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On Best Multiplier Approximation of k-Monotone of $f \in L_{p,\lambda_n}[-\pi, \pi]$

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Abstrac:

The aim of this paper is to obtain the degree of the best multiplier approximation of monotone unbounded periodic functions in L_{p,λ_n} −space.

Keywords: Multiplier Integral, Multiplier Averaged Modulus of Smoothness, Multiplier Norm.

Introduction:

There are several researchers and specialists have worked in the field of approximation theory and have obtained results.For example; In 1982 [1] , V.A.Popov studied and got several results about one-sided approximation of periodic function. Also, in 1989 [2] , V.H.Hristove, obtained some results of best one-sided approximation by interpolating polynomial of periodic functions. In 2004 [3] N.M.Kassim had studied the monotone and comonotone approximation. In 2004 [4] L.Leindler analyzed the topic regarding the degree of approximation and got results. In 2014 [5] S.K.Jassim and Zoboon had studied the approximation of unbounded functions by utilizing the trigonometric polynomials in locally-Global space $L_{p,\delta,\omega}$.

In this work, the Jackson polynomial will be utilized to study and analyze the degree of the best multiplier approximation of monotone unbounded periodic functions in L_{p,λ_n} −space.

1. DEFINITONS and CONCEPTS:

Definition (1.1) [7] (A **Multiplier Convergence)**

A series $\sum_{n=0}^{\infty} a_n$ is called a multiplier convergent series if there is a convergent sequence of real numbers $\{\lambda_n\}_{n=0}^{\infty}$ such that $\sum_{n=0}^{\infty} a_n \lambda_n \leq \infty$ and $\{\lambda_n\}_{n=0}^{\infty}$ is called multiplier for the convergence.

Definition (1.2) (Multiplier Intergal)

For any real valued function $f \in L_{p,\lambda_n}(X)$, where $X = [-\pi, \pi]$,if there is a sequence $\{\lambda_n\}_{n=0}^{\infty}$,such that:

$$\int_{-\pi}^{\pi} f(x)\lambda_n dx < \infty, \quad (1.1)$$

then f is called a Multiplier integrable function, λ_n, is called a Multiplier integrable sequence.

Definition (1.3) (The Multiplier Norm)

Let $f \in L_{p,\lambda_n}(X)$, where $X = [-\pi, \pi]$ then: $\| f \|_{p,\lambda_n}$, is given by the below definite integral:

$$\| f \|_{p,\lambda_n} = [\int_{-\pi}^{\pi} |(\lambda_n f)(x)|^p dx]^{\frac{1}{p}}. \quad (1.2)$$

Definition (1.4) [6]

Let $f \in L_p[a,b]$, where $1 \le p \le \infty$, then the integral modulus(L_p -modulus or p -modulus) of order k of the function f is the following function of $\delta \in [0,(b-a)/k]$:

$$\omega_k(f;\delta)_{L_p} = \sup_{0 \le h \le \delta} \{ \int_a^{b-kh} \left|\Delta_h^k f(x)\right|^p dx \}^{1/p} \quad (1.3)$$

Definition (1.5)

Let $f \in L_{p,\lambda_n}(X)$, where $X = [-\pi, \pi]$, $1 \le p < \infty$, then:

The Multiplier integral modulus of order k of the function f where $0 \le \delta \le b - ak$, is defined by:

$$\omega_k(f,\delta)_{p,\lambda_n} = \sup_{h \in [0,\delta]} (\int_a^{b-kh} | \triangle_h^k (\lambda_n f)(x)|^p dx)^{\frac{1}{p}}, \quad (1.4)$$

where

$$\triangle_h^k (\lambda_n f)(x) = \sum_{m=i}^{k} (-1)^{m+k} \binom{k}{m} (\lambda_n f)(x+mh); \binom{k}{m} = \frac{k!}{m!(k-m)!}. \quad (1.5)$$

Definition (1.6) [6]

Let $f \in L_p(X)$; where $X = [a,b]$ and $1 \le p \le \infty$. The local modulus of smoothness of the function f of order k at a point $x \in [a,b]$ is the following function of $\delta \in [0,(b-a)/k]$:

$$\omega_k(f,x;\delta) = \sup\{\left|\Delta_h^k f(t)\right| : t, t+kh \in [x - \frac{k\delta}{2}, x + \frac{k\delta}{2}] \cap [a,b]\} \quad (1.6)$$

Definition (1.7)

If $f \in L_{p,\lambda_n}(X)$, where $X = [-\pi, \pi]$, $1 \le p < \infty$, then the multiplier local modulus of smoothness of a function f of order k at a point$x \in [a,b]$,

$$0 \le \delta \le \frac{b-a}{k},$$

is defined by:

$$\omega_k(f,x,\delta)_{p,\lambda_n} = \sup_{h \in [0,\delta]} \{\triangle_h^k (\lambda_n f)(t): t, t+kh \in [x - \tfrac{k\delta}{2}, x + \tfrac{k\delta}{2}] \cap [a,b]\}. \quad (1.7)$$

Definition (1.8) [6]

The averaged modulus of smoothness of order k (or τ -modulus) of the function $f \in M[a,b]$ is the following function of $\delta \in [0,(b-a)/k]$:

$$\tau_k(f;\delta)_p = \|\omega_k(f,.,;\delta)\|_{L_p} = [\int_a^b (\omega_k(f,x;\delta))^p dx]^{1/p} \quad (1.8)$$

Definition (1.9)

If $f \in L_{p,\lambda_n}(X)$, where $X = [-\pi,\pi]$,$1 \le p < \infty$, then the multiplier averaged modulus of smoothness of order k of $f \in L_{p,\lambda_n}(X)$, where $X = [-\pi,\pi]$, is defined by:

$$\tau_k(f,\delta)_{p,\lambda_n} = \| \omega_k(f,.,\delta) \|_{p,\lambda_n} = (\int_a^b [\omega_k(\lambda_n f,x,\delta)]^p dx)^{\frac{1}{p}}. (1.9)$$

Definition (1.10)[6]

If $f \in L_p(X), X = [a,b]$, then:

$$E_n(f)_p = \inf\{\|f-P_n\|_p : P_n \in P\} \quad (1.10)$$

Such that $E_n(f)_p$ is called the degree of the best monotone multiplier approximationof f by polynomial P_n.

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Definition (1.11)

If $f \in L_{p,\lambda_n}(X), X = [-\pi,\pi]$, then:

$$E_n(f)_{p,\lambda_n} = \inf\{\|f-S_n\|_{p,\lambda_n} : S_n \in P\} \quad (1.11)$$

Such that $E_n(f)_{p,\lambda_n}$ is called the degree of the bestmonotone multiplier approximationof f by polynomial S_n.

Definition (1.12) [6]

If $f \in L_p(X)$, Then the best one-sided approximation of f by means of trigonometric polynomails of order n in $L_p(X)$ is given by:

$$\tilde{E}_n(f)_{p,\lambda_n} = \inf\{\|P-Q\|_{L_p} : P,Q \in T, Q(x) \le f(x) \le P(x); \forall x\} \quad (1.12)$$

Definition (1.13)

If $f \in L_p(X)$, $X = [-\pi,\pi]$, then:

$$\tilde{E}_n(f)_{p,\lambda_n} = \inf\{\|S_n - G_n\|_{P,\lambda_n} : S_n, G_n \in T, G_n(x) \le f(x) \le S_n(x); \forall x\} \quad (1.13)$$

Such that $\tilde{E}_n(f)_{p,\lambda_n}$ is called the degree of the best one-sided monotone multiplier approximation of f by polynomials S_n and G_n.

In the next section, significant lemmas will be proved.

2. AUXILIARY LEMMAS:

Let Jackson polynomial such that:

$$\varphi_{r,n}(t) = (n\sin\frac{\pi}{4n})^{2r}[\frac{\sin nt}{n\sin(t/2)}]^{2r};\ r,n\in\mathbb{N}, \qquad (2.1)$$

Thus, $\varphi_{r,n}(t)$ is trigonometric polynomial of degree $r(2n-1)$.

Lemma (2.1)

$\varphi_{r,n}(t)$ has the following properties:

(1) $\varphi_{r,n}(t)\geq 1$, for every $|t|\leq\frac{\pi}{2n}$.

(2) $\varphi_{r,n}(t)\leq c(r)$.

(3) $\sup\{\varphi_{r,n}(t): t\in[\frac{m\pi}{n},\frac{(m+1)\pi}{n}]\}\leq c(r)m^{-2r}; m=1,2,...,n-1$.

(4) $\sup\{\varphi_{r,n}(t): t\in[\frac{(m-1)\pi}{n},\frac{m\pi}{n}]\}\leq c(r)|m|^{-2r}; m=-n+1,-n+2,...,-1$.

(5) $\|\varphi_{r,n}(t)\|_{p,\lambda_n}\leq c(r)(\frac{1}{n})^{1/p}; 1\leq p<\infty; r>\frac{1}{p}$.

Where $\varphi_{r,n}(t) = (n\sin\frac{\pi}{4n})^{2r}[\frac{\sin nt}{n\sin(t/2)}]^{2r}$ for every $r,n\in\mathbb{N}$, is a trigonometric polynomial of degree $r(2n-1)$.

Proof:

To show that (1) holds, i.e., for every for every $|t|\leq\frac{\pi}{2n}$, we get $\varphi_{r,n}(t)\geq 1$, where

$$\varphi_{r,n}(t) = (n\sin\frac{\pi}{4n})^{2r}[\frac{\sin nt}{n\sin(t/2)}]^{2r};\ r,n\in\mathbb{N},$$

Let $t=\frac{\pi}{2n}$, then

$$\varphi_{r,n}(t) = (n\sin\frac{\pi}{4n})^{2r}[\frac{\sin n(\frac{\pi}{2n})}{n\sin(\frac{\pi/2n}{2})}]^{2r} = [n\sin\frac{\pi}{4n}]^{2r}\frac{[\sin(\pi/2)]^{2r}}{[n\sin\frac{\pi}{4n}]^{2r}} = [\sin(\pi/2)]^{2r} = 1 \qquad (2.2)$$

Then for $t=\frac{\pi}{2n}$, $\varphi_{r,n}(t)=1$. For $|t|<\frac{\pi}{2n}$, $t\in(-\frac{\pi}{2n},\frac{\pi}{2n})$, $\varphi_{r,n}(t)>1$. Which then yields that $\varphi_{r,n}(t)\geq 1$ for every $|t|\leq\frac{\pi}{2n}$. To show that (3) works for every t, it is sufficient to show that

$\sup\{\varphi_{r,n}(t): t\in[\frac{m\pi}{n},\frac{(m+1)\pi}{n}]\}\le c(r)m^{-2r}; m=1,2,...,n-1$. where $\varphi_{r,n}(t)=(n\sin\frac{\pi}{4n})^{2r}[\frac{\sin nt}{n\sin(t/2)}]^{2r}$; $r,n\in\mathbb{N}$.

Since $\sin nt\le 1$, then

$$\varphi_{r,n}(t)\le[n\sin(\frac{\pi}{4n})]^{2r}\frac{1}{[n\sin(t/2)]^{2r}}$$

Which then yields that $\varphi_{r,n}(t)\le c(r)\frac{1}{[n\sin(t/2)]^{2r}}$

Hence, we get: $\varphi_{r,n}(t)\le c(r)\frac{1}{n^{2r}\sin^{2r}(m\pi/2n)}$

For every $t\in[\frac{m\pi}{n},\frac{(m+1)\pi}{n}]$. Then

$$\varphi_{r,n}(t)\le c(r)\pi^{2r}\frac{1}{n^{2r}2^{2r}(\frac{m\pi}{2n})^{2r}}=c(r)\pi^{2r}\frac{1}{n^{2r}2^{2r}\frac{m^{2r}\pi^{2r}}{2^{2r}n^{2r}}}=c(r)\frac{1}{m^{2r}}=c(r)m^{-2r} \qquad (2.3)$$

Hence $\varphi_{r,n}(t)\le c(r)m^{-2r}$ for $m=1,2,...,n-1$. Next, to show (2), from above $\varphi_{r,n}(t)\le c(r)m^{-2r}$, we have $\varphi_{r,n}(t)\le c(r)$. To show that (4) works, in the same way (3) we get:

$$\varphi_{r,n}(t)\le c(r)|m|^{-2r} \text{ for every } m=-n+1,-n+2,...,-1.$$

To show that (5) holds, it is sufficient to show that:

$$\|\varphi_{r,n}(t)\|_{P,\lambda_n}\le c(r)(\frac{1}{n})^{1/p} \text{ for } 1\le p<\infty,\ r>\frac{1}{2p}$$

Now

$$\|\varphi_{r,n}(t)\|_{P,\lambda_n}=[\int_{-\pi}^{\pi}|\lambda_n\varphi_{r,n}(t)|^p dt]^{1/p} \qquad (2.4)$$

$$\|\varphi_{r,n}(t)\|_{P,\lambda_n}^P=\int_{-\pi}^{\pi}\lambda_n\varphi_{r,n}^p(t)dt=\int_{-\pi}^{\pi/n}\lambda_n\varphi_{r,n}^p(t)dt+\int_{\pi/n}^{\pi}\lambda_n\varphi_{r,n}^p(t)dt$$

$$\|\varphi_{r,n}(t)\|_{P,\lambda_n}^P=2\int_0^{\pi/n}\lambda_n\varphi_{r,n}^p(t)dt+\int_{\pi/n}^{\pi}\lambda_n\varphi_{r,n}^p(t)dt\le 2\frac{\pi}{n}c(r)^p \qquad (2.5)$$

Since $|\varphi_{r,n}(t)|\le c(r)$, then

$$\|\varphi_{r,n}(t)\|_{P,\lambda_n}\le\frac{c(r)^p}{n} \qquad (2.6)$$

By taking the root $\frac{1}{p}$. Hence $\|\varphi_{r,n}(t)\|_{P,\lambda_n}^P\le c(r)(\frac{1}{n})^{1/p}$.

The proof is completed.

Now let $r,n\in\mathbb{N}$, and let $N=r(2n-1)$

$$S_n^{\pm}(f,x)=f(x)\pm\int_{-\pi}^{\pi}\varphi_{r,n}(x-t)dt\sum_{m=0}^{k}(-1)^{k-m}\binom{k}{m}\sup\{f(x+mh):x,x+mh\in[-\pi,\pi]\}\lambda_n \qquad (2.7)$$

Lemma (2.2)

Let $f\in L_{P,\lambda_n}(X)$, where $X=[-\pi,\pi]$ then $S_n^-\le f(x)\le S_n^+$.

Proof:

Since $S_n^{\pm}(f,x)=f(x)\pm\int_{-\pi}^{\pi}\varphi_{r,n}(x-t)dt\sum_{m=0}^{k}(-1)^{k-m}\binom{k}{m}\cdot\sup\{f(x+mh):x,x+mh\in[-\pi,\pi]\}\lambda_n$

Where $\varphi_{r,n}(t)$ is a trigonometric polynomial of degree $r(2n-1)$ and

$$\varphi_{r,n}(t)=(n\sin\frac{\pi}{4n})^{2r}[\frac{\sin nt}{n\sin(t/2)}]^{2r};\ r,n\in\mathbb{N},$$

First, we want to show that $f(x)\ge S_n^-$. Start with:

$$f(x)-[S_n^-(f,x)]=f(x)-[f(x)-\int_{-\pi}^{\pi}\varphi_{r,n}(x-t)dt\sum_{m=0}^{k}(-1)^{k-m}\binom{k}{m}\cdot\sup\{f(x+mh):x,x+mh\in[-\pi,\pi]\}\lambda_n]$$

$$f(x)-[S_n^-(f,x)]=\int_{-\pi}^{\pi}\varphi_{r,n}(x-t)dt\sum_{m=0}^{k}(-1)^{k-m}\binom{k}{m}\sup\{f(x+mh):x,x+mh\in[-\pi,\pi]\}\lambda_n$$

$$f(x)-[S_n^-(f,x)]\ge\sum_{m=0}^{k}(-1)^{k-m}\binom{k}{m}\sup\{f(x+mh):x,x+mh\in[-\pi,\pi]\}\lambda_n\ge 0$$

Then $f(x)-[S_n^-(f,x)]\ge 0$ and hence $f(x)\ge[S_n^-(f,x)]\ldots(1)$

Similarly,

$$S_n^+(f,x)-f(x)=f(x)+\int_{-\pi}^{\pi}\varphi_{r,n}(x-t)dt\sum_{m=0}^{k}(-1)^{k-m}\binom{k}{m}\sup\{f(x+mh):x,x+mh\in[-\pi,\pi]\}\lambda_n-f(x)$$

$$S_n^+(f,x)-f(x)=\int_{-\pi}^{\pi}\varphi_{r,n}(x-t)dt\sum_{m=0}^{k}(-1)^{k-m}\binom{k}{m}\sup\{f(x+mh):x,x+mh\in[-\pi,\pi]\}\lambda_n$$

$$S_n^+(f,x)-f(x)\ge\sum_{m=0}^{k}(-1)^{k-m}\binom{k}{m}\sup\{f(x+mh):x,x+mh\in[-\pi,\pi]\}\ge 0$$

$S_n^+(f,x)-f(x)\ge 0$ and then we get $S_n^+(f,x)\ge f(x)$ $\ldots(2)$

From (1) and (2) we get: $S_n^-\le f(x)\le S_n^+$.

3. MAIN RESULTS

Theorem:

If $f\in L_{P,\lambda_n}(X)$, where $X=[-\pi,\pi]$ then $E_n(f)_{P,\lambda_n}\le\left\|S_n^+-S_n^-\right\|_{P,\lambda_n}\le c(r,n)\tau_k(f,\delta)_{P,\lambda_n}$

Proof:

First, we will show that $E_n(f)_{P,\lambda_n} \le c(r,n)\tau_k(f,\delta)_{P,\lambda_n}$. Let $r,n \in \mathbb{N}$,and $N = r(2n-1)$.

Since $S_n^{\pm}(f,x) = f(x) \pm \int_{-\pi}^{\pi} \varphi_{r,n}(x-t)dt \sum_{m=0}^{k}(-1)^{k-m}\binom{k}{m}\cdot \sup\{f(x+mh): x, x+mh \in [-\pi,\pi]\}\lambda_n$

Where $\varphi_{r,n}(t)$ is a trigonometric polynomial of degree $r(2n-1)$ and

$$\varphi_{r,n}(t) = (n\sin\frac{\pi}{4n})^{2r}[\frac{\sin nt}{n\sin(t/2)}]^{2r};\ r,n \in \mathbb{N}.$$

Now

$$S_n^+ - S_n^- = 2\int_{-\pi}^{\pi} \varphi_{r,n}(x-t)dt \sum_{m=0}^{k}(-1)^{k-m}\binom{k}{m}\sup\{f(x+mh): x, x+mh \in [-\pi,\pi]\}\lambda_n \tag{3.1}$$

$$\left\|S_n^+ - S_n^-\right\|_{P,\lambda_n} \le c(r,n)\int_{-\pi}^{\pi} \sum_{m=0}^{k}(-1)^{k-m}\binom{k}{m}\sup\{f(x+mh): x, x+mh \in [-\pi,\pi]\}\lambda_n dx \tag{3.2}$$

$$\left\|S_n^+ - S_n^-\right\|_{P,\lambda_n} \le c(r,n)\int_{-\pi}^{\pi} \omega(f,\lambda_n,x,\delta)dx, \delta = \frac{1}{n} \tag{3.3}$$

And then

$$c(r,n)\int_{-\pi}^{\pi} \omega(f,\lambda_n,x,\delta)dx = c(r,n)\tau_k(f,\delta)_{P,\lambda_n} \tag{3.4}$$

$$\left\|S_n^+ - S_n^-\right\|_{P,\lambda_n} \le c(r,n)\tau_k(f,\delta)_{P,\lambda_n} \tag{3.5}$$

To prove the other direction, i.e., prove $\tau_k(f,\delta)_{P,\lambda_n} \le \frac{c}{n}\sum_{i=1}^{n} E_i(f)_{P,\lambda_n}$.

Since $\tilde{E}_n(f)_{p,\lambda_n} = \left\|S_n^+ - S_n^-\right\|_{P,\lambda_n}$, then:

$$\Delta_h^k(\lambda_n f)(x) = \sum_{m=0}^{k}(-1)^{m}\binom{k}{m}(\lambda_n f)(t+(k-m)h),\ x \in [-\pi,\pi],\ t, t+kh \in [x-\frac{k\delta}{2}, x+\frac{k\delta}{2}]$$

From lemma (2) we have $S_n^- \le f(x) \le S_n^+$. Hence:

$$\Delta_h^k(\lambda_n f)(x) \le \sum_{i=0}^{k/2}\binom{k}{2i}S_n^+(t+(k-2i)h) - \sum_{i=0}^{k-1/2}\binom{k}{2i+1}S_n^-(t+(k-2i-1)h) \tag{3.6}$$

Then

$$\Delta_h^k(\lambda_n S_n^+)(t) - \sum_{i=0}^{k-1/2}\{\binom{k}{2i+1}S_n^-(t+(k-2i-1)h) - S_n^+(t+(k-2i-1)h) - (S_n^+(x)-S_n^-(x)\} + \sum_{i=0}^{k-1/2}\binom{k}{2i+1}(S_n^+(x)-S_n^-(x))$$

$$\le \Delta_h^k\lambda_n S_n^+(t) + \omega_k(S_n^+ - S_n^-,\delta)_{P,\lambda_n} + 2^k(S_n^+(x)-S_n^-(x))$$

By using the property: $\omega_k(f,\delta)_{P,\lambda_n} \le c(k)\omega_1(f,\delta)_{P,\lambda_n}$

$$\le \Delta_h^k(\lambda_n S_n)(t) + 2^k\omega_1(S_n^+ - S_n^-,\delta)_{P,\lambda_n} + 2^k(S_n^+(x)-S_n^-(x))$$

$$= \Delta_h^k(\lambda_n S_n)(t) + 2^k k\delta\left\|S_n^+ - S_n^-\right\|_{P,\lambda_n} + 2^k(S_n^+(x)-S_n^-(x))$$

By taking the norm for both sides we get:

$$\omega_k(f,\delta)_{P,\lambda_n} \le \omega_k(S_n^+,\delta)_{P,\lambda_n} + 2^k\left\|S_n^+ - S_n\right\|_{P,\lambda_n} \tag{3.7}$$

Then

$$\omega_k(f,\delta)_{P,\lambda_n} \le \omega_k(S_n^+,\delta)_{P,\lambda_n} + 2^k \tilde{E}_n(f)_{p,\lambda_n} \tag{3.8}$$

By utilizing the fact that

$$\int_a^b f(x)dx = \frac{b-a}{n}\sum_{i=1}^{n} f(x_i) \tag{3.9}$$

And by taking the norm for both sides we get:

$$\tau_k(f,\delta)_{P,\lambda_n} \le \tau_k(S_n^+,\delta)_{P,\lambda_n} + \frac{2^k 2\pi}{n}\sum_{i=1}^{n} \tilde{E}_n(f(x_i)_{p,\lambda_n} \tag{3.10}$$

By using the property:

$$\tau_k(f_1+f_2,\delta)_{P,\lambda_n} \le \tau_k(f_1,\delta)_{P,\lambda_n} + \tau_k(f_2,\delta)_{P,\lambda_n} \tag{3.11}$$

We get:

$$\tau_k(S_n^+,\delta)_{P,\lambda_n} \le \sum_{i=1}^{n} \tau_k((S_{2^i}^+ - S_{2^{i-1}}^+),\delta) + \tau_k((S_1^+ - S_0^+),\delta) \tag{3.12}$$

And then

$$\tau_k(f,\delta)_{P,\lambda_n} \le \sum_{i=1}^{n} \tau_k((S_{2^i}^+ - S_{2^{i-1}}^+),\delta) + \frac{2^{k+1}\pi}{n}\tilde{E}_n(f(x)_{p,\lambda_n} + \tau_k((S_1^+ - S_0^+),\delta)_{P,\lambda_n} \tag{3.13}$$

But

$$\tau_k((S_{2^i}^+ - S_{2^{i-1}}^+),\delta)_{P,\lambda_n} \le \delta^k \left\|(S_{2^i}^+ - S_{2^{i-1}}^+)^k\right\|_{P,\lambda_n}$$

$$\le \delta^k 2^{ik}\left\|S_{2^i}^+ - S_{2^{i-1}}^+\right\|_{P,\lambda_n} \le \delta^k 2^{ik}\left\|(S_{2^i}^+ - f) + (f - S_{2^{i-1}}^+)\right\|_{P,\lambda_n}$$

$$\le \delta^k 2^{ik}\left\|S_{2^i}^+ - f\right\|_{P,\lambda_n} + \delta^k 2^{ik}\left\|S_{2^{i-1}}^+ - f\right\|_{P,\lambda_n} \le \delta^k 2^i\left\|S_{2^i}^+ - S_{2^i}^-\right\|_{P,\lambda_n} + \delta^k 2^i\left\|S_{2^{i-1}}^+ - S_{2^{i-1}}^-\right\|_{P,\lambda_n} \tag{3.14}$$

As $S_n^- \le f(x) \le S_n^+$.

$$\tau_k((S_{2^i}^+ - S_{2^{i-1}}^+),\delta)_{P,\lambda_n} \le \delta^k 2^{ik}\left\|S_{2^i}^+ - S_{2^i}^-\right\|_{P,\lambda_n} + \delta^k 2^{ik}\left\|S_{2^{i-1}}^+ - S_{2^{i-1}}^-\right\|_{P,\lambda_n} = 2\delta^k 2^{ik}\tilde{E}_{2^{i-1}}(f)_{p,\lambda_n} \tag{3.15}$$

By utilizing 2.33 we get:

$$\tau_k(f,\delta)_{P,\lambda_n} \le \sum_{i=1}^{n} \tau_k((S_{2^i}^+ - S_{2^{i-1}}^-),\delta)_{P,\lambda_n} + \frac{2^{k+1}\pi}{n}\tilde{E}_n(f)_{p,\lambda_n} + \tau_k((S_1^+ - S_0^-),\delta)_{P,\lambda_n}$$

$$\le 2\delta^k 2^{ik}\tilde{E}_{2^{i-1}}(f)_{p,\lambda_n} + \frac{2^{k+1}\pi}{n}\tilde{E}_n(f)_{p,\lambda_n} + 2\delta^k 2^k \tilde{E}_0(f)_{p,\lambda_n} \tag{3.16}$$

Then

$$\tau_k(f,\delta)_{P,\lambda_n} \le \frac{c(k)}{n}\sum_{i=1}^{n} E_i(f)_{P,\lambda_n} \tag{3.17}$$

The proof is completed.

CONCLUSION:

The aim of this paper is to obtain of the degree of the best multiplier approximation of monotone unbounded periodic functions, $f \in L_{p,\lambda_n}$ −space on the closed interval $[-\pi, \pi]$ in terms of averaged multiplier modulus smoothness $\tau(f,\delta)_{p,\lambda_n}$.

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