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Existence of Hermation Solutions for CertainTypes of Operator Equations

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Abstract:

In this paper,we present the conditions that must be met in order to obtain a general solution to the bounded operator equation $\mathfrak{P}X\mathfrak{P}^* + \mathfrak{P}X^*\mathfrak{P}^* = \mathcal{A}$, where $\mathfrak{P}$ is a closed range operator and X is any operator on the Hilbert space H. We only propose what is necessary and sufficient. Several theorems are proved by the existence conditions. that provide explicit forms for solutions and conditions for their existence. The general explicit solution of the operator equation is one of the main results, as well as bounded solutions for cases whereP has a pseudo-inverse operator, The authors also provide proofs for each of the theorems.

Keyword: Operator Equation, Hermationsolution, pseudo-inverse operator, bounded operator , closed range operator.

1-Introduction

Let $B(H)$ be the bounded linear operators on a complex Hilbert space of infinite dimensions. We investigate the solvability of some operator equations in this paper. which have direct and indirect applications in quantum mechanics.to Many mathematical researchers have presented many studies on the concept of Operator equations define any equation in which the variable and components are defined operators on functional spaces, and have a significant impact on calculus, integral equations, and control theory[1][2]. in Salim Dawood Mohsen studied General Positive Adjoin table Operator Equation Solutions via Generalized Inverse,The explicit solution of some operator equations $AX + X^*A^* = B$,is given by Dragan S. in 2007 ,via Moor-person inverse, also given Dragan S. developed the general explicit solution to this adjointable operator equation in 2008. [12], given conditions that are both necessary and sufficient toget the general explicit solution another types of adjointable operator equation[4].another researcher founded the positive solution of operator equations such as M. Laura Arias and M. Celeste Gonzalez, in 2010 studied the existence of a positive operator equation solution $AXB = C$ under the conditions A and B have closed Range [9]. In 2010 Qing-Wen Wangand Chang-ZhouDong given the general positive adjointable solution operator equation system, [11]. In [10] studied the solution of Lyapunov operator equation $A^{*}XB+BXA=W$.In [8], S. Mecheri1 studied the On the Operator Equation $AXB-XD=E$ .In [7] studied On Hermation solutions to an adjoin table operator equation system. [5] in 2007 given the general positive and common Hermation In 2005, Bounded linear operator equation solutions $AX=C$ and $XB=D$ were also provided. Zhang Xian, [13] introduced general common Hermation nonnegative matrix equation solutions, in this work, we introduce the generalized to matrix equation appear in [13] and give the general common solutions for the operator equation.

In this paper, we give the general solution for operator equation for the formula

$\mathfrak{P}X\mathfrak{P}^{*}+\mathfrak{P}X^{*}\mathfrak{P}^{*}=\mathcal{A}$……..(1.1)

In the section when introduce the equation(1.1), where $\mathcal{A}$ and $\mathfrak{P}$ bounded linear operators define on Hilbert spaces H.but X unknown operator we must to find
Where exists Xsubstation the most equation that set to be X is a solution of dis equation moreover $X=X^{*}$ in this case we set X is Hermation solution.

Definition (1.1), [3]: An operator $\mathfrak{P}: H\longrightarrow K$ is called bounded operator if for each $M\subseteq X$ is a bounded set then $\mathfrak{P}(M)$ is a bounded set in Y, otherwise $\mathfrak{P}$ is unbounded.

In general we denoted as $B(H,K)$ is a set of all bounded and linear operators, also $B(H)$ is set of all operators with bounds on$H$m another face of definition of bounded linear operator which is given in the following proposition.

Proportion (1.2), [6]: The operator $\mathfrak{P}: H\longrightarrow K$ is called bounded operator if there is positive real m then $\|\mathfrak{P}(x)\|\leq m\|x\|$.
Special kinds of invertible operator, one can have from the down definition

Definition (1.3),[7]: Let $\mathfrak{P}$be a Hilbert space H bounded linear operator into a Hilbert space K, an operator $\mathfrak{P}^{+}$ on K into H is called pseudo inverse some time Moore-Penrose inverse of $\mathfrak{P}$ if satisfies $\mathfrak{P}=\mathfrak{P}\mathfrak{P}^{+}\mathfrak{P}, \mathfrak{P}^{+}=\mathfrak{P}^{+}\mathfrak{P}\mathfrak{P}^{+}$ and $\mathfrak{P}\mathfrak{P}^{+}$, $\mathfrak{P}^{+}\mathfrak{P}$ are self-adjoint.

Remark (1.4), [7]: Let $\mathfrak{P}: H\longrightarrow K$ be a bounded linear operator so we have
1) If $\mathfrak{P}$ has pseudo inverse then it is unique.
2) $\mathfrak{P}$ has pseudo inverse if and only if $\mathcal{A}$ has a close range
3)$\mathfrak{P}^{*}=\mathfrak{P}^{*}\mathfrak{P}\mathfrak{P}^{+}$
Definition (1.5), [7]: Let $\mathfrak{P}: H\longrightarrow K$ be a bounded linear operator, and H and K be two Hilbert spaces. The operator $\mathfrak{P}^{*}: K\longrightarrow H$, with the condition $<\mathfrak{P}x,y>=<x,\mathfrak{P}^{*}y>$, for each $x\in H,\ y\in K$.

Definition (1.6), [7]: The operator $\mathfrak{P}: H\longrightarrow K$ is called Hermationif satisfy $\mathfrak{P}=\mathfrak{P}^{*}$.

2- Solution of operator equation
In this pace of article, we introduce the solution and Hermation solution for bounded linear operator equation (1.1).

Theorem (2.1): Let$\mathfrak{P} \in B(H,K)$be closed range operators, then $X_0 = \frac{1}{2}\mathfrak{P}^+\mathcal{A}(\mathfrak{P}^*)^+ \in B(H)$is a practical solution of the operator equation(1.1) if and only if $\mathfrak{P}\mathfrak{P}^+\mathcal{A}\mathfrak{P}\mathfrak{P}^+ = \mathcal{A}$ and $\mathcal{A} \in B(H)$ is a self-adjoint operator.

Proof:Let A is a self-adjoint operator and we want to prove $X_0 = \frac{1}{2}\mathfrak{P}^+\mathcal{A}(\mathfrak{P}^*)^+ \in B(H)$ be a solution of (1.1),to do this replaces the bounded operator equation(1.1) left sidethen we can have;

$$\mathfrak{P}X_0\mathfrak{P}^* + \mathfrak{P}X_0{}^*\mathfrak{P}^* = \mathfrak{P}\left[\frac{1}{2}\mathfrak{P}^+\mathcal{A}(\mathfrak{P}^*)^+\right]\mathfrak{P}^* + \mathfrak{P}\left[\frac{1}{2}\mathfrak{P}^+\mathcal{A}(\mathfrak{P}^*)^+\right]^*\mathfrak{P}^*$$

$= \frac{1}{2}\mathfrak{P}\mathfrak{P}^+\mathcal{A}(\mathfrak{P}^*)^+\mathfrak{P}^* + \frac{1}{2}\mathfrak{P}\mathfrak{P}^+\mathcal{A}^*(\mathfrak{P}^*)^+\mathfrak{P}^*$and by using the condition $[(\mathfrak{P}\mathfrak{P}^+)^* = \mathfrak{P}\mathfrak{P}^+]$ from definition of pseudo inverse of operator, thus can get,

$\mathfrak{P}X_0\mathfrak{P}^* + \mathfrak{P}X_0{}^*\mathfrak{P}^* = \frac{1}{2}\mathfrak{P}\mathfrak{P}^+\mathcal{A}\mathfrak{P}\mathfrak{P}^+ + \frac{1}{2}\mathfrak{P}\mathfrak{P}^+\mathcal{A}^*\mathfrak{P}\mathfrak{P}^+ = \mathfrak{P}\mathfrak{P}^+\left[\frac{1}{2}\mathcal{A} + \frac{1}{2}\mathcal{A}^*\right]\mathfrak{P}\mathfrak{P}^+$, also from hypothesis $\mathcal{A}$ is a self-adjoint operator, so get $\mathfrak{P}X_0\mathfrak{P}^* + \mathfrak{P}X_0{}^*\mathfrak{P}^* = \mathfrak{P}\mathfrak{P}^+A\mathfrak{P}\mathfrak{P}^+ = \mathcal{A}$, one can have $\mathfrak{P}X_0\mathfrak{P}^* + \mathfrak{P}X_0{}^*\mathfrak{P}^* = \mathcal{A}$,therefore;$X_0 = \frac{1}{2}\mathfrak{P}^+\mathcal{A}(\mathfrak{P}^*)^+$is a bounded solution of(1.1).

Conversely, suppose $X_0 = \frac{1}{2}\mathfrak{P}^+\mathcal{A}(\mathfrak{P}^*)^+$ is a bounded operator equation solution (1.1), thenwe get$\mathfrak{P}X_0{}^*\mathfrak{P}^* + \mathfrak{P}X_0\mathfrak{P}^* = \mathcal{A}$, at first $[\mathfrak{P}X\mathfrak{P}^* + \mathfrak{P}X^*\mathfrak{P}^*]^* = \mathcal{A}^*$, this implies to $\mathfrak{P}X_0{}^*\mathfrak{P}^* + \mathfrak{P}X_0\mathfrak{P}^* = \mathcal{A}^*$, thus $\mathcal{A}$ is self adjoint operator that is $\mathcal{A} = \mathcal{A}^*$, also, one can have

$\mathfrak{P}X_0{}^*\mathfrak{P}^* + \mathfrak{P}X_0\mathfrak{P}^* = \mathfrak{P}\left[\frac{1}{2}\mathfrak{P}^+\mathcal{A}(\mathfrak{P}^*)^+\right]^*\mathfrak{P}^* + \mathfrak{P}\left[\frac{1}{2}\mathfrak{P}^+\mathcal{A}(\mathfrak{P}^*)^+\right]\mathfrak{P}^*$, implies that

$=\frac{1}{2}\mathfrak{P}\mathfrak{P}^+\mathcal{A}^*(\mathfrak{P}^*)^+\mathfrak{P}^* + \frac{1}{2}\mathfrak{P}\mathfrak{P}^+\mathcal{A}(\mathfrak{P}^*)^+\mathfrak{P}^* = \mathcal{A}$

and by using the condition $[(\mathfrak{P}\mathfrak{P}^+)^* = \mathfrak{P}\mathfrak{P}^+]$ from definition of pseudo inverse of operator, thus can get,

$\mathfrak{P}X_0{}^*\mathfrak{P}^* + \mathfrak{P}X_0\mathfrak{P}^* = \frac{1}{2}\mathfrak{P}\mathfrak{P}^+\mathcal{A}^*\mathfrak{P}\mathfrak{P}^+ + \frac{1}{2}\mathfrak{P}\mathfrak{P}^+\mathcal{A}\mathfrak{P}\mathfrak{P}^+ = \mathfrak{P}\mathfrak{P}^+\left[\frac{1}{2}\mathcal{A}^* + \frac{1}{2}\mathcal{A}\right]\mathfrak{P}\mathfrak{P}^+$,

But $\mathcal{A} = \mathcal{A}^*$, thus, $\mathfrak{P}X_0{}^*\mathfrak{P}^* + \mathfrak{P}X_0\mathfrak{P}^* = \mathfrak{P}\mathfrak{P}^+\mathcal{A}\mathfrak{P}\mathfrak{P}^+$, also $\mathfrak{P}X_0{}^*\mathfrak{P}^* + \mathfrak{P}X_0\mathfrak{P}^* = \mathcal{A}$, then we obtain $\mathfrak{P}\mathfrak{P}^+\mathcal{A}\mathfrak{P}\mathfrak{P}^+ = \mathcal{A}$.

Now, from above theorem we can get the following remark.

Remark (2.2):$X_0 = \frac{1}{2}B^+\mathcal{A}(B^*)^+ \in B(H)$, be a solution of equation$\mathfrak{P}X\mathfrak{P}^* - \mathfrak{P}X^*\mathfrak{P}^* = \mathcal{A}$if and only if$\mathfrak{P}\mathfrak{P}^+\mathcal{A}\mathfrak{P}\mathfrak{P}^+ = \mathcal{A}$and $\mathcal{A} \in B(K)$is a skew-adjoint operator.
Next, the following theorem shows the formula of bounded general solution of (1.1).

Theorem (2.3): Let $\mathfrak{P} \in B(H,K)$be closed range operator and $z \in B(K)$ is a skew-adjoint then $\mathcal{A} \in B(K)$ is a selfadjoint operator if and only if $\mathfrak{P}\mathfrak{P}^+\mathcal{A}\mathfrak{P}\mathfrak{P}^+ = \mathcal{A}$ and $X \in B(H)$is an operator equation (1.1).

Proof: Let $X = X_0 + \mathfrak{P}^+z(\mathfrak{P}^*)^+$is a solution of equation (1.1), to do this replaces the bounded operator equation(1.1) left side also by using z is a skew - adjoint then we get;

$\mathfrak{P}X\mathfrak{P}^* + \mathfrak{P}X^*\mathfrak{P}^* = \mathfrak{P}[X_0 + \mathfrak{P}^+z(\mathfrak{P}^*)^+]\mathfrak{P}^* + \mathfrak{P}[X_0 + \mathfrak{P}^+z(\mathfrak{P}^*)^+]^*\mathfrak{P}^*$, and substation by $\mathcal{A}$, this lead to next step

$$\mathfrak{P}X\mathfrak{P}^* + \mathfrak{P}X^*\mathfrak{P}^* = \mathfrak{P}\left[\frac{1}{2}\mathfrak{P}^+\mathcal{A}(\mathfrak{P}^*)^+ + \mathfrak{P}^+z(\mathfrak{P}^*)^+\right]\mathfrak{P}^* + \mathfrak{P}\left[\frac{1}{2}\mathfrak{P}^+\mathcal{A}(\mathfrak{P}^*)^+ + \mathfrak{P}^+z(\mathfrak{P}^*)^+\right]^*\mathfrak{P}^*$$

So one can have; $\mathfrak{P}X\mathfrak{P}^* + \mathfrak{P}X^*\mathfrak{P}^* = \frac{1}{2}(\mathfrak{P}\mathfrak{P}^+\mathcal{A}(\mathfrak{P}^*)^+\mathfrak{P}^*) + \mathfrak{P}\mathfrak{P}^+z(\mathfrak{P}^*)^+\mathfrak{P}^*$

$$+\frac{1}{2}(\mathfrak{P}\mathfrak{P}^+\mathcal{A}^*(\mathfrak{P}^*)^+\mathfrak{P}^*) + \mathfrak{P}\mathfrak{P}^+z^*(\mathfrak{P}^*)^+\mathfrak{P}^*$$

and by using the condition $[(\mathfrak{P}\mathfrak{P}^+)^* = \mathfrak{P}\mathfrak{P}^+]$ from definition of pseudo inverse of operator, thus can get

$$\mathfrak{P}X\mathfrak{P}^* + \mathfrak{P}X^*\mathfrak{P}^* = \frac{1}{2}(\mathfrak{P}\mathfrak{P}^+\mathcal{A}\mathfrak{P}\mathfrak{P}^+) + \mathfrak{P}\mathfrak{P}^+Z\mathfrak{P}\mathfrak{P}^+ + \frac{1}{2}(\mathfrak{P}\mathfrak{P}^+\mathcal{A}^*\mathfrak{P}\mathfrak{P}^+) + \mathfrak{P}\mathfrak{P}^+z^*\mathfrak{P}\mathfrak{P}^+$$

Since $Z \in B(K)$ is a skew-adjoint, thus,

$\mathfrak{P}X\mathfrak{P}^* + \mathfrak{P}X^*\mathfrak{P}^* = \frac{1}{2}\mathfrak{P}\mathfrak{P}^+\mathcal{A}\mathfrak{P}\mathfrak{P}^+ + \mathfrak{P}\mathfrak{P}^+(-z^*)\mathfrak{P}\mathfrak{P}^+ + \frac{1}{2}\mathfrak{P}\mathfrak{P}^+\mathcal{A}^*\mathfrak{P}\mathfrak{P}^+ + \mathfrak{P}\mathfrak{P}^+z^*\mathfrak{P}\mathfrak{P}^+,$

so one can have $\mathfrak{P}X\mathfrak{P}^* + \mathfrak{P}X^*\mathfrak{P}^* = \frac{1}{2}\mathfrak{P}\mathfrak{P}^+\mathcal{A}\mathfrak{P}\mathfrak{P}^+ + \frac{1}{2}\mathfrak{P}\mathfrak{P}^+\mathcal{A}^*\mathfrak{P}\mathfrak{P}^+$

$= \mathfrak{P}\mathfrak{P}^+\left[\frac{1}{2}\mathcal{A} + \frac{1}{2}\mathcal{A}^*\right]\mathfrak{P}\mathfrak{P}^+$, also from hypothesis $\mathcal{A}$ is a self-adjoint operator, so get $\mathfrak{P}X\mathfrak{P}^* + \mathfrak{P}X^*\mathfrak{P}^* = \mathfrak{P}\mathfrak{P}^+\mathcal{A}\mathfrak{P}\mathfrak{P}^+ = \mathcal{A}$, one can have $\mathfrak{P}X\mathfrak{P}^* + \mathfrak{P}X^*\mathfrak{P}^* = \mathcal{A}$, therefore; $X = X_0 + \mathfrak{P}^+z(\mathfrak{P}^*)^+$ is bounded solution of equation(1.1).

Conversely; assume that $X = X_0 + \mathfrak{P}^+z(\mathfrak{P}^*)^+$ is a solution of (1.1),then we get $\mathfrak{P}X^*\mathfrak{P}^* + \mathfrak{P}X\mathfrak{P}^* = \mathcal{A}$, at first $[\mathfrak{P}X\mathfrak{P}^* + \mathfrak{P}X^*\mathfrak{P}^*]^* = \mathcal{A}^*$, this implies to $\mathfrak{P}X^*\mathfrak{P}^* + \mathfrak{P}X\mathfrak{P}^* = \mathcal{A}^*$, thus $\mathcal{A}$ is self adjoint operator that is $\mathcal{A} = \mathcal{A}^*$, also since $\mathfrak{P}X^*\mathfrak{P}^* + \mathfrak{P}X\mathfrak{P}^* = \mathcal{A}$, and by using theorem (2.1) implies that the next step

$$\mathfrak{P}[X_0 + \mathfrak{P}^+z(\mathfrak{P}^*)^+]^*\mathfrak{P}^* + \mathfrak{P}[X_0 + \mathfrak{P}^+z(\mathfrak{P}^*)^+]\mathfrak{P}^* = \mathcal{A}$$

$= \mathfrak{P}\left[\frac{1}{2}\mathfrak{P}^+\mathcal{A}(\mathfrak{P}^*)^+ + \mathfrak{P}^+Z(\mathfrak{P}^*)^+\right]^*\mathfrak{P}^* + \mathfrak{P}\left[\frac{1}{2}\mathfrak{P}^+\mathcal{A}(\mathfrak{P}^*)^+ + \mathfrak{P}^+Z(\mathfrak{P}^*)^+\right]\mathfrak{P}^* = \mathcal{A} =$
$\frac{1}{2}\mathfrak{P}\mathfrak{P}^+\mathcal{A}^*(\mathfrak{P}^*)^+\mathfrak{P}^* + \mathfrak{P}\mathfrak{P}^+Z^*(\mathfrak{P}^*)^+\mathfrak{P}^* + \frac{1}{2}\mathfrak{P}\mathfrak{P}^+\mathcal{A}(\mathfrak{P}^*)^+\mathfrak{P}^* + \mathfrak{P}\mathfrak{P}^+Z(\mathfrak{P}^*)^+\mathfrak{P}^* = \mathcal{A},$

and by using the condition $[(\mathfrak{P}\mathfrak{P}^+)^* = \mathfrak{P}\mathfrak{P}^+]$ from definition of pseudo inverse of operator, which means $\mathfrak{P}\mathfrak{P}^+$ is self-adjoint, so can get,

$\frac{1}{2}(\mathfrak{P}\mathfrak{P}^+\mathcal{A}^*\mathfrak{P}\mathfrak{P}^+) + \mathfrak{P}\mathfrak{P}^+z^*\mathfrak{P}\mathfrak{P}^+ + \frac{1}{2}\mathfrak{P}\mathfrak{P}^+\mathcal{A}\mathfrak{P}\mathfrak{P}^+ + \mathfrak{P}\mathfrak{P}^+z\mathfrak{P}\mathfrak{P}^+ = \mathcal{A}$, Since $z \in B(K)$ is a skew-adjoint,Thus,

$$\frac{1}{2}\mathfrak{P}\mathfrak{P}^+\mathcal{A}^*\mathfrak{P}\mathfrak{P}^+ + \mathfrak{P}\mathfrak{P}^+z^*\mathfrak{P}\mathfrak{P}^+ + \frac{1}{2}\mathfrak{P}\mathfrak{P}^+\mathcal{A}\mathfrak{P}\mathfrak{P}^+ + \mathfrak{P}\mathfrak{P}^+(-z^*)\mathfrak{P}\mathfrak{P}^+ = \mathcal{A}$$

$$\frac{1}{2}\mathfrak{P}\mathfrak{P}^+\mathcal{A}^*\mathfrak{P}\mathfrak{P}^+ + \frac{1}{2}\mathfrak{P}\mathfrak{P}^+\mathcal{A}\mathfrak{P}\mathfrak{P}^+ = \mathcal{A}$$

$\mathfrak{P}\mathfrak{P}^+\left[\frac{1}{2}\mathcal{A}^* + \frac{1}{2}\mathcal{A}\right]\mathfrak{P}\mathfrak{P}^+ = \mathcal{A}$, but $\mathcal{A} = \mathcal{A}^*$, thus, $\mathcal{A} = \mathfrak{P}\mathfrak{P}^+\mathcal{A}\mathfrak{P}\mathfrak{P}^+$

Remark(2.4): It easy to check that $X = \frac{1}{2}\mathfrak{P}^{-1}\mathcal{A}(\mathfrak{P}^*)^{-1}$ is a solution of equation (1.1), when $\mathfrak{P}$ is invertible operator.

Now, we introduce the bounded general solution for bounded operator equation (1.1), when $\mathfrak{P}$ has pseudo inverse operator.

Theorem(2.5): Let $\mathfrak{P} \in B(H)$ and Fhas closed range operator,where$F = \mathfrak{P}^+\mathfrak{P}$,$M = \mathfrak{P}^+\mathcal{A}(\mathfrak{P}^*)^+andW \in B(H)$be arbitrary operator,then the bounded operator equation (1.1),has a solution; $X = M\mathrm{F} - \frac{1}{2}F^*\mathrm{M} + \mathrm{W}(I - FF^+) \in B(H)$ if and only if $\mathcal{A} \in B(K)$is a self-adjoint operator and$(I - F^+F)M(I - F^+F) = 0$.

Proof:we multiply the bounded operator equation (1.1) thus, by $(\mathfrak{P}^+)$ from the left and $((\mathfrak{P}^*)^+)$ from the right,;$\mathfrak{P}^+\mathfrak{P}X\mathfrak{P}^*(\mathfrak{P}^*)^+ + \mathfrak{P}^+\mathfrak{P}X^*\mathfrak{P}^*(\mathfrak{P}^*)^+ = \mathfrak{P}^+\mathcal{A}(\mathfrak{P}^*)^+$and we claim$X = M\mathrm{F} - \frac{1}{2}F^*\mathrm{M} + \mathrm{W}(I - FF^+)$, is solution of bounded operator equation (1.1) . To do this substitute in the left side of operator equation$FXF + FX^*F = M$ then we get;

$= F\left[MF - \frac{1}{2}F^*M + W(I - FF^+)\right]F + F\left[MF - \frac{1}{2}F^*M + W(I - FF^+)\right]^* F$ so one get$= FMFF - \frac{1}{2}FF^*MF + FW(I - FF^+)F + FF^*M^*F - \frac{1}{2}FM^*FF + F(I - F^+F)W^*F$

$= FMFF - \frac{1}{2}FF^*MF + FW(F - FF^+F) + FF^*M^*F - \frac{1}{2}FM^*FF + (F - FF^+F)W^*F$ and by using the condition $[FF^+F = F]$ from definition of pseudo inverse of operator, thus can get,

$FXF + FX^*F = FMF\mathrm{F} - \frac{1}{2}FF^*MF + FF^*M^*F - \frac{1}{2}FM^*FF,$

This leadto the following equations have from substation

$$FXF + FX^*F = \mathfrak{P}^+\mathfrak{P}\mathfrak{P}^+\mathcal{A}(\mathfrak{P}^*)^+\mathfrak{P}^+\mathfrak{P}\mathfrak{P}^+\mathfrak{P} - \frac{1}{2}\mathfrak{P}^+\mathfrak{P}(\mathfrak{P}^+\mathfrak{P})^*\mathfrak{P}^+\mathcal{A}(\mathfrak{P}^*)^+\mathfrak{P}^+\mathfrak{P}$$
$$+\mathfrak{P}^+\mathfrak{P}(\mathfrak{P}^+\mathfrak{P})^*(\mathfrak{P}^+\mathcal{A}(\mathfrak{P}^*)^+)^*\mathfrak{P}^+\mathfrak{P} - \frac{1}{2}\mathfrak{P}^+\mathfrak{P}\,(\mathfrak{P}^+\mathcal{A}(\mathfrak{P}^*)^+)^*\mathfrak{P}^+\mathfrak{P}\mathfrak{P}^+\mathfrak{P}$$

But from definition of Moore-Penrose inverse exactly have$[(\mathfrak{P}^+\mathfrak{P})^* = \mathfrak{P}^+\mathfrak{P}]$, thus we can rewrite the equations as formula

$$FXF + FX^*F = \mathfrak{P}^+\mathcal{A}(\mathfrak{P}^*)^+\mathfrak{P}^+\mathfrak{P} - \frac{1}{2}\mathfrak{P}^+\mathcal{A}(\mathfrak{P}^*)^+\mathfrak{P}^+\mathfrak{P}$$
$+\mathfrak{P}^+\mathcal{A}^*(\mathfrak{P}^*)^+\mathfrak{P}^+\mathfrak{P} - \frac{1}{2}\mathfrak{P}^+\mathcal{A}^*(\mathfrak{P}^*)^+\mathfrak{P}^+\mathfrak{P}$, via some properties, we will have

$$FXF + FX^*F = \mathfrak{P}^+\mathcal{A}(\mathfrak{P}^*)^+\mathfrak{P}^*(\mathfrak{P}^*)^+ - \frac{1}{2}\mathfrak{P}^+\mathcal{A}(\mathfrak{P}^*)^+\mathfrak{P}^*(\mathfrak{P}^*)^+$$
$+\mathfrak{P}^+\mathcal{A}^*(\mathfrak{P}^*)^+\mathfrak{P}^*(\mathfrak{P}^*)^+ - \frac{1}{2}\mathfrak{P}^+\mathcal{A}^*(\mathfrak{P}^*)^+\mathfrak{P}^*(\mathfrak{P}^*)^+$, and from definition of pseudo inverse of operator, we have the condition$[(\mathfrak{P}^*)^+ = (\mathfrak{P}^*)^+\mathfrak{P}^*(\mathfrak{P}^*)^+]$, so get next equation

$$FXF + FX^*F = \mathfrak{P}^+\mathcal{A}(\mathfrak{P}^*)^+ - \frac{1}{2}\mathfrak{P}^+\mathcal{A}(\mathfrak{P}^*)^+ + \mathfrak{P}^+\mathcal{A}^*(\mathfrak{P}^*)^+ - \frac{1}{2}\mathfrak{P}^+\mathcal{A}^*(\mathfrak{P}^*)^+$$
$$= \frac{1}{2}\mathfrak{P}^+\mathcal{A}(\mathfrak{P}^*)^+ + \frac{1}{2}\mathfrak{P}^+\mathcal{A}^*(\mathfrak{P}^*)^+$$

$= \mathfrak{P}^+[\frac{1}{2}\mathcal{A} + \frac{1}{2}\mathcal{A}^*](\mathfrak{P}^*)^+$,also from hypothesis $\mathcal{A}$ is a self-adjoint operator, so get,$\mathfrak{P}^+\mathcal{A}(\mathfrak{P}^*)^+ = M$thus, $X = M\mathrm{F} - \frac{1}{2}F^*\mathrm{M} + \mathrm{W}(I - FF^+)$ is general solution of equation(1.1)

Conversely, let X be a solution of equation (1.1), then its satisfy the equation (1.1), we get $\mathfrak{P}X^*\mathfrak{P}^* + \mathfrak{P}X\mathfrak{P}^* = \mathcal{A}$, at first $[\mathfrak{P}X\mathfrak{P}^* + \mathfrak{P}X^*\mathfrak{P}^*]^* = \mathcal{A}^*$, this implies to $\mathfrak{P}X^*\mathfrak{P}^* + \mathfrak{P}X\mathfrak{P}^* = \mathcal{A}^*$, so $\mathcal{A}$ is self-adjoint operator, and by substation in the right side of equation(1.1), one get the following

$$\mathfrak{P}X^*\mathfrak{P}^* + \mathfrak{P}X\mathfrak{P}^* = \mathfrak{P}\left[MF - \frac{1}{2}F^*M + W(I - FF^+)\right]^* \mathfrak{P}^* + \mathfrak{P}\left[MF - \frac{1}{2}F^*M + W(I - FF^+)\right]\mathfrak{P}^* = \mathfrak{P}F^*M^*\mathfrak{P}^* - \frac{1}{2}\mathfrak{P}M^*F\mathfrak{P}^* + \mathfrak{P}(I - F^+F)W^*\mathfrak{P}^*$$

$+\mathfrak{P}MF\mathfrak{P}^* - \frac{1}{2}\mathfrak{P}F^*M\mathfrak{P}^* + \mathfrak{P}W(I - FF^+)\mathfrak{P}^*$, now multiple by $(\mathfrak{P}^*)^+$from right and $\mathfrak{P}^+$from left, then get the equation form

$$\mathfrak{P}^+\mathcal{A}^*(\mathfrak{P}^*)^+ = \mathfrak{P}^+\mathfrak{P}F^*M^*\mathfrak{P}^*(\mathfrak{P}^*)^+ - \frac{1}{2}\mathfrak{P}^+\mathfrak{P}M^*F\mathfrak{P}^*(\mathfrak{P}^*)^+ + \mathfrak{P}^+\mathfrak{P}(I - F^+F)w^*\mathfrak{P}^*(\mathfrak{P}^*)^+$$
$$+\mathfrak{P}^+\mathfrak{P}MF\mathfrak{P}^*(\mathfrak{P}^*)^+ - \frac{1}{2}\mathfrak{P}^+\mathfrak{P}F^*M\mathfrak{P}^*(\mathfrak{P}^*)^+ + \mathfrak{P}^+\mathfrak{P}w(I - FF^+)\mathfrak{P}^*(\mathfrak{P}^*)^+$$
$$= \mathfrak{P}^+\mathcal{A}(\mathfrak{P}^*)^+\mathfrak{P}^+\mathfrak{P} - \frac{1}{2}\mathfrak{P}^+\mathcal{A}(\mathfrak{P}^*)^+\mathfrak{P}^+\mathfrak{P} + \mathfrak{P}^+\mathcal{A}^*(\mathfrak{P}^*)^+\mathfrak{P}^+\mathfrak{P}$$
$$-\frac{1}{2}\mathfrak{P}^+\mathcal{A}^*(\mathfrak{P}^*)^+\mathfrak{P}^+\mathfrak{P}$$

$= \mathfrak{P}^+\mathcal{A}(\mathfrak{P}^*)^+ - \frac{1}{2}\mathfrak{P}^+\mathcal{A}(\mathfrak{P}^*)^+ + \mathfrak{P}^+\mathcal{A}^*(\mathfrak{P}^*)^+ - \frac{1}{2}\mathfrak{P}^+\mathcal{A}^*(\mathfrak{P}^*)^+$, this lead to the next reduces equation $\mathfrak{P}^+\mathcal{A}^*(\mathfrak{P}^*)^+ = \frac{1}{2}\mathfrak{P}^+\mathcal{A}(\mathfrak{P}^*)^+ + \frac{1}{2}\mathfrak{P}^+\mathcal{A}^*(\mathfrak{P}^*)^+$, lead to,

$\mathfrak{P}^+\mathcal{A}^*(\mathfrak{P}^*)^+ = \mathfrak{P}^+\mathcal{A}(\mathfrak{P}^*)^+$, Then $\mathcal{A}^* = \mathcal{A}$, therefore; $\mathcal{A}$ is self-adjoint operator

Also; to show that the condition $(I - F^+F)M(I - F^+F) = 0$,

at first $(I - F^+F)(FXF + FX^*F)(I - F^+F)$ so one can have,

$(I - F^+F)M(I - F^+F) = (I - F^+F)[FXF + FX^*F - FXFF^+F - FX^*FF^+F]$,

but we known this $[FF^+F = F]$ by this condition, one can see

$(I - F^+F)M(I - F^+F) = (I - F^+F)[FXF + FX^*F - FXF - FX^*F] = 0$

Remark(2.6): In case$\mathfrak{P}^{-1}\mathcal{A}^*(\mathfrak{P}^*)^{-1}$ is a solution of equation (1.1), when $\mathfrak{P}$ is invertible operator, We get the solution be the same and only we need a condition self-adjoint.

Now, The following theorem shows the general solution of bounded operator equation (1.1) , when $\mathfrak{P}$ has pseudo inverse operator.

Theorem (2.7): Let $\mathfrak{P} \in B(H, K)$ and F have closed range operators. such that $F = \mathfrak{P}^+\mathfrak{P}$, $M = \mathfrak{P}^+\mathcal{A}(\mathfrak{P}^*)^+$also, $\mathfrak{P}^*(\mathfrak{P}\mathfrak{P}^+) = (\mathfrak{P}^+\mathfrak{P})\mathfrak{P}^*$where $Z, y, W \in B(H)$,arearbitraryoperators Then The bounded operator equation(1.1)has explicit general solution of the form $X = MF - \frac{1}{2}F^*M + (I - FF^+)w\mathfrak{P}\mathfrak{P}^+ +$

$(I-\mathfrak{P}\mathfrak{P}^{+})((\mathfrak{P}^{+})^{*}\mathcal{A}\mathfrak{P}^{+})+Z(I-\mathfrak{P}\mathfrak{P}^{+})$ if and only if $\mathcal{A}\in B(K)$ is a self-adjoint operator and $(I-F^{+}F)M(I-F^{+}F)=0$

Proof: let be any solution of equation (1.1) Then it's satisfy it so,

$\mathfrak{P}X\mathfrak{P}^{*}+\mathfrak{P}X^{*}\mathfrak{P}^{*}=\mathcal{A}$, thus $[\mathfrak{P}X\mathfrak{P}^{*}+\mathfrak{P}X^{*}\mathfrak{P}^{*}=\mathcal{A}]^{*}$, then $\mathcal{A}=\mathcal{A}^{*}$ and

$(\mathrm{I}-F^{+}F)M(\mathrm{I}-F^{+}F)=0$, so one have $(\mathrm{I}-F^{+}F)(FXF+FX^{*}F)(\mathrm{I}-F^{+}F)$, this lead to

$=(I-F^{+}F)[FXF+FX^{*}F-FXF-FX^{*}F]=0$, Therefore, $(I-F^{+}F)M(I-F^{+}F)=0$

Conversely; since $R(\mathfrak{P})$ is closed we get; $K=R(\mathfrak{P}^{*})\oplus N(\mathfrak{P})$ and $H=R(\mathfrak{P})\oplus N(\mathfrak{P}^{*})$, we have, $\mathfrak{P}=\begin{bmatrix}R(\mathfrak{P}^{*})\\N(\mathfrak{P})\end{bmatrix}\longrightarrow\begin{bmatrix}R(\mathfrak{P})\\N(\mathfrak{P}^{*})\end{bmatrix}$ be the matrix form $\mathfrak{P}=\begin{bmatrix}\mathfrak{P}_1&0\\0&0\end{bmatrix}$, and , $\mathfrak{P}^{*}=\begin{bmatrix}\mathfrak{P}_1^{*}&0\\0&0\end{bmatrix}$

Where, $\mathfrak{P}_1=R(\mathfrak{P}^{*})\to R(\mathfrak{P})$ is an invertible, $\mathfrak{P}^{+}=\begin{bmatrix}\mathfrak{P}_1^{-1}&0\\0&0\end{bmatrix}$, also we known

$(I-\mathfrak{P}^{+}\mathfrak{P})\mathcal{A}(I-\mathfrak{P}^{+}\mathfrak{P})=0$, this lead to the down equation of matrices

$$\left[\begin{bmatrix}\mathrm{I}&0\\0&\mathrm{I}\end{bmatrix}-\begin{bmatrix}\mathfrak{P}_1^{-1}&0\\0&0\end{bmatrix}\begin{bmatrix}\mathfrak{P}_1&0\\0&0\end{bmatrix}\right]\begin{bmatrix}\mathcal{A}_1&\mathcal{A}_2\\\mathcal{A}_3&A_4\end{bmatrix}\left[\begin{bmatrix}I&0\\0&I\end{bmatrix}-\begin{bmatrix}\mathfrak{P}_1^{-1}&0\\0&0\end{bmatrix}\begin{bmatrix}\mathfrak{P}_1&0\\0&0\end{bmatrix}\right]=\begin{bmatrix}0&0\\0&0\end{bmatrix}$$

$$\left[\begin{bmatrix}I&0\\0&I\end{bmatrix}-\begin{bmatrix}I&0\\0&0\end{bmatrix}\right]\begin{bmatrix}\mathcal{A}_1&\mathcal{A}_2\\\mathcal{A}_3&\mathcal{A}_4\end{bmatrix}\left[\begin{bmatrix}I&0\\0&I\end{bmatrix}-\begin{bmatrix}I&0\\0&0\end{bmatrix}\right]=\begin{bmatrix}0&0\\0&0\end{bmatrix}$$

$\begin{bmatrix}0&0\\0&I\end{bmatrix}\begin{bmatrix}\mathcal{A}_1&\mathcal{A}_2\\\mathcal{A}_3&\mathcal{A}_4\end{bmatrix}\begin{bmatrix}0&0\\0&I\end{bmatrix}=\begin{bmatrix}0&0\\0&0\end{bmatrix}$, and this implies $\begin{bmatrix}0&0\\0&\mathcal{A}_4\end{bmatrix}=\begin{bmatrix}0&0\\0&0\end{bmatrix}$

Therefore ; $\mathcal{A}_4=0$ and hence $\mathcal{A}=\begin{bmatrix}\mathcal{A}_1&\mathcal{A}_2\\\mathcal{A}_3&\mathcal{A}_4\end{bmatrix}=\begin{bmatrix}\mathcal{A}_1&\mathcal{A}_2\\\mathcal{A}_3&0\end{bmatrix}$ but $\begin{bmatrix}\mathcal{A}_1&\mathcal{A}_2\\\mathcal{A}_3&0\end{bmatrix}=\begin{bmatrix}\mathcal{A}_1^{*}&\mathcal{A}_3^{*}\\\mathcal{A}_2^{*}&0\end{bmatrix}$ thus $\mathcal{A}=\begin{bmatrix}\mathcal{A}_1&\mathcal{A}_2\\\mathcal{A}_2^{*}&0\end{bmatrix}$ as $(\mathfrak{P}^{+}\mathfrak{P})\mathfrak{P}^{*}=\mathfrak{P}^{*}(\mathfrak{P}\mathfrak{P}^{+})$

$\left[\begin{bmatrix}\mathfrak{P}_1^{-1}&0\\0&0\end{bmatrix}\begin{bmatrix}\mathfrak{P}_1&0\\0&0\end{bmatrix}\right]\begin{bmatrix}\mathfrak{P}_1^{*}&0\\0&0\end{bmatrix}=\begin{bmatrix}\mathfrak{P}_1^{*}&0\\0&0\end{bmatrix}\left[\begin{bmatrix}\mathfrak{P}_1&0\\0&0\end{bmatrix}\begin{bmatrix}\mathfrak{P}_1^{-1}&0\\0&0\end{bmatrix}\right]$, so one can have, $\begin{bmatrix}\mathfrak{P}_1^{*}&0\\0&0\end{bmatrix}=\begin{bmatrix}\mathfrak{P}_1^{*}&0\\0&0\end{bmatrix}$

Let X has the form ; $X=\begin{bmatrix}X_1&X_2\\X_3&X_4\end{bmatrix}$, and since the equation $\mathfrak{P}X\mathfrak{P}^{*}+\mathfrak{P}X^{*}\mathfrak{P}^{*}=\mathcal{A}$ can be written as; $\begin{bmatrix}\mathfrak{P}_1&0\\0&0\end{bmatrix}\begin{bmatrix}X_1&X_2\\X_3&X_4\end{bmatrix}\begin{bmatrix}\mathfrak{P}_1^{*}&0\\0&0\end{bmatrix}+\begin{bmatrix}\mathfrak{P}_1&0\\0&0\end{bmatrix}\begin{bmatrix}{X_1}^{*}&{X_3}^{*}\\{X_2}^{*}&{X_4}^{*}\end{bmatrix}\begin{bmatrix}\mathfrak{P}_1^{*}&0\\0&0\end{bmatrix}=\begin{bmatrix}\mathcal{A}_1&\mathcal{A}_2\\\mathcal{A}_2^{*}&0\end{bmatrix}$, so

$\begin{bmatrix}\mathfrak{P}_1X_1\mathfrak{P}_1^{*}&0\\0&0\end{bmatrix}+\begin{bmatrix}\mathfrak{P}_1{X_1}^{*}\mathfrak{P}_1^{*}&0\\0&0\end{bmatrix}=\begin{bmatrix}\mathcal{A}_1&\mathcal{A}_2\\\mathcal{A}_2^{*}&0\end{bmatrix}$, thus $\begin{bmatrix}\mathfrak{P}_1X_1\mathfrak{P}_1^{*}+\mathfrak{P}_1{X_1}^{*}\mathfrak{P}_1^{*}&0\\0&0\end{bmatrix}=\begin{bmatrix}\mathcal{A}_1&\mathcal{A}_2\\\mathcal{A}_2^{*}&0\end{bmatrix}$, and this implies that $\mathfrak{P}_1X_1\mathfrak{P}_1^{*}+\mathfrak{P}_1{X_1}^{*}\mathfrak{P}_1^{*}=\mathcal{A}_1$ since $\mathfrak{P}_1$ is closed range operator and $\mathcal{A}$ is a self-adjoint then by using theorem(2.5) this equation have a solutions these solutions have the form :

$X = M\mathrm{F} - \frac{1}{2}F^*\mathrm{M} + \mathrm{W}(I - FF^+)$, where $F = \begin{bmatrix} \mathfrak{P}^+\mathfrak{P} & 0 \\ 0 & 0 \end{bmatrix}$, $M = \begin{bmatrix} \mathfrak{P}^+\mathcal{A}(\mathfrak{P}^*)^+ & 0 \\ 0 & 0 \end{bmatrix}$

Hence $X = \begin{bmatrix} M\mathrm{F} - \frac{1}{2}F^*M + \mathrm{W}(I - FF^+) & X_2 \\ 0 & X_4 \end{bmatrix}$

Let $W: \begin{bmatrix} R(\mathfrak{P}^*) \\ N(\mathfrak{P}) \end{bmatrix} \longrightarrow \begin{bmatrix} R(\mathfrak{P}^*) \\ N(\mathfrak{P}) \end{bmatrix}$ is of the form $W = \begin{bmatrix} w_1 & w_2 \\ 0 & w_4 \end{bmatrix}$

$Z: \begin{bmatrix} R(\mathfrak{P}^*) \\ N(\mathfrak{P}) \end{bmatrix} \longrightarrow \begin{bmatrix} R(\mathfrak{P}^*) \\ N(\mathfrak{P}) \end{bmatrix}$; is of the form $= \begin{bmatrix} y_1 & X_2 \\ y_3 & X_4 \end{bmatrix}$, where w_1, y_3 are arbitrary then,

$$X = MF - \frac{1}{2}F^*\mathrm{M} + (I - FF^+)w\mathfrak{P}\mathfrak{P}^+ + (I - \mathfrak{P}\mathfrak{P}^+)((\mathfrak{P}^+)^*A\mathfrak{P}^+) + Z(I - \mathfrak{P}\mathfrak{P}^+)$$

Now, $MF = \begin{bmatrix} \mathfrak{P}^+\mathcal{A}(\mathfrak{P}^*)^+ & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \mathfrak{P}^+\mathfrak{P} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \mathfrak{P}^+\mathcal{A}(\mathfrak{P}^*)^+\mathfrak{P}^+\mathfrak{P} & 0 \\ 0 & 0 \end{bmatrix}$

And $-\frac{1}{2}F^*\mathrm{M} = -\frac{1}{2}\begin{bmatrix} (\mathfrak{P}^+\mathfrak{P})^* & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \mathfrak{P}^+\mathcal{A}(\mathfrak{P}^*)^+ & 0 \\ 0 & 0 \end{bmatrix} = -\frac{1}{2}\begin{bmatrix} (\mathfrak{P}^+\mathfrak{P})^*\mathfrak{P}^+\mathcal{A}(\mathfrak{P}^*)^+ & 0 \\ 0 & 0 \end{bmatrix}$

$(I - FF^+)W\mathfrak{P}\mathfrak{P}^+ = \left[\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} - \begin{bmatrix} \mathfrak{P}^+\mathfrak{P} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} (\mathfrak{P}^+\mathfrak{P})^+ & 0 \\ 0 & 0 \end{bmatrix}\right]\begin{bmatrix} w_1 & w_2 \\ 0 & w_4 \end{bmatrix}\begin{bmatrix} \mathfrak{P}_1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \mathfrak{P}_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}$

$$= \left[\begin{bmatrix} w_1 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} \mathfrak{P}^+\mathfrak{P}(\mathfrak{P}^+\mathfrak{P})^+w_1 & 0 \\ 0 & 0 \end{bmatrix}\right] = \begin{bmatrix} w_1\langle I - \mathfrak{P}^+\mathfrak{P}(\mathfrak{P}^+\mathfrak{P})^+\rangle & 0 \\ 0 & 0 \end{bmatrix}$$

Thus $(I - \mathfrak{P}\mathfrak{P}^+)((\mathfrak{P}^+)^*\mathcal{A}\mathfrak{P}^+) = \left[\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} - \begin{bmatrix} \mathfrak{P}_1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \mathfrak{P}_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}\right]\begin{bmatrix} (\mathfrak{P}_1^{-1})^* & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \mathcal{A}_1 & \mathcal{A}_2 \\ \mathcal{A}_3 & \mathcal{A}_4 \end{bmatrix}\begin{bmatrix} \mathfrak{P}_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}$

$$= \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} (\mathfrak{P}_1^*)^{-1}\mathcal{A}_1\mathfrak{P}_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Next, $Z(I - \mathfrak{P}\mathfrak{P}^+) = \begin{bmatrix} y_1 & X_2 \\ y_3 & X_4 \end{bmatrix}\left[\begin{bmatrix} \mathrm{I} & 0 \\ 0 & \mathrm{I} \end{bmatrix} - \begin{bmatrix} \mathfrak{P}_1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \mathfrak{P}_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}\right]$

$$= \begin{bmatrix} y_1 & X_2 \\ y_3 & X_4 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & \mathrm{I} \end{bmatrix} = \begin{bmatrix} 0 & X_2 \\ 0 & X_4 \end{bmatrix}$$

Therefore; $MF - \frac{1}{2}F^*\mathrm{M} + (I - FF^+)w\mathfrak{P}\mathfrak{P}^+ + (I - \mathfrak{P}\mathfrak{P}^+)((\mathfrak{P}^+)^*\mathcal{A}\mathfrak{P}^+) + Z(I - \mathfrak{P}\mathfrak{P}^+)$

$$\begin{bmatrix} \mathfrak{P}^+\mathcal{A}(\mathfrak{P}^*)^+\mathfrak{P}^+\mathfrak{P} & 0 \\ 0 & 0 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} (\mathfrak{P}^+\mathfrak{P})^*\mathfrak{P}^+\mathcal{A}(\mathfrak{P}^*)^+ & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} w_1\langle I - \mathfrak{P}^+\mathfrak{P}(\mathfrak{P}^+\mathfrak{P})^+\rangle & 0 \\ 0 & 0 \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & X_2 \\ 0 & X_4 \end{bmatrix}$$

$= \begin{bmatrix} MF - \frac{1}{2}F^*M + W(I - FF^+) & X_2 \\ 0 & X_4 \end{bmatrix} = X$, so we get the unknown operator X, form

$X = MF - \frac{1}{2}F^*M + (I - FF^+)w\mathfrak{P}\mathfrak{P}^+ + (I - \mathfrak{P}\mathfrak{P}^+)((\mathfrak{P}^+)^*\mathcal{A}\mathfrak{P}^+) + Z(I - \mathfrak{P}\mathfrak{P}^+)$, is general a boundedsolution of (1.1).

Now, the following theorem introduces the Hermation solution of operator equation (1.1) .

Theorem (2.8): Let$\mathfrak{P} \in B(H,K)$ be closed range If $\mathfrak{P}\mathfrak{P}^+\mathcal{A}\mathfrak{P}\mathfrak{P}^+ = \mathcal{A}$ is Hermation operator, Then the equation (1.1) has a Hermation solution $X_0 \in B(H)$. Where $\mathcal{A} \in B(K)$ is Hermation operator.

Proof:suppose $\mathcal{A} \in B(K)$ is Hermation operator then,

$X_0 = \frac{1}{2}[(\mathfrak{P}^+\mathcal{A}(\mathfrak{P}^*)^+)^* + (I - \mathfrak{P}^+\mathfrak{P})(\mathfrak{P}^+\mathcal{A}(\mathfrak{P}^*)^+)]$ is aHermation and to do this replaces the left the operator equation's side(1.1). then we get;

$$\mathfrak{P}X_0\mathfrak{P}^* + \mathfrak{P}X_0{}^*\mathfrak{P}^*$$

$$= \mathfrak{P}\left[\frac{1}{2}\langle(\mathfrak{P}^+\mathcal{A}(\mathfrak{P}^*)^+)^* + (I - \mathfrak{P}^+\mathfrak{P})(\mathfrak{P}^+\mathcal{A}(\mathfrak{P}^*)^+)\rangle\right]\mathfrak{P}^* + \mathfrak{P}\left[\frac{1}{2}\langle(\mathfrak{P}^+\mathcal{A}(\mathfrak{P}^*)^+)^* + (I - \mathfrak{P}^+\mathfrak{P})(\mathfrak{P}^+\mathcal{A}(\mathfrak{P}^*)^+)\rangle\right]^*\mathfrak{P}^*$$

So, one can have

$$\mathfrak{P}X_0\mathfrak{P}^* + \mathfrak{P}X_0{}^*\mathfrak{P}^* = \frac{1}{2}\mathfrak{P}\mathfrak{P}^+\mathcal{A}^*(\mathfrak{P}^*)^+\mathfrak{P}^* + \frac{1}{2}\mathfrak{P}(I - \mathfrak{P}^+\mathfrak{P})(\mathfrak{P}^+\mathcal{A}(\mathfrak{P}^*)^+)\mathfrak{P}^* + \frac{1}{2}\mathfrak{P}\mathfrak{P}^+\mathcal{A}(\mathfrak{P}^*)^+\mathfrak{P}^* + \frac{1}{2}\mathfrak{P}\mathfrak{P}^+\mathcal{A}^*(\mathfrak{P}^*)^+(I - \mathfrak{P}^+\mathfrak{P})\mathfrak{P}^*$$

Since $[(\mathfrak{P}\mathfrak{P}^+)^* = \mathfrak{P}\mathfrak{P}^+]$,$[\mathfrak{P}^* = \mathfrak{P}^+\mathfrak{P}\mathfrak{P}^*]$ and $[\mathfrak{P}\mathfrak{P}^+\mathfrak{P} = \mathfrak{P}]$

$= \frac{1}{2}\mathfrak{P}\mathfrak{P}^+\mathcal{A}^*\mathfrak{P}\mathfrak{P}^+ + \frac{1}{2}\mathfrak{P}\mathfrak{P}^+\mathcal{A}\mathfrak{P}\mathfrak{P}^+$, also from hypothesis $\mathcal{A} \in B(K)$ is Hermation operatorthen, so get $\mathfrak{P}X_0\mathfrak{P}^* + \mathfrak{P}X_0{}^*\mathfrak{P}^* = \frac{1}{2}\mathfrak{P}\mathfrak{P}^+\mathcal{A}\mathfrak{P}\mathfrak{P}^+ + \frac{1}{2}\mathfrak{P}\mathfrak{P}^+\mathcal{A}\mathfrak{P}\mathfrak{P}^+$

thus, $\mathfrak{P}X_0\mathfrak{P}^* + \mathfrak{P}X_0{}^*\mathfrak{P}^* = \mathfrak{P}\mathfrak{P}^+\mathcal{A}\mathfrak{P}\mathfrak{P}^+$, also$\mathfrak{P}X_0\mathfrak{P}^* + \mathfrak{P}X_0{}^*\mathfrak{P}^* = \mathcal{A}$,Therefore ;

X_0 is Hermation solution of equation(1.1) .

Now, we show that is X_0 is Hermation solution of equation (1.1) .

$$X_0^* = \left[\frac{1}{2}\langle(\mathfrak{P}^+\mathcal{A}(\mathfrak{P}^*)^+)^* + (I - \mathfrak{P}^+\mathfrak{P})(\mathfrak{P}^+\mathcal{A}(\mathfrak{P}^*)^+)\rangle\right]^*$$

$= \frac{1}{2}[\mathfrak{P}^+\mathcal{A}(\mathfrak{P}^*)^+ + \mathfrak{P}^+\mathcal{A}^*(\mathfrak{P}^*)^+(I - \mathfrak{P}^+\mathfrak{P})]$so one can have,

$X_0^* = \frac{1}{2}[\mathfrak{P}^+\mathcal{A}(\mathfrak{P}^*)^+ + \mathfrak{P}^+\mathcal{A}^*(\mathfrak{P}^*)^+ - \mathfrak{P}^+\mathcal{A}^*(\mathfrak{P}^*)^+\mathfrak{P}^+\mathfrak{P}]$ since$[(\mathfrak{P}^*)^+ = (\mathfrak{P}^*)^+\mathfrak{P}^*(\mathfrak{P}^*)^+]$

$$= \frac{1}{2}\mathfrak{P}^+\mathcal{A}^*(\mathfrak{P}^*)^+ + \frac{1}{2}\mathfrak{P}^+\mathcal{A}(\mathfrak{P}^*)^+ - \frac{1}{2}\mathfrak{P}^+\mathcal{A}^*(\mathfrak{P}^*)^+$$

$= \frac{1}{2}(\mathfrak{P}^+\mathcal{A}(\mathfrak{P}^*)^+)^* + \frac{1}{2}\mathfrak{P}^+\mathcal{A}(\mathfrak{P}^*)^+ - \frac{1}{2}\mathfrak{P}^+\mathcal{A}^*(\mathfrak{P}^*)^+$, $\mathcal{A} \in B(K)$ is Hermation operatorthen,

so get;

$$X_0^* = \frac{1}{2}(\mathfrak{P}^+\mathcal{A}(\mathfrak{P}^*)^+)^* + \frac{1}{2}\mathfrak{P}^+\mathcal{A}(\mathfrak{P}^*)^+ - \frac{1}{2}\mathfrak{P}^+\mathcal{A}(\mathfrak{P}^*)^+$$

and by using the condition $[\mathfrak{P}^+\mathfrak{P}\mathfrak{P}^+ = \mathfrak{P}^+]$ from definition of pseudo inverse of operator, thus can get,

$$X_0^* = \frac{1}{2}(\mathfrak{P}^+\mathcal{A}(\mathfrak{P}^*)^+)^* + \frac{1}{2}\mathfrak{P}^+\mathcal{A}(\mathfrak{P}^*)^+ - \frac{1}{2}\mathfrak{P}^+\mathfrak{P}\mathfrak{P}^+\mathcal{A}(\mathfrak{P}^*)^+$$
$$= \frac{1}{2}[(\mathfrak{P}^+\mathcal{A}(\mathfrak{P}^*)^+)^* + (I - \mathfrak{P}^+\mathfrak{P})(\mathfrak{P}^+\mathcal{A}(\mathfrak{P}^*)^+)]$$
$$= X_0$$

Then$X_0 = X_0^*$, so X_0 is Hermation solution equation (1.1)

Conversely; assume that X_0 is Hermation solution of equation (1.1) Then we get

$\mathfrak{P}X_0{}^*\mathfrak{P}^* + \mathfrak{P}X_0\mathfrak{P}^* = \mathcal{A}$, at first $[\mathfrak{P}X\mathfrak{P}^* + \mathfrak{P}X^*\mathfrak{P}^*]^* = \mathcal{A}^*$, this implies to $\mathfrak{P}X_0{}^*\mathfrak{P}^* + \mathfrak{P}X_0\mathfrak{P}^* = \mathcal{A}^*$, thus $\mathcal{A}$ is Hermation operator that is $\mathcal{A} = \mathcal{A}^*$, also

$$\mathfrak{P}X^*\mathfrak{P}^* + \mathfrak{P}X\mathfrak{P}^* =$$
$$\mathfrak{P}\left[\frac{1}{2}\langle(\mathfrak{P}^+\mathcal{A}(\mathfrak{P}^*)^+)^* + (I - \mathfrak{P}^+\mathfrak{P})(\mathfrak{P}^+\mathcal{A}(\mathfrak{P}^*)^+)\rangle\right]^*\mathfrak{P}^*$$
$$+ \mathfrak{P}\left[\frac{1}{2}\langle(\mathfrak{P}^+\mathcal{A}(\mathfrak{P}^*)^+)^* + (I - \mathfrak{P}^+\mathfrak{P})(\mathfrak{P}^+\mathcal{A}(\mathfrak{P}^*)^+)\rangle\right]\mathfrak{P}^* = \mathcal{A}$$
$$= \frac{1}{2}\mathfrak{P}\mathfrak{P}^+\mathcal{A}(\mathfrak{P}^*)^+\mathfrak{P}^* + \frac{1}{2}\mathfrak{P}\mathfrak{P}^+\mathcal{A}^*(\mathfrak{P}^*)^+(I - \mathfrak{P}^+\mathfrak{P})\mathfrak{P}^*$$
$$+ \frac{1}{2}\mathfrak{P}\mathfrak{P}^+\mathcal{A}^*(\mathfrak{P}^*)^+\mathfrak{P}^* + \frac{1}{2}\mathfrak{P}(I - \mathfrak{P}^+\mathfrak{P})(\mathfrak{P}^+\mathcal{A}(\mathfrak{P}^*)^+) = \mathcal{A}$$

Since $[(\mathfrak{P}\mathfrak{P}^+)^* = \mathfrak{P}\mathfrak{P}^+]$,$[\mathfrak{P}^* = \mathfrak{P}^+\mathfrak{P}\mathfrak{P}^*]$ and $[\mathfrak{P}\mathfrak{P}^+\mathfrak{P} = \mathfrak{P}]$
$= \frac{1}{2}\mathfrak{P}\mathfrak{P}^+\mathcal{A}\mathfrak{P}\mathfrak{P}^+ + \frac{1}{2}\mathfrak{P}\mathfrak{P}^+\mathcal{A}^*\mathfrak{P}\mathfrak{P}^+ = \mathcal{A}$, but $\mathcal{A} = \mathcal{A}^*$,

thus, $\mathfrak{P}X_0{}^*\mathfrak{P}^* + \mathfrak{P}X_0\mathfrak{P}^* = \mathfrak{P}\mathfrak{P}^+\mathcal{A}\mathfrak{P}\mathfrak{P}^+$, also$\mathfrak{P}X_0{}^*\mathfrak{P}^* + \mathfrak{P}X_0\mathfrak{P}^* = \mathcal{A}$,then we obtain

$\mathfrak{P}\mathfrak{P}^+\mathcal{A}\mathfrak{P}\mathfrak{P}^+ = \mathcal{A}$, and since X_0 is a solution of (1.1) then satisfy it, so $\mathfrak{P}X^*\mathfrak{P}^* + \mathfrak{P}X\mathfrak{P}^* = \mathcal{A}$ this implies that $\mathfrak{P}X_0\mathfrak{P}^* + \mathfrak{P}X_0{}^*\mathfrak{P}^* = \mathcal{A}^*$, but from hypothesis, we get $\mathfrak{P}X_0{}^*\mathfrak{P}^* + \mathfrak{P}X_0\mathfrak{P}^* = \mathcal{A}^*$, thus $\mathcal{A} = \mathcal{A}^*$, Therefore; $\mathcal{A} \in B(H)$ is Hermation .

Now, the following theorem gives the general Hermation solution of operator equation (1.1) .

Theorem(2.9): Let$\mathfrak{P} \in B(H, K)$ be closed range If $\mathfrak{P}\mathfrak{P}^+\mathcal{A}\mathfrak{P}\mathfrak{P}^+ = \mathcal{A}$ is Hermation operator. Then the form of general Hermation solution of operator equation (1.1) is $X = X_0 + (I - \mathfrak{P}^+\mathfrak{P})S(I - \mathfrak{P}^+\mathfrak{P})$, Where $\mathcal{A} \in B(K)$ is Hermation operator and $S \in B(H)$ is arbitrary operator.

Proof : Suppose $\mathcal{A} \in B(K)$ is Hermation operator then, $X = X_0 + (I - \mathfrak{P}^+\mathfrak{P})S(I - \mathfrak{P}^+\mathfrak{P})$ is a Hermation also, to do This replaces the bounded operator equation's(1.1)left side,then we get:

$$\mathfrak{P}X\mathfrak{P}^* + \mathfrak{P}X^*\mathfrak{P}^* = \mathfrak{P}[X_0 + (I - \mathfrak{P}^+\mathfrak{P})S(I - \mathfrak{P}^+\mathfrak{P})]\mathfrak{P}^* + \mathfrak{P}[X_0 + (I - \mathfrak{P}^+\mathfrak{P})S(I - \mathfrak{P}^+\mathfrak{P})]^*\mathfrak{P}^*$$
$$= \mathfrak{P}\left[\frac{1}{2}\langle(\mathfrak{P}^+\mathcal{A}(\mathfrak{P}^*)^+)^* + (I - \mathfrak{P}^+\mathfrak{P})(\mathfrak{P}^+\mathcal{A}(\mathfrak{P}^*)^+)\rangle + (I - \mathfrak{P}^+\mathfrak{P})S(I - \mathfrak{P}^+\mathfrak{P})\right]\mathfrak{P}^*$$
$$+ \mathfrak{P}\left[\frac{1}{2}\langle(\mathfrak{P}^+\mathcal{A}(\mathfrak{P}^*)^+)^* + (I - \mathfrak{P}^+\mathfrak{P})(\mathfrak{P}^+\mathcal{A}(\mathfrak{P}^*)^+)\rangle + (I - \mathfrak{P}^+\mathfrak{P})S(I - \mathfrak{P}^+\mathfrak{P})\right]^*\mathfrak{P}^*$$

$$=\frac{1}{2}\mathfrak{P}\mathfrak{P}^{+}\mathcal{A}^{*}(\mathfrak{P}^{*})^{+}\mathfrak{P}^{*}+\frac{1}{2}\mathfrak{P}(I-\mathfrak{P}^{+}\mathfrak{P})(\mathfrak{P}^{+}\mathcal{A}(B^{*})^{+})B^{*}+\mathfrak{P}(I-\mathfrak{P}^{+}\mathfrak{P})S(I-\mathfrak{P}^{+}\mathfrak{P})\mathfrak{P}^{*}$$
$$+\frac{1}{2}\mathfrak{P}\mathfrak{P}^{+}\mathcal{A}(\mathfrak{P}^{*})^{+}\mathfrak{P}^{*}+\frac{1}{2}\mathfrak{P}\mathfrak{P}^{+}\mathcal{A}^{*}(\mathfrak{P}^{*})^{+}(I-\mathfrak{P}^{+}\mathfrak{P})\mathfrak{P}^{*}+\mathfrak{P}(I-\mathfrak{P}^{+}\mathfrak{P})S(I-\mathfrak{P}^{+}\mathfrak{P})\mathfrak{P}^{*}$$

Since $[(\mathfrak{P}\mathfrak{P}^{+})^{*}=\mathfrak{P}\mathfrak{P}^{+}]$,$[\mathfrak{P}^{*}=\mathfrak{P}^{+}\mathfrak{P}\mathfrak{P}^{*}]$ and $[\mathfrak{P}\mathfrak{P}^{+}\mathfrak{P}=\mathfrak{P}]$
$=\frac{1}{2}\mathfrak{P}\mathfrak{P}^{+}\mathcal{A}^{*}\mathfrak{P}\mathfrak{P}^{+}+\frac{1}{2}\mathfrak{P}\mathfrak{P}^{+}\mathcal{A}\mathfrak{P}\mathfrak{P}^{+}$, A is Hermation operatorthen, so get
$$=\frac{1}{2}\mathfrak{P}\mathfrak{P}^{+}\mathcal{A}\mathfrak{P}\mathfrak{P}^{+}+\frac{1}{2}\mathfrak{P}\mathfrak{P}^{+}\mathcal{A}\mathfrak{P}\mathfrak{P}^{+}=\mathfrak{P}\mathfrak{P}^{+}\mathcal{A}\mathfrak{P}\mathfrak{P}^{+}$$
So, $\mathfrak{P}X\mathfrak{P}^{*}+\mathfrak{P}X^{*}\mathfrak{P}^{*}=\mathcal{A}$Therefore X is general solution of equation (1.1) .

Now, we show that is X is general Hermation solution of equation (1.1)

$$X^{*}=[X_{0}+(I-\mathfrak{P}^{+}\mathfrak{P})S(I-\mathfrak{P}^{+}\mathfrak{P})]^{*}$$
$$=\left[\frac{1}{2}\langle(\mathfrak{P}^{+}\mathcal{A}(\mathfrak{P}^{*})^{+})^{*}+(I-\mathfrak{P}^{+}\mathfrak{P})(\mathfrak{P}^{+}\mathcal{A}(\mathfrak{P}^{*})^{+})\rangle+(I-\mathfrak{P}^{+}\mathfrak{P})S(I-\mathfrak{P}^{+}\mathfrak{P})\right]^{*}$$
$$=\frac{1}{2}\mathfrak{P}^{+}\mathcal{A}(\mathfrak{P}^{*})^{+}+\frac{1}{2}\mathfrak{P}^{+}\mathcal{A}^{*}(\mathfrak{P}^{*})^{+}(I-\mathfrak{P}^{+}\mathfrak{P})+(I-\mathfrak{P}^{+}\mathfrak{P})S(I-\mathfrak{P}^{+}\mathfrak{P})$$
$$=\frac{1}{2}\mathfrak{P}^{+}\mathcal{A}(\mathfrak{P}^{*})^{+}+\frac{1}{2}\mathfrak{P}^{+}\mathcal{A}^{*}(\mathfrak{P}^{*})^{+}-\frac{1}{2}\mathfrak{P}^{+}\mathcal{A}^{*}(\mathfrak{P}^{*})^{+}\mathfrak{P}^{+}\mathfrak{P}+(I-\mathfrak{P}^{+}\mathfrak{P})S(I-\mathfrak{P}^{+}\mathfrak{P})$$
But $[(\mathfrak{P}^{*})^{+}=(\mathfrak{P}^{*})^{+}\mathfrak{P}^{*}(\mathfrak{P}^{*})^{+}]$
$$=\frac{1}{2}\mathfrak{P}^{+}\mathcal{A}(\mathfrak{P}^{*})^{+}+\frac{1}{2}\mathfrak{P}^{+}\mathcal{A}^{*}(\mathfrak{P}^{*})^{+}-\frac{1}{2}\mathfrak{P}^{+}\mathcal{A}^{*}(\mathfrak{P}^{*})^{+}+(I-\mathfrak{P}^{+}\mathfrak{P})S(I-\mathfrak{P}^{+}\mathfrak{P})$$
$$=\frac{1}{2}(\mathfrak{P}^{+}\mathcal{A}(\mathfrak{P}^{*})^{+})^{*}+\frac{1}{2}\mathfrak{P}^{+}\mathcal{A}(\mathfrak{P}^{*})^{+}-\frac{1}{2}\mathfrak{P}^{+}\mathcal{A}^{*}(\mathfrak{P}^{*})^{+}+(I-\mathfrak{P}^{+}\mathfrak{P})S(I-\mathfrak{P}^{+}\mathfrak{P})$$
But $\mathcal{A}$ is Hermation operatorthen, so get
$$X^{*}=\frac{1}{2}(\mathfrak{P}^{+}\mathcal{A}(\mathfrak{P}^{*})^{+})^{*}+\frac{1}{2}\mathfrak{P}^{+}\mathcal{A}(\mathfrak{P}^{*})^{+}-\frac{1}{2}\mathfrak{P}^{+}\mathcal{A}(\mathfrak{P}^{*})^{+}+(I-\mathfrak{P}^{+}\mathfrak{P})S(I-\mathfrak{P}^{+}\mathfrak{P})$$
And by using the condition $[\mathfrak{P}^{+}\mathfrak{P}\mathfrak{P}^{+}=\mathfrak{P}^{+}]$ from definition of pseudo inverse of operator, thus can get,

$$X^{*}=\frac{1}{2}(\mathfrak{P}^{+}\mathcal{A}(\mathfrak{P}^{*})^{+})^{*}+\frac{1}{2}\mathfrak{P}^{+}\mathcal{A}(\mathfrak{P}^{*})^{+}-\frac{1}{2}\mathfrak{P}^{+}\mathfrak{P}\mathfrak{P}^{+}\mathcal{A}(\mathfrak{P}^{*})^{+}+(I-\mathfrak{P}^{+}\mathfrak{P})S(I-\mathfrak{P}^{+}\mathfrak{P})$$
$$=\frac{1}{2}[(\mathfrak{P}^{+}\mathcal{A}(\mathfrak{P}^{*})^{+})^{*}+(I-\mathfrak{P}^{+}\mathfrak{P})(\mathfrak{P}^{+}\mathcal{A}(\mathfrak{P}^{*})^{+})]+(I-\mathfrak{P}^{+}\mathfrak{P})S(I-\mathfrak{P}^{+}\mathfrak{P})$$
$$=X_{0}+(I-\mathfrak{P}^{+}\mathfrak{P})S(I-\mathfrak{P}^{+}\mathfrak{P})$$
$$=X$$
Then$X=X^{*}$ so X is Hermation solution equation (1.1)

Conversely; assume that X is Hermation solution of equation (1.1) Then we get

$\mathfrak{P}X^{*}\mathfrak{P}^{*}+\mathfrak{P}X\mathfrak{P}^{*}=\mathcal{A}$, at first $[\mathfrak{P}X\mathfrak{P}^{*}+\mathfrak{P}X^{*}\mathfrak{P}^{*}]^{*}=\mathcal{A}^{*}$, this implies to $\mathfrak{P}X^{*}\mathfrak{P}^{*}+\mathfrak{P}X\mathfrak{P}^{*}=\mathcal{A}^{*}$, thus $\mathcal{A}$ is Hermation operator that is $\mathcal{A}=\mathcal{A}^{*}$, also
$$\mathfrak{P}X^{*}\mathfrak{P}^{*}+\mathfrak{P}X\mathfrak{P}^{*}=\mathfrak{P}[X_{0}+(I-\mathfrak{P}^{+}\mathfrak{P})S(I-\mathfrak{P}^{+}\mathfrak{P})]^{*}\mathfrak{P}^{*}+\mathfrak{P}[X_{0}+(I-\mathfrak{P}^{+}\mathfrak{P})S(I-\mathfrak{P}^{+}\mathfrak{P})]\mathfrak{P}^{*}$$

$$= \mathfrak{P}\left[\frac{1}{2}[(\mathfrak{P}^+\mathcal{A}(\mathfrak{P}^*)^+)^* + (I-\mathfrak{P}^+\mathfrak{P})(\mathfrak{P}^+\mathcal{A}(\mathfrak{P}^*)^+)] + (I-\mathfrak{P}^+\mathfrak{P})S(I-\mathfrak{P}^+\mathfrak{P})\right]^*\mathfrak{P}^* + \mathfrak{P}\left[\frac{1}{2}\langle(\mathfrak{P}^+\mathcal{A}(\mathfrak{P}^*)^+)^* + (I-\mathfrak{P}^+\mathfrak{P})(\mathfrak{P}^+\mathcal{A}(\mathfrak{P}^*)^+)\rangle + (I-\mathfrak{P}^+\mathfrak{P})S(I-\mathfrak{P}^+\mathfrak{P})\right]\mathfrak{P}^* = \mathcal{A}$$

This lead to,

$$= \frac{1}{2}\mathfrak{P}\mathfrak{P}^+\mathcal{A}(\mathfrak{P}^*)^+\mathfrak{P}^* + \frac{1}{2}\mathfrak{P}\mathfrak{P}^+\mathcal{A}^*(\mathfrak{P}^*)^+(I-\mathfrak{P}^+\mathfrak{P})\mathfrak{P}^* + \mathfrak{P}(I-\mathfrak{P}^+\mathfrak{P})S(I-\mathfrak{P}^+\mathfrak{P})\mathfrak{P}^* + \frac{1}{2}\mathfrak{P}\mathfrak{P}^+\mathcal{A}^*(\mathfrak{P}^*)^+\mathfrak{P}^* + \frac{1}{2}\mathfrak{P}(I-\mathfrak{P}^+\mathfrak{P})(\mathfrak{P}^+\mathcal{A}(\mathfrak{P}^*)^+)\mathfrak{P}^* + \mathfrak{P}(I-\mathfrak{P}^+\mathfrak{P})S(I-\mathfrak{P}^+\mathfrak{P})\mathfrak{P}^* = \mathcal{A}$$

$= \frac{1}{2}\mathfrak{P}\mathfrak{P}^+\mathcal{A}^*\mathfrak{P}\mathfrak{P}^+ + \frac{1}{2}\mathfrak{P}\mathfrak{P}^+\mathcal{A}\mathfrak{P}\mathfrak{P}^+$, but $\mathcal{A} = \mathcal{A}^*$,

thus, $\mathfrak{P}X^*\mathfrak{P}^* + \mathfrak{P}X\mathfrak{P}^* = \mathfrak{P}\mathfrak{P}^+\mathcal{A}\mathfrak{P}\mathfrak{P}^+$, also $\mathfrak{P}X^*\mathfrak{P}^* + \mathfrak{P}X\mathfrak{P}^* = \mathcal{A}$,then we obtain

$\mathfrak{P}\mathfrak{P}^+A\mathfrak{P}\mathfrak{P}^+ = \mathcal{A},$ and since X is a solution of (1.1) then satisfy it, so $\mathfrak{P}X^*\mathfrak{P}^* + \mathfrak{P}X\mathfrak{P}^* = \mathcal{A}$ this implies that $\mathfrak{P}X\mathfrak{P}^* + \mathfrak{P}X^*\mathfrak{P}^* = \mathcal{A}^*$, but from hypothesis, we get $\mathfrak{P}X^*\mathfrak{P}^* + \mathfrak{P}X\mathfrak{P}^* = \mathcal{A}^*$, thus $\mathcal{A} = \mathcal{A}^*$, Therefore; $\mathcal{A} \in B(H)$ is Hermation .

3. Conclusion

This work, presented practical and general solution of bounded operator equation(1.1), when the operator$\mathfrak{P}$ is noninvertible operator, so we instead of that we use the pseudo inverse of this operator, also discovered the explicit Hermation solutions of operator equation (1.1).

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