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Direct Solution of General Fourth-Order Ordinary Differential Equations Using the Diagonally Implicit Runge-Kutta Method

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Abstract:

The diagonally implicit Runge-Kutta method (DIRKT), fourth-and fifth-order, is developed in this study to solve general fourth-order ordinary differential equations (ODEs). DIRKTO5 and DIRKTO4, respectively, are the names of these methods. Both methods consist of three stages. A collection of test problems were utilized to support the methodologies. The numerical outcomes demonstrated that the suggested approaches performed better than the implicit Runge-Kutta (RK) methods in terms of precision and the quantity of the evaluations of function.

Keywords:initial value problem, general fourth-order ODEs, implicit Runge-Kuttatype methods

1-Introduction

Numerous engineering and applied fields contain general 4th-order ODEs, such as neural networks [1],beam theory [2,3], electric circuits [4],and fluid dynamics [5]. This article is devoted to build a numerical technique to solve the general ODEs of 4th-order with the following form

$$q^{(4)} = f(t, q(t), q'(t)), \quad t \geq t_0, \tag{1}$$

with initial conditions

$$q(t_0) = q_0, \qquad q'(t_0) = q'_0, \quad q''(t_0) = q''_0, \quad q'''(t_0) = q'''_0.$$

The importance of implicit methods stems from their superiority over explicit methods for the same stage number, in terms of the high levels of accuracy that can be attained.This makes it easier to address stiff problems.DIRKT technique is a bottom triangular matrix with at least one nonzero diagonal element. In order to solve (1), there are two approaches that can be applied. The first approach is to transform (1) into a first-order problem before applying any RK strategy. As a result, various implicit RK approaches have been developed [6]. The other approach is the Runge-Kutta Type (RKT) method to directly solve (1). Many authors provided an effective implicit RK method for solving systems of second-order ODEs (see [7,8,9]). Later in [10] the authors developed a Runge-Kutta Nystrom approach for periodic IVPs that is embedded diagonally implicit. In [11], for stiff oscillatory problems, two kinds of three-stage diagonally implicit Runge-Kutta (DIRK) techniques were developed.Furthermore, the convergent of the DIRK approach using the Lipschitz condition was given by [12]. In [13], the authors derived RK type methods for solving 4th-order ODEs. Recently, Third-order ODEs can be solved using the DIRK-type approach of the fifth-order as proposed in [14]. This paper's primary goal is to derive the DIRKT method for solving general ODEs of the fourth-order.

This paper has the following structure: In section 2, the concept of derivation DIRKT approaches to resolve problem (1) is covered.Section 3 describes the order conditions for the DIRKT method. The construction of three-stage DIRKT methods for orders 4 and 5 respectively are given in section 4. The efficiency of the suggested methods is evaluated in section 5 in comparison to current implicit RK methods. In section 6, a conclusion is offered.

2-Methodology of proposed methods

The following is the general form of the s-stage DRIKT algorithm for resolving (1):

$$q_{n+1} = q_n + h\,q'_n + \frac{h^2}{2}q''_n + \frac{h^3}{6}q'''_n + h^4\sum_{i=1}^{s} b_i f(t_n + c_j h\,, Q_i, Q'_i), \quad (2)$$

$$q'_{n+1} = q'_n + h\,q''_n + \frac{h^2}{2}q'''_n + h^3\sum_{i=1}^{s} b'_i\, f\big(t_n + c_j h, Q_i, Q'_i\big), \quad (3)$$

$$q''_{n+1} = q''_n + h\,q'''_n + h^2\sum_{i=1}^{s} b''_i\, f\big(t_n + c_j h, Q_i, Q'_i\big), \quad (4)$$

$$q'''_{n+1} = q'''_n + h\sum_{i=1}^{s} b'''_i\, f\big(t_n + c_j h, Q_i, Q'_i\big), \quad (5)$$

$$Q_j = q_n + h\,c_i q'_n + \frac{h^2}{2}c_i^2 q''_n + \frac{h^3}{6}c_i^3 q'''_n + h^4\sum_{j=1}^{s} a_{ij} f\big(t_n + c_j h, Q_j, Q'_j\big), \quad (6)$$

$$Q'_j = q'_n + h\,c_i q''_n + \frac{h^2}{2}c_i^2 q'''_n + h^3\sum_{j=1}^{s} \bar{a}_{ij} f\big(t_n + c_j h, Q_j, Q'_j\big), \quad (7)$$

The coefficients $b_i, b'_i, b''_i, b'''_i, a_{i,j}, \bar{a}_{ij}$ and c_i of DIRKT method are real numbers. When $a_{i,j} \neq 0$ for $i < j$, the technique is diagonally implicit.By using the Butcher tableau, the DIRKT approach is illustrated as in Table 1:

Table 1. The coefficients of DIRKT method

c_1	a_{11}	a_{12}	...	a_{1s}	$\bar{a}_{11}\bar{a}_{12}$	...	$\bar{a}_{1s}$	
c_2	$a_{21}a_{22}$		...	a_{2s}	$\bar{a}_{21}\bar{a}_{22}$	...	$\bar{a}_{2s}$	
c_3	$a_{31}a_{32}$		...	a_{3s}	$\bar{a}_{31}\bar{a}_{32}$	...	$\bar{a}_{3s}$	
$\vdots$			::::			::::		
c_s	$a_{s1}a_{s2}$		...	a_{ss}	$\bar{a}_{s1}\bar{a}_{s2}$	...	$\bar{a}_{ss}$	
	b_1b_2	...	b_s					
	$b'_1b'_2$	...	b'_s					
	$b''_1b''_2$	...	b''_s					
	$b'''_1b'''_2$	...	b'''_s					

3-DIRKT method order conditions

Ghawadri[15], derived the following ordering of algebraic criteria for DIRKT up to order 6:

order 1: $\sum b'''_i = 1,$

order 2: $\sum b'''_i c_i = \frac{1}{2}, \sum b''_i = \frac{1}{2},$

order 3: $\sum b'''_i c_i^2 = \frac{1}{3}, \sum b''_i c_i = \frac{1}{6}, \sum b'_i = \frac{1}{6},$

order 4: $\sum b'''_i c_i^3 = \frac{1}{4}, \sum b'''_i \overline{a_{ij}} = \frac{1}{24}, \sum b''_i c_i^2 = \frac{1}{12}, \sum b'_i c_i = \frac{1}{24}, \sum b_i = \frac{1}{24},$

order 5: $\sum b'''_i c_i^4 = \frac{1}{5}, \sum b'''_i a_{ij} = \frac{1}{120}, \sum b'''_i \bar{a}_{ij} c_j = \frac{1}{120}, \sum b'''_i c_i \bar{a}_{ij} = \frac{1}{30},$

$$\sum b''_i c_i^3 = \frac{1}{20}, \sum b''_i a_{ij} = \frac{1}{120}, \sum b'_i c_i^2 = \frac{1}{60}, \sum b_i c_i = \frac{1}{120},$$

order 6: $\sum b'''_i c_i^5 = \frac{1}{6}, \sum b'''_i a_{ij} c_j = \frac{1}{720}, \sum b'''_i \bar{a}_{ij} c_i^2 = \frac{1}{360}, \sum b'''_i c_i^2 \bar{a}_{ij} = \frac{1}{36},$

$$\sum b'''_i c_i \bar{a}_{ij} c_j = \frac{1}{144}, \sum b'''_i c_i a_{ij} = \frac{1}{144}, \sum b''_i c_i^4 = \frac{1}{30}, \sum b''_i \bar{a}_{ij} c_j = \frac{1}{720},$$

$$\sum b''_i c_i \bar{a}_{ij} = \frac{1}{180}, \sum b''_i \bar{a}_{ij} = \frac{1}{720}, \sum b'_i c_i^3 = \frac{1}{120}, \sum b_i c_i^2 = \frac{1}{360}, \sum b'_i \bar{a}_{ij} = \frac{1}{720},$$

4-The construction of the DIRKT methods

We proceed to build implicit DIRKT techniques in accordance with the order criteriadescribed in the preceding section. Following is the definition of the global error of the r order DIRKT approach:

$$\left\|\tau_g^{(r+1)}\right\|_2=\left(\sum_{i=}^{n_r+1}\left(\tau_i^{(r+1)}\right)^2+\sum_{i=}^{n'_r+1}\left(\tau_i'^{(r+1)}\right)^2+\sum_{i=}^{n''_r+1}\left(\tau_i''^{(r+1)}\right)^2+\sum_{i=}^{n'''_r+1}\left(\tau_i'''^{(r+1)}\right)^2\right)^{\frac{1}{2}}(8)$$

Wherethe local truncation error norms for y, y', y'' and y''',respectively are$\tau^{(r+1)}, \tau'^{(r+1)}, \tau''^{(r+1)}$ and $\tau'''^{(r+1)}$.

4.1-DIRKT method of three-stage fourth-order

Using algebraic conditions until order 4, a three-stage DIRKT approach of the fourth order shall be derived. The consequencing system is made up of 11 nonlinear equations and 22 unknown variables that must be solved,the solutions must be determined in term of $\alpha=\bar{a}_{11}=\bar{a}_{22}=\bar{a}_{33}, \gamma=a_{11}=a_{22}=a_{33}, a_{21}, a_{31}, a_{32}, b_1, b_2, b_3, b'_1$ and $\bar{a}_{32}$ are got as follows:

$$\gamma=-\frac{1}{2}\bar{a}_{32}+\frac{1}{24}, c_2=\frac{1}{2}-\frac{1}{6}\sqrt{3}, c_3=\frac{1}{2}+\frac{1}{6}\sqrt{3}, c_1=1,$$

$$b'_2=-\frac{1}{48}(1+\sqrt{3})(24\, b'_1-\sqrt{3}-1), b'_3=\frac{1}{48}(\sqrt{3}-1)(24\, b'_1+\sqrt{3}-1),$$

The global error in ten free parameters given by

$$\left\|\tau_g^{(5)}\right\|_2=\Big(-\frac{1}{40}\bar{a}_{32}-\frac{1}{1080}b'_1-\frac{1}{60}b_1-\frac{1}{120}b_2-\frac{1}{120}b_3-\frac{1}{120}a_{31}-\frac{1}{120}a_{21}+\frac{7}{4800}$$
$$-\frac{1}{120}a_{32}-\frac{1}{20}\gamma-\frac{1}{60}\alpha+\frac{3}{4}\gamma^2-\frac{1}{12}\sqrt{3}\bar{a}_{32}\gamma+\frac{1}{3}\sqrt{3}b_1b_3-\frac{1}{3}\sqrt{3}b_1b_2$$
$$-\frac{1}{360}\sqrt{3}\bar{a}_{32}+\frac{3}{4}\gamma\,\bar{a}_{32}+\frac{1}{4}\bar{a}_{32}^2-\frac{1}{24}\sqrt{3}\bar{a}_{32}^2+\frac{49}{36}(b'_1)^2+\frac{1}{360}\sqrt{3}b_2$$
$$-\frac{1}{360}\sqrt{3}b_3+b_1^2+b_1b_2+b_1b_3+\frac{1}{3}b_2^2-\frac{1}{6}\sqrt{3}\,b_2^2+\frac{1}{3}b_1b_3+\frac{1}{3}b_3^2+a_{21}\alpha$$
$$+\frac{1}{6}\sqrt{3}\,b_3^2+\frac{1}{4}a_{21}^2+\frac{1}{2}a_{21}a_{31}+\frac{1}{2}a_{21}a_{32}+\alpha^2+a_{31}\alpha+a_{32}\alpha+\frac{1}{4}a_{31}^2$$
$$+\frac{1}{2}a_{31}a_{32}+\frac{1}{4}a_{32}^2\Big)^{1/2}. \qquad (9)$$

By using minimize command in Maple for equation (9) we find:

$$\alpha=-\frac{4}{10}, \gamma=-\frac{2}{1000}, a_{21}=\frac{3}{10}, a_{31}=\frac{3}{10}, a_{32}=\frac{3}{10}, b_1=-\frac{2}{10}, b_2=\frac{7}{10}, b_3=\frac{6}{100},$$

$$b'_1=\frac{2}{1000}, \bar{a}_{32}=\frac{9}{100},$$

and the global error is$\tau_g^{(5)}=0.0078567420148$.

The three-stage fourth-order Implicit DIRKT technique coefficients, stated as DIRKTO4, are written as given in Table 2:

Table 2. Coefficients of DIRKTO4 approach

1	$-\frac{4}{10}$			$-\frac{2}{1000}$		
$\frac{1}{2}-\frac{\sqrt{3}}{6}$	$\frac{3}{10}$	$-\frac{4}{10}$	0	0	$-\frac{2}{1000}$	0
$\frac{1}{2}+\frac{\sqrt{3}}{6}$	$\frac{3}{10}$	$\frac{3}{10}$	$-\frac{4}{10}$	0	$\frac{9}{100}$	$-\frac{2}{1000}$
	$-\frac{2}{10}$	$\frac{7}{10}$	$\frac{6}{100}$			
	$\frac{2}{1000}$	$\frac{2}{10}$	$\frac{2}{100}$			
	0	$\frac{1}{4}+\frac{\sqrt{3}}{12}$	$\frac{1}{4}-\frac{\sqrt{3}}{12}$			
	0	$\frac{1}{2}$	$\frac{1}{2}$			

4.2-DIRKT technique of three-stage fifth-order

Order conditions until order 5 are utilized to construct the three-stage DIRKT method of order five.The solution family in terms of$\alpha, \gamma, a_{21}, a_{31}, b_1,$ and $\bar{a}_{32}$are shown after simultaneously solving the system, which has 19 nonlinear equations as a consequence,

$$c_1 = 1, c_2 = \frac{2}{5} - \frac{\sqrt{6}}{10}, c_3 = \frac{2}{5} + \frac{\sqrt{6}}{10}, b_1' = 0, b_2' = \frac{1}{12} + \frac{\sqrt{6}}{48}, b_3' = \frac{1}{12} - \frac{\sqrt{6}}{48},$$

$$b_2 = \frac{1}{720}(-1+\sqrt{6})(360\, b_1 + 4\sqrt{6} + 9), b_3 = -\frac{1}{720}(1+\sqrt{6})(360\, b_1 - 4\sqrt{6} + 9),$$

$$b_1'' = 0, b_2'' = \frac{1}{4} + \frac{\sqrt{6}}{36}, b_3'' = \frac{1}{4} - \frac{\sqrt{6}}{36}, b_1''' = 0, b_2''' = \frac{4}{9} - \frac{\sqrt{6}}{36}, b_3''' = \frac{4}{9} + \frac{\sqrt{6}}{36},$$

$$\bar{a}_{32} = \bar{a}_{32}, \gamma = \frac{1}{15} - \frac{17}{30}\bar{a}_{32} - \frac{11}{90}\sqrt{6}\bar{a}_{32},$$

$$\bar{a}_{21} = -\frac{9}{250} + \frac{131}{250}\bar{a}_{32} + \frac{8}{125}\sqrt{6}\bar{a}_{32} - \frac{21}{1000}\sqrt{6},$$

$$\bar{a}_{31} = -\frac{9}{250} - \frac{3}{10}\bar{a}_{32} + \frac{1}{5}\sqrt{6}\bar{a}_{32} + \frac{21}{1000}\sqrt{6},$$

The global error normin seven free parameters is given as follows:

$$\left\|\tau_g^{(6)}\right\|_2 = \frac{1}{7200}\Big(162397 - 432000\,\alpha - 2528376\bar{a}_{32} - 4255200\,\gamma + 44000a_{21}\sqrt{6}$$

$$-44000a_{31}\sqrt{6} - 35200a_{32}\sqrt{6} - 645336\bar{a}_{32}\sqrt{6} - 204000a_{21} - 204000a_{31}$$

$$-163200a_{32} + 1152000\,\alpha a_{32}\sqrt{6} - 4320000\,\alpha a_{21}\sqrt{6} + 4320000\,\alpha a_{31}\sqrt{6}$$

$$-2000000a_{21}a_{32}\sqrt{6} + 1168000a_{31}a_{32}\sqrt{6} + 8620800\,\gamma\,\bar{a}_{32}\sqrt{6} + 25920000\alpha^2$$

$$+33120000\,\alpha a_{21} + 33120000\,\alpha a_{31} + 18432000\,\alpha a_{32} + 13400000{a_{21}}^2 - 8640b_1$$

$$+22000000a_{21}a_{31} + 10000000a_{21}a_{32} + 13400000{a_{31}}^2 + 12688000a_{31}a_{32}$$

$$+4611200{a_{32}}^2 + 15074640\bar{a}_{32}^2 + 38332800\bar{a}_{32}\gamma + 34560000\gamma^2 + 4665600{b_1}^2$$

$$-2400000{a_{21}}^2\sqrt{6} + 2400000{a_{31}}^2\sqrt{6} + 563200{a_{32}}^2\sqrt{6} + 5219040{\bar{a}_{32}}^2\sqrt{6}\Big)^{\frac{1}{2}}. \quad (10)$$

Minimizing the error in equation (10) with respect to the free parameters, we get:

$$\alpha = -0.308879354517, \gamma = -0.022705099286, a_{21} = 0.444231886189,$$

$$a_{31} = -0.101488284190, a_{32} = 0.511312280359, b_1 = 0.000925924733,$$

$$\bar{a}_{32} = 0.097975426143.$$

and the global error$\tau_g^{(6)} = 0.00748935976$. For the optimized value, we select $\alpha = -\frac{3}{10}$, $\gamma = -\frac{2}{100}, a_{21} = \frac{4}{10}, a_{31} = -\frac{1}{10}, a_{32} = \frac{5}{10}, b_1 = \frac{9}{10000}$, and $\bar{a}_{32} = \frac{9}{100}$

Finally, coefficients of the 5th-order Implicit DIRKT method with 3-stage denoted by DIRKTO5 can be represented in Table 3:

Table 3. The coefficients of DIRKTO5 Method

1	$-\frac{3}{10}$	0	0	$-\frac{2}{100}$	0	0
$\frac{2}{5}-\frac{\sqrt{6}}{10}$	$\frac{4}{10}$	$-\frac{3}{10}$	0	$\frac{279}{25000}-\frac{381\sqrt{6}}{25000}$	$-\frac{2}{100}$	0
$\frac{2}{5}+\frac{\sqrt{6}}{10}$	$-\frac{1}{10}\frac{5}{10}$		$-\frac{3}{10}$	$-\frac{63}{1000}+\frac{39\sqrt{6}}{1000}\frac{9}{100}$	$-\frac{2}{100}$	
	$\frac{9}{10000}$	$\frac{1223}{60000}+\frac{1331\sqrt{6}}{180000}$	$\frac{1223}{60000}-\frac{1331\sqrt{6}}{180000}$			
	0	$\frac{1}{12}+\frac{\sqrt{6}}{48}$	$\frac{1}{12}-\frac{\sqrt{6}}{48}$			
	0	$\frac{1}{4}+\frac{\sqrt{6}}{36}$	$\frac{1}{4}-\frac{\sqrt{6}}{36}$			
	$\frac{1}{9}$	$\frac{4}{9}-\frac{\sqrt{6}}{36}$	$\frac{4}{9}+\frac{\sqrt{6}}{36}$			

5-Numerical results

To assess the superiority of the suggested DIRKT approaches to the well-known implicit RK methods in the published papers, a set of experimental tests is addressed in this section. The techniques used for comparison are as follows:

(a)

- **DIRKTO5:**3-stage 5th-order DIRKT method derived in the present paper.
- **Radau 1:**5th-order RK approach with 3-stage proposed in[16].
- **Radau IA:**5th-order RK approach with 3-stage given in[17].

(b)

- **DIRKTO4:**3-stage 4th-order DIRKT method derived in the present paper.
- **RKLIIIB4:** 4th-order implicit RK approach with 3-stage derived in [16].
- **DIRKN4:**4th-order implicit RKapproach with 3-stage derived in [18].

Problem 1. [15] (Inhomogeneous linear problem)

$$q^{(4)}(x) = q' - \cos(t), \qquad q(0) = -\frac{1}{2}, q'(0) = \frac{1}{2}, q''(0) = \frac{1}{2}, q'''(0) = -\frac{1}{2},$$

Exact solution is $q(t) = \frac{1}{2}\sin(t) - \frac{1}{2}\cos(t)$.

Problem 2. [15] (Homogeneous linear problem)

$$q^{(4)} = q^2 + (q')^2 + \sin(t) - 1, \quad q(0) = 0, \qquad q'(0) = 1, \qquad q''(0) = 0\,, \qquad q'''(0) = -1$$

Exact solution is $q(t) = \sin(t)$.

Problem 3. [15] (Homogeneous linear system)

$$q_1^{(4)}(t) = -\frac{e^{3t}}{4} q_4', \qquad q_1(0) = 1, q_1'(0) = -1, q_1''(0) = 1, q_1'''(0) = -1,$$

$$q_2^{(4)}(t) = -16\, e^{-t} q_1', \qquad q_2(0) = 1, q_2'(0) = -2, q_2''(0) = 4, q_2'''(0) = -8,$$

$$q_3^{(4)}(t) = -\frac{81\, e^{-t}}{2} q_2', \qquad q_3(0) = 1, q_3'(0) = -3, q_3''(0) = 9, q_3'''(0) = -27,$$

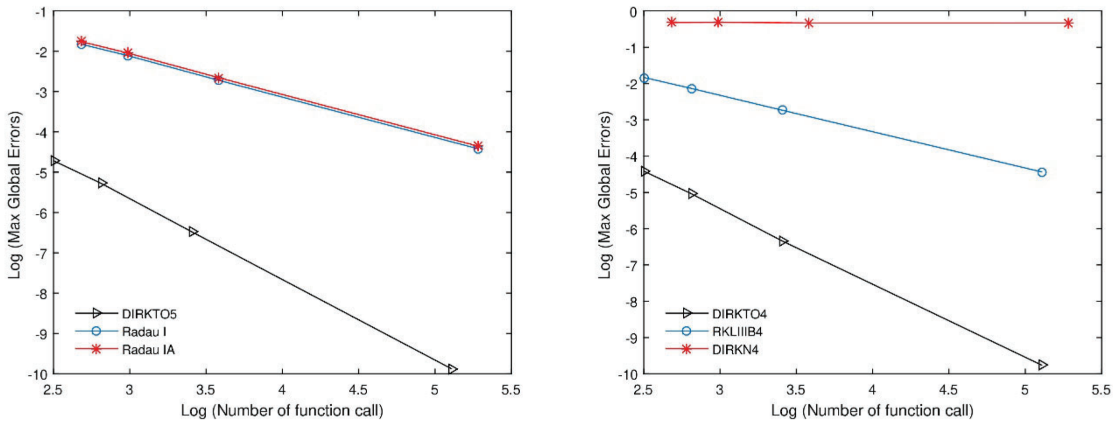
$$q_4^{(4)}(t) = -\frac{356\, e^{-t}}{4} q_3', \qquad q_4(0) = 1, q_4'(0) = -4, q_4''(0) = 16, q_4'''(0) = -64,$$

Exact solution is $q_1(t) = e^{-t},\ q_2(t) = e^{-2t},\ q_3(t) = e^{-3t},\ q_4(t) = e^{-4t}$.

Problem 4. [15] (Inhomogeneous nonlinear problem)

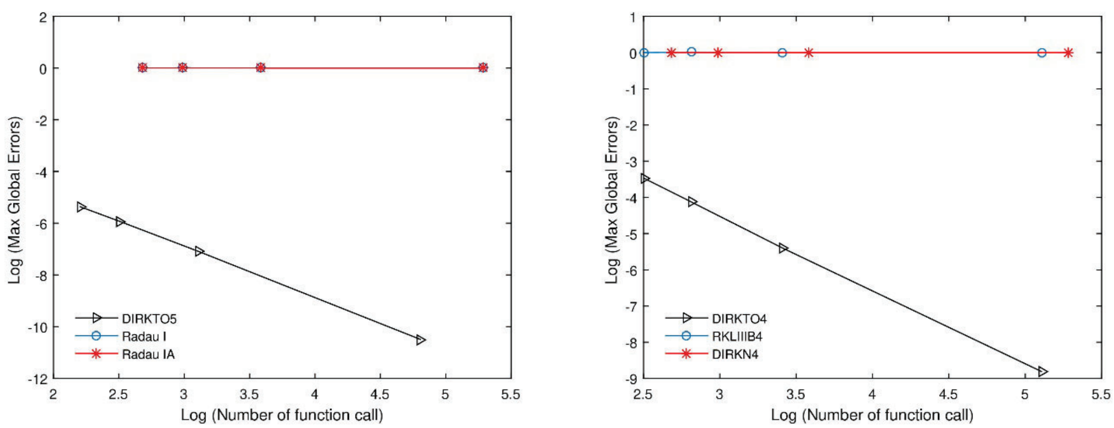
$$q^{(4)}(t) = -\frac{15\, q'}{8\, q^6}, \qquad q(0) = 1, q'(0) = \frac{1}{2}, q''(0) = -\frac{1}{4}, q'''(0) = \frac{3}{8},$$

Exact solution is $q(t) = \sqrt{t+1}$.



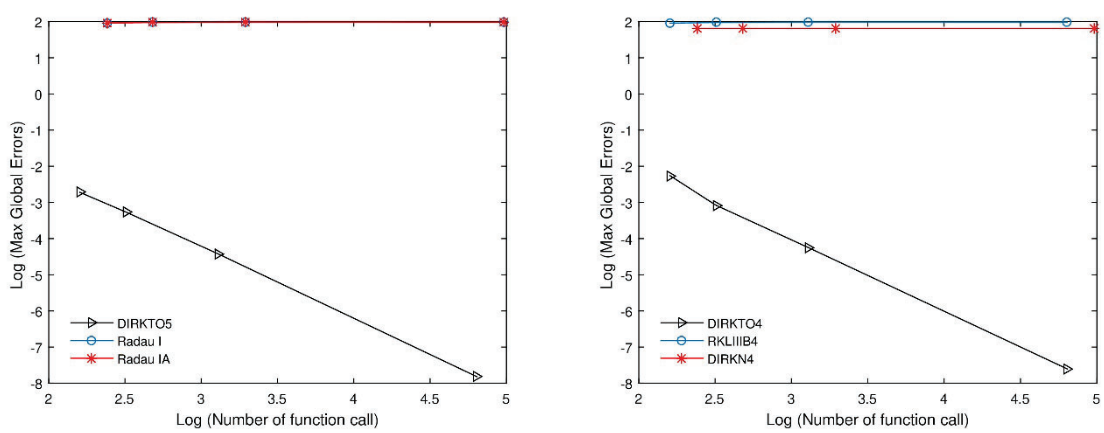
(i) Comparisons for methods (a) (ii) Comparisons for methods (b)

Figure 1. Competence graphs for problem 1**.**



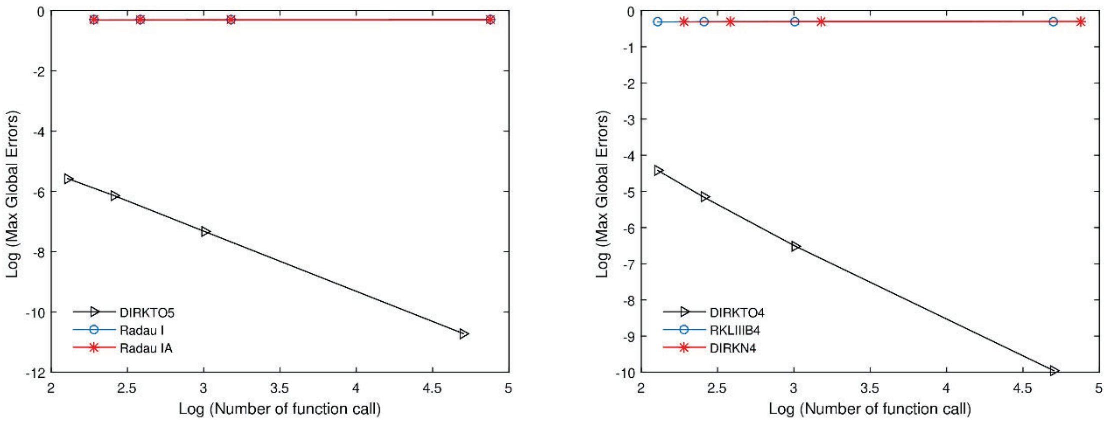
(i) Comparisons for methods (a) (ii) Comparisons for methods (b)

Figure 2. Competence graphs for problem 2.



(i) Comparisons for methods (a) (ii) Comparisons for methods (b)

Figure 3. Competence graphs for problem 3.



(i) Comparisons for methods (a) (ii) Comparisons for methods (b)

Figure 4. Competence graphs for problem 4**.**

Figures 1-4 show the performance of DIRKT techniques in terms of the maximum global vs the function evaluations each method requires.For this criterion, the decimal logarithm measure is employed. The DIRKTO5 and DIRKTO4 methods require fewer function evaluations than other implicit RK approaches already in use for the same order. This is because there are four times as many equations when the problems are converted into a system of first-order ODEs.

6-Conclusion

The integration of ODEs using the novel fifth-order DIRKTO5 and fourth-order DIRKTO4 methods with three stages has been discussed in this paper. In comparison to the implicit RK methods currently used in the scientific literature, numerical findings demonstrate that the suggested approaches are much more effective in terms of the number of function evaluations while solving the generic 4th-order ODEs.

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