[\mathcal{S}\left\{\frac{\partial^2 \sum_{n=0}^{\infty} u_n}{\partial x^2}\right\} - \mathcal{S}\left\{\sum_{n=0}^{\infty} \mathcal{A}_n\right\}\right]\right\} \quad (27)$$

As a result, the approximate solution may be derived using Eq.(15)

$$u_0(x,t) = u(x,0)$$

$$u_1(x,t) = \mathcal{S}^{-1}\left\{(1-a+aw)\left[\mathcal{S}\{\mathcal{A}_0\} - \mathcal{S}\left\{\frac{\partial^2 u_0}{\partial x^2}\right\}\right]\right\}$$

$$u_2(x,t) = \mathcal{S}^{-1}\left\{(1-a+aw)\left[\mathcal{S}\{\mathcal{A}_1\} - \mathcal{S}\left\{\frac{\partial^2 u_1}{\partial x^2}\right\}\right]\right\}$$

By the above algorithms, we obtain:

$$u_0(x,t) = x$$
$$u_1(x,t) = -x(1-a+at)$$
$$u_2(x,t) = x\,(2a^2 - 4a + a^2 t^2 - 4a^2 t + 4at + 2) \quad (28)$$

and so on.

Therefore, the series solution $u(x,t)$ of Eq.(17) is given by

$$u(x,t) = x - x(1-a+at) + x\,(2a^2 - 4a + a^2 t^2 - 4a^2 t + 4at + 2) - \cdots \quad (29)$$

If we put $a \to 1$ in Eq.(22) , we get the exact solution

$$u(x,t) = x - xt + xt^2 - \cdots = x\sum_{k=0}^{\infty}(-t)^k = \frac{x}{1+t} \quad (30)$$

From Eq.(25) and Eq.(30), The outcome of utilizing DJM to approximate the solution of the given issue Eq.(21), is the same as that achieved using SDM.

If we put $a \to 1$ in Eq.(25) and Eq.(30), we get the exact solution

$$u(x,t) = x - xt + xt^2 - \cdots = x\sum_{k=0}^{\infty}(-t)^k = \frac{x}{1+t} \quad (31)$$

Example 2. Consider the following heat-like equation in the Caputo-Fabrizio sense

$${}^{\mathcal{CF}}_{0}\mathcal{D}_t^{a}\, u(x,y,t) = \frac{\partial^2 u(x,y,t)}{\partial x^2} + \frac{\partial^2 u(x,y,t)}{\partial y^2} \ , \quad 0 < a \le 1 \quad (32)$$

where $0 \le x, y \le 2\pi$, $t > 0$, with the initial condition $u(x,y,0) = sin(x)sin(y)$.

First step by using DJM:

Taking the integral of Caputo-Fabrizio to both sides of Eq.(32), we obtain

$$u(x,y,t) = u(x,y,0) + {}^{CF}\mathcal{J}_t^a\left[\frac{\partial^2 u(x,y,t)}{\partial x^2} + \frac{\partial^2 u(x,y,t)}{\partial y^2}\right] \quad (33)$$

As a result of Eq.(10), the following approximate solution may be obtained:

$$u_1(x,y,t) = u(x,y,0)$$
$$u_1(x,y,t) = {}^{CF}\mathcal{J}_t^a\left[\frac{\partial^2 u_0(x,y,t)}{\partial x^2} + \frac{\partial^2 u_0(x,y,t)}{\partial y^2}\right]$$
$$u_2(x,y,t) = {}^{CF}\mathcal{J}_t^a\left[\frac{\partial^2 u_1(x,y,t)}{\partial x^2} + \frac{\partial^2 u_1(x,y,t)}{\partial y^2}\right]$$
$$\vdots$$

By the above algorithms, we obtain:

$$u_0(x,y,t) = sin(x)sin(y)$$
$$u_1(x,y,t) = -2(1-a+at)sin(x)sin(y)$$
$$u_2(x,y,t) = 4sin(x)sin(y)\left[(1-2a+a^2) + (2a-2a^2)t + \frac{1}{2}a^2t^2\right] \quad (34)$$

and so on.

Therefore, the series solution $u(x,y,t)$ of Eq.(32) is given by

$$u(x,y,t) = sin(x)sin(y) - 2(1-a+at)sin(x)sin(y) + 4sin(x)sin(y)\left[(1-2a+a^2) + (2a-2a^2)t + \tfrac{1}{2}a^2t^2\right] \quad (35)$$

Second step by using SDM:
We've achieved this by using the Sumudu transform differentiation feature.

$$\mathcal{S}\{{}^{CF}_{0}\mathcal{D}_t^a\, u(x,y,t)\} = (1-a+aw)\left[\mathcal{S}\left\{\frac{\partial^2 u(x,y,t)}{\partial x^2}\right\} + \mathcal{S}\left\{\frac{\partial^2 u(x,y,t)}{\partial y^2}\right\}\right], \quad (36)$$

by using the inverse Sumudu transform to both sides of (36), we obtain

$$u(x,y,t) = u(x,y,0) + \mathcal{S}^{-1}\left\{(1-a+aw)\left[\mathcal{S}\left\{\frac{\partial^2 u(x,y,t)}{\partial x^2}\right\} + \mathcal{S}\left\{\frac{\partial^2 u(x,y,t)}{\partial y^2}\right\}\right]\right\}, \quad (37)$$

Hence, from Eq.(25) and Eq.(28), we give the components as follows:

$$u_0(x,y,t) = \sin(x)\sin(y)$$
$$u_1(x,y,t) = \mathcal{S}^{-1}\left\{(1-a+aw)\left[\mathcal{S}\left\{\frac{\partial^2 u_0(x,y,t)}{\partial x^2}\right\} + \mathcal{S}\left\{\frac{\partial^2 u_0(x,y,t)}{\partial y^2}\right\}\right]\right\}$$
$$= -2\sin(x)\sin(y)(1-a+at)$$
$$u_2(x,y,t) = \mathcal{S}^{-1}\left\{(1-a+aw)\left[\mathcal{S}\left\{\frac{\partial^2 u_0(x,y,t)}{\partial x^2}\right\} + \mathcal{S}\left\{\frac{\partial^2 u_0(x,y,t)}{\partial y^2}\right\}\right]\right\}$$

$$= 4\sin(x)\sin(y)\left[(1-2a+a^2)+(2a-2a^2)t+\frac{1}{2}a^2t^2\right]$$

$$\vdots$$

and so on.

Therefore, the approximate solution $u(x,t)$ of Eq.(29) is given by

$$u(x,y,t) = \sin(x)\sin(y) - 2\sin(x)\sin(y)(1-a+at) + 4\sin(x)\sin(y)\left[(1-2a+a^2)+(2a-2a^2)t+\tfrac{1}{2}a^2t^2\right]+\cdots \quad (38)$$

From Eq.(35) and Eq.(38), The outcome of utilizing DJM to approximate the solution of the given issue Eq.(21), is the same as that achieved using SDM.

If we put $a \to 1$ in Eq.(35) and Eq.(38), we get the exact solution

$$u(x,y,t) = \sin(x)\sin(y)\left(1-2t+\frac{(2t)^2}{2!}-\cdots\right) = \sin(x)\sin(y)\,e^{-2t} \quad (39)$$

6 Conclusions:

The DJM and SDM have been used to solve the nonlinear Burgers equation, heat-like equation, and linked nonlinear fractional Burger's equation with Caputo-Fabrizio fractional operator analytically. The examples indicate that DJM's findings are quite similar to those of SDM, and that all of the computations may be done with simple manipulations. In applied research, these strategies may be used to tackle a variety of linear and nonlinear fractional issues.

References:

[1] G. Aguilar et al., Homotopy perturbation transform method for nonlinear differential equations involving to fractional operator with exponential kernel, Advances in Difference Equations (2017) 2017:68

[2] M. Al-Refai, K. Pal, New Aspects of Caputo-Fabrizio Fractional Derivative, Progress in Fractional Differentiation and Applications, 5(2) (2019) 157-166 .

[3] D. Baleanu, et al., Approximate Analytical Solutions of Goursat Problem within Local Fractional Operators, Journal of Nonlinear Science and Applications, 9 (2016) 4829-4837.

[4] D. Baleanu, H. K. Jassim, A Modification Fractional Homotopy Perturbation Method for Solving Helmholtz and Coupled Helmholtz Equations on Cantor Sets, Fractal and Fractional, 3(30) (2019) 1-8.

[5] D. Baleanu, et al., Solving Helmholtz Equation with Local Fractional Derivative Operators, Fractal and Fractional, 3(43) (2019) 1-13.

[6] D. Baleanu, et al., Approximate Solutions of the Damped Wave Equation and Dissipative Wave Equation in Fractal Strings, Fractal and Fractional, 3(26) (2019) 1-12.

[7] D. Baleanu, et al., A Modification Fractional Variational Iteration Method for solving Nonlinear Gas Dynamic and Coupled KdV Equations Involving Local Fractional Operators, Thermal Science, 22(1) (2018) S165-S175.

[8] Belgacem, Fethi Bin Muhammed, and Ahmed Abdullatif Karaballi. Sumudu transform fundamental properties investigations and applications.(International Journal of Stochastic Analysis 2006 (2006).

[9] M. Caputo, M. Fabrizio, A new Definition of Fractional Derivative without Singular Kernel, Progress in Fractional Differentiation and Applications, 1(2)(2015) 73-85.

[10] V. Gill, K. Modi, and Y. Singh, Analytic solutions of fractional differential equation associated with RLC electrical circuit, Journal of Statistics and Management Systems, 21(4) (2018) 575-582.

[11] V. G. Gupta, P. Kumar, Approximate solutions of fractional linear and nonlinear differential equations using Laplace homotopy analysis method. Int. J. Nonlinear Sci. 19(2)(2015) 113-120

[12] H. Jafari, M. Ghorbani, S. Ghasempour, A note on exact solutions for nonlinear integral equations by a modified homotopy perturbation method. New Trends Math. Sci. 1(2), 22-26 (2013)

[13] H. Jafari, H. K. Jassim, J. Vahidi, Reduced Differential Transform and Variational Iteration Methods for 3D Diffusion Model in Fractal Heat Transfer within Local Fractional Operators, Thermal Science, 22 (2018) S301-S307.

[14] H. Jafari, H. K. Jassim, S. P. Moshokoa, V. M. Ariyan and F. Tchier, Reduced differential transform method for partial differential equations within local fractional derivative operators, Advances in Mechanical Engineering, 8 (2016) 1-6.

[15] A. Shaikh et al. Analysis of differential equations involving Caputo-Fabrizio fractional operator and its applications to reaction-diffusion equations Advances in Difference Equations (2019) 2019:178

[16] H. K. Jassim, M. A. Hussain, On approximate solutions for fractional system of differential equations with Caputo-Fabrizio fractional operator, Journal of Mathematics and Computer Science, 23 (2021), no. 1, 58--66