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Some Results Topoi by Topological Groupoids

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Abstract:

In this paper, we discuss the action for topological groupoids. In our work, some definitions and propositions are introduced and in addition to the basic construction of topological groupoid which are needed and we studied the different properties of categories and the properties of topological groupoid, and we define concepts such as topos, , functor and sheave.

Key words : Category , Power object **,** Sheaf , Functor.

1. Introduction:

We study in this paper topological groupoid proposition and new relation can be applied in algebraic topology. In fact, using topological groupoid, we provide symmetry between their principal action rule. The category C contain for:

(i) The class for objects.(ii) If $\mathfrak{r} \in$ morphism (S, L) with domain S and range L, we write $\mathfrak{r}: S \to L$ for all arranged pair of things S and L. (iii) A function that associates two morphisms $\mathfrak{r}: S \to L$ and $\mathfrak{r}_1: L \to H$ their composite $\mathfrak{r}_1 o \mathfrak{r} : S \to H$ for all ordered triple of objects S, L and H. This satisfies the following axioms.

(1) The associative axiom: let $\mathfrak{r}: S \to L$, $\mathfrak{r}_1: L \to H$, $\mathfrak{r}_2: H \to K$ then $\mathfrak{r}_2(\mathfrak{r}_1\mathfrak{r}) = (\mathfrak{r}_2\mathfrak{r}_1)\,\mathfrak{r}$.(2) The identity axiom of all objects L there is the morphism $I_L: L \to L$ where let $\mathfrak{r}: S \to L$, implies $I_L\mathfrak{r} = \mathfrak{r}$, and if $\mathfrak{r}_2: L \to H$, then $\mathfrak{r}_2 I_L = \mathfrak{r}_2$[10] . A groupoid be the pair of sets (N, M) where be get.(1) onto functions $\alpha: N \to M$, $\beta: N \to M$ they are called respectively, a source function, a target function. (2) one-to-one function $w: M \to N$ known as the object inclusion with $\alpha o w = I_M$, $\beta o w = I_M$ where $I_M: M \to M$.(3) A partial composition law λ in N , a compositional rule for the set $N * N$ is defined as $N * N = \{(n_1, n_2) \in N \times N | \alpha(n_1) = \beta(n_2)\}$ "fiber product of β and α over M" s.t :

(1) $\lambda(n.\lambda(n_1 , n_2))\, \lambda(\lambda(n , n_1), n_2)$, $\forall (n, n_1), (n_1, n_2) \in N * N$.

(2)$\alpha(\lambda(n_1,n_2))$=$\alpha(n_2)$, $\beta(\lambda(n_1, n_2))$=$\beta(n_1)$ for each $(n_1, n_2) \in N * N$.

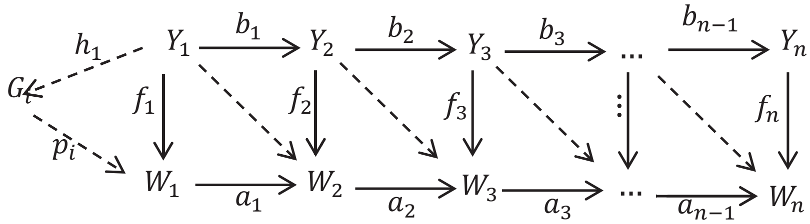
(3) $\lambda\,(n_1, w(\alpha(n_1))) = n_1$ and $\lambda\,(w(\beta(n_1)), n_1) = n_1$, for all $n_1 \in N$.
(iv) A bijection $\delta: N \to N$ known as the inversion of N satisfying:
(a) $\alpha\,(\delta(n_1)) = \beta(n_1), \beta(\delta\,(n_1)) = \alpha(n_1))$, for all $n_1 \in N$.
(b) $\lambda(\delta(n_1), n_1) = w(\alpha(n_1)), \lambda(n_1, \delta(n_1)) = w(\beta(n_1))$, for all $n_1 \in N$.
We they note $\delta(n_1) = (n_1)^{-1}$, known an inverse for element $n_1 \in N$,$w(x) = x$ known a unit for element on N associated into an element $x \in M$. We will take notes $(n_1, n_2) = n_1 n_2$. We say that N is a groupoid on M. We call say this is N be the groupoid in M [7]. Let C be the category with finite limits. The Power object of the object $a \in C$ is an object γ^a and a monomorphism M_a where $M_a \subseteq a \times \gamma^a$, such that for every object b and every monomorphism $\grave{r} \subseteq a \times b$ there is a unique morphism $X_{\grave{r}}: b \longrightarrow \gamma^a$ such that $\grave{r}$ is a Pullback [3]. Topos is the category C has a next two qualities (1) Each limit applied to a finite index. (2) All object does have the power object that is plays a role for a power collection into set theory [1]. A category for set $\grave{C}$ is the topos, and play a role similar to the one-point space's role in topology. The topos morphism, in particular, is the point of the topos C. $\rho: \grave{C} \to C$. It is provided by a functor. $\rho^*: C \to \grave{C}$. which is commutative with finite limits and colimits [6]. Asheaf is a bundle with some additional topological structure. Let Z be the topological space, with U it is collection for open set. The sheaf over Z be the pair (Y, P) such that Y is the topological space, $P: Y \to Z$ is the continuous map that is a local homeomorphism [2]. Let S be the topological space with K its collection for open subsets K becomes the post category when inclusions are made in a specific order, with arrows denoting only the inclusions that are part of the U subset of the V[9]. The stack or presheaf over S is the contravariant functor from K to collection. As a result, a stack F assigns the function to every inclusion U subset of V and the set $F(V)$ to every open V. $F_U^V: \mathrm{F(V)} \to \mathrm{F(U)}$ such that (1) $F_U^U = \mathrm{id}_U$ (2) If $U \subseteq V \subseteq K$, then Commutes i.e $F_U^K = F_U^V o\, F_V^K$ commutes i.e $F_U^K = F_U^V o\, F_V^K$ [5]. A continuous map $f: Y \to W$ for topological spaces be known locally connected. Suppose f be the open function, Y has a basis for open collection denoted by B_a with a property that for every $w \in W$, as well as any basic open set $A \in B_a$, a fibre, $A_w = f^{-1}(w) \cap A$ is connected or empty[4].

2. Action by the groupoid:

In this part we talk about the most important results that have been reached

Proposition (2.1):

(1) Let $f_1, f_{2,\dots,} f_n$ be locally connected maps then the composition of $f_n\, o \dots o\, f_2\, o\, f_1$ such that $a_i\, o f_i = f_{i+1} o b_i\ \ i = 1, \dots., n-1$ is locally conected where $n \in N$.
(2) In a pullback square(1)



… (1)

if f_1 is locally connected, then so $f_2, f_3, \dots, f_n$ such that the projection formula $a_i^* f_{i+1}^* = f_i^* b_i^*$ $(i = 1,2, \dots, n)$.

(3) Any local homeomorphism is locally connected .

(4) If the composition $f_i = p_i \, o \, h_i: Y_i \to G_i \to W_i$ is locally connected and $p_i: G_i \to W_i, (i = 1, \dots, n)$ is the local homeomorphism implies h_i $i = 1, \dots, n$ is locally connected.

Proof

One first proof that for a locally connected. $f_i : Y_i \to W_i$ with basis B_a, $f_i(E_i) \cap A$ be connected for each open $A \in B_a$ and any connected subset $E \subset f_i(A)$ .implies , if $h_i : W_i \to K_i, i = 1, \dots, n$ is another a locally connected map with basis B_{a1} for W_i, the set $A \cap f_i^{-1}(B)$, for $A \in B_a$ and $B \in B_{a1}$ with $B \in f_i(A)$, from a basis for y_i witnessing that $h_i \, o \, f_i$ is a locally connected.

Proposition (2.2):

For any a locally connected maps $g \, o \, f: Y \to K$ where there exists a unique (up to homeomorphism) factorization. $Y \overset{q}{\to} \grave{\pi}(g \circ f) \overset{p}{\to} K$ where $p: \grave{\pi}(g \circ f) \to K$ is a local homeomorphism , $q: Y \to \grave{\pi}(g \circ f)$ be the locally connected function with connected fibres.

Proof

we define a space $\grave{\pi}\,(gof)$: a points are pairs (k, q) such that $k \in K$ & q be the connected component for $(gof)^{-1}(k)$. We define a topology onto $\grave{\pi}(gof)$, let B_a be a collection for all these open sets $A \subset Y$ into which A_K be connected or empty $(\forall k \in K)$. Implies B_a is the basis for Y. A basic open sets for $\grave{\pi}(gof)$ are presently defined as the sets. A^*

$= \{(k, [A_k] \,| k \in (gof)\,(A)\}$, where $[A_K]$ is a connected component for $(gof)^{-1}\,(k)$ that contains $A_k = (gof)^{-1}(k) \cap A$.and A ranges over all elements for B_a. Seeing this as a basis, suppose $(k, q) \in B^* \cap A^*$. Thus $B, A \in B_a$ and $B_k \cap A_k$ supset of q. Since q be connected, there is a chain for basic open sets, $B = A_\circ, A_1, \dots, A_n = A$ in Y with a property

that$A_j \cap A_{j+1} \cap \dot{q} \neq \emptyset$ $(j = 0, \dots, n-1)$.Now let
$L = (A_o \cup \dots \cup A_n) \cap (gof)^{-1} \, (\cap_{j=0}^{n-1} (gof)(A_j \cap A_{j+1})$ .Then $L \in \mathrm{B}_a$, and$(k, q) \in L^* \subset A^* \cap B^*$. Now $(gof): Y \to K$ factors to the map $c: Y \to \grave{\pi}\,(gof), c\,(y) = ((gof)\,(y), [y])$

where $[y]$ is a component of $(gof)^{-1}\,(gof)(y)$ containing y, and as the map $\sigma: \grave{\pi}(gof) \longrightarrow K, \sigma(k, q) = q$. This map q restricts to the homeomorphism $A^* \to (gof)\,(A)$ for all $A \in L$, hence be the local homeomorphism. Thus c be the locally connected map by (in page3

Proposition (2.1)(4)), a fibres for c are evidently connected, since $c^{-1}(k,q) = q \subset (gof)^{-1}(k)$.

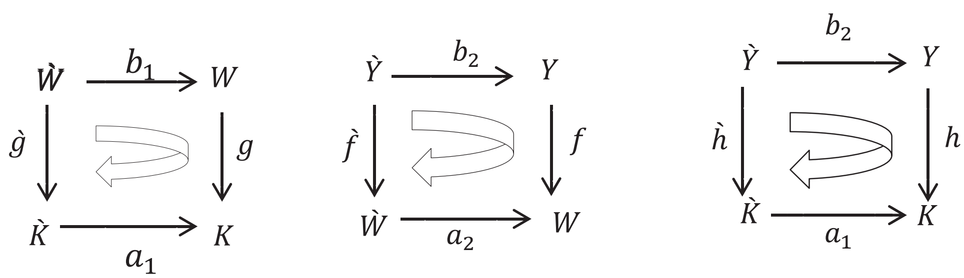
Theorem (2.3):

(1) Let $h = (gof): Y \to K$ where $f: Y \to W, g: W \to K$ are the locally connected map. Then a pullback functor for sheaves

$h^* = (gof)^*: sh(K) \to sh(Y)$ has the left adjoint .

$h_1 = (gof): sh(Y) \to sh(K)$

(2) For any pullback square for topological spaces



and h locally connected, a projection formula $a^* \grave{h}_1 = h_1 b^*$

Proof

we identify a category for sheaves G over Y with that for local homeomorphism $\mathrm{p}: G \to Y$.

(1) For the local homeomorphism $\mathrm{p}: G \to Y$, a composite $h\, o\, \mathrm{p}$ be locally connected by(in page 3 proposition (2.1),(1),(3)), so factors uniquely as $h\, o\mathrm{p} =$

$(G \to \grave{\pi}\, (h\, o\mathrm{p}) \to K)$ as into(in page 3 proposition (2.2)) define $h_1(G)$ to be a sheaf

$\grave{\pi}\, (h\, o\mathrm{p}) \to K$. Thus, by construction, a stalk for $h_1(G)$ at k into a set for connected components for $(h\, o\mathrm{p})^{-1}(k)$, $h_1\,(G)_k = \grave{\pi}\,(h\mathrm{p})^{-1}\,(k))$.

For adjointness , $q: E \to K$ is the sheaf on K, and suppose

$s: G \to (h)^*(E) = E \times_K Y$ is the map . Then $\bar{s} = \grave{\pi}_1 o\, s\; : G \to E$ be the map over k, i.e., $q o \bar{s} = h\, op$. Since a fibres for q are discrete, $\bar{s}$ is constant onto a connected components for all fibre $(h\mathrm{p}))^{-1}\,(k)$, hence factors uniquely as the map $h_1(G) \to E$.

(3) For the sheaf G on Y, an adjointness for part (1) provides the canonical map $h_1\, b^*\,(G) \to a^* h_1\,(G)$.

Proposition (2.4):

The fibres of a source and target maps $\alpha, \beta: N \to M$ are enumeration spaces for a point (p,m) $\in M$ there are homeomorphisms, there are homeomorphisms $\alpha^{-1}\times \alpha^{-1}$((p, m), (p, m)) $\cong E_n\ (\alpha_p)\times E_n\ (\alpha_p) \cong \beta^{-1}\times\beta^{-1}$ ((p, m), (p, m)) .

Proof

It suffices into prove that of a source map. ((p, m), (p, m)) $\in M\times M$. Note first that any Point $((\mathrm{p},\mathrm{m}),(\mathrm{p},\mathrm{m})) \overset{\theta}{\rightarrow} ((q,t),(q,t))$ into $N\times N$ is equivalent into a Paint $\big((\mathrm{p},\mathrm{m}),(\mathrm{p},\mathrm{m})\big) \overset{id}{\rightarrow} ((p,\theta_{\propto}^{-1}\ o\ t),(p,\theta_{\propto}^{-1}\ o\ t))$.Other words, all equivalence class has a representation of for a from $\big((\mathrm{p},\mathrm{m}),(\mathrm{p},\mathrm{m})\big) \overset{id}{\rightarrow} \big((\mathrm{p},\mathrm{t}),(\mathrm{p},\mathrm{t})\big)$, a evident map $j_{(p,m)}\times j_{(p,m)}$: $E_n(\alpha_p)\times E_n(\alpha_p) \rightarrow N\times N$

defined by $t\times t \rightarrow [(p,m)\times(p,m) \overset{id}{\rightarrow} ((p,t)\times(p,t)]$ is a bijection in $\alpha^{-1}\times\alpha^{-1}$ $((\mathrm{p},\mathrm{m}),(\mathrm{p},\mathrm{m}))$. Now let the basic open set $U\times U$ into $N\times N$

$U\times U = \{\big((\mathrm{p},\mathrm{m}),(\mathrm{p},\mathrm{m})\big) \overset{\theta}{\rightarrow} \big((q,t),(q,t)\big) | m(i)\times m(i) \in t_p\times t_p, m(j)\times m(j) \in C_q\times C_q$, $\theta\ (m(i)\times m(i)) = (t(j)\times t(j))\}$ … (1)

Representing equivalence classes in to a the form (1), we can also write $U\times U = \{\big((\mathrm{p},\mathrm{m}),(\mathrm{p},\mathrm{m})\big) \overset{\theta}{\rightarrow} \big((p,t),(p,t)\big) |\ (m(i)\times m(i)) \in t_p\times t_p$

$t(j)\times t(j) \in C_q\times C_q, (m(i)\times m(i) = t(j)\times t(j)\}$. Thus, of $(\mathrm{p},\mathrm{m}),(\mathrm{p},\mathrm{m})$ fixed and for

$\alpha_1\times\alpha_1 = m(i_1), m(i_1),\dots \alpha_n\times\alpha_n = m(i_n), m(i_n)$, we see that $j_{(\mathrm{p,m})}^{-1}(U)\times j_{(\mathrm{p,m})}^{-1}(U) = \{t\times t \in E_n(\alpha_p)\times E_n(\alpha_p) | \alpha_1\times\alpha_1 = \big(t(i_1)\times t(i_1)\big), \dots \alpha_n\times\alpha_n = \big(t(i_n)\times t(i_n)\big)\}$,…(2)

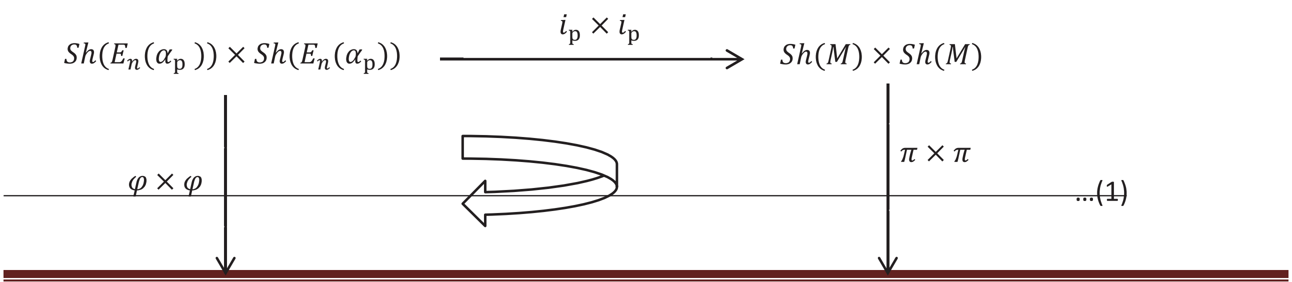
if $(\alpha_1,\dots,\alpha_n)\times(\alpha_1,\dots,\alpha_n) \in \big(t_p\ \cap\ C_p\big)\times\big(t_p\ \cap\ C_p\big)$ otherwise, it's empty. But a right-hand side for (2) exactly describes the standard basic open set into an enumeration space $E_n(\alpha_p)\times E_n(\alpha_p)$.

Preposition (2.5) :[8]

The source map $\alpha: N \rightarrow M$ and the target map $\beta: N \rightarrow M$ are locally connected.

Preposition (2.6):

The functor $\pi^*\times\pi^*: C\times C \rightarrow Sh(M)\times Sh(M)$ does have the left adjoining $\pi_1\times\pi_1$. For each point $\mathrm{p}\times\mathrm{p}: D\times D \rightarrow C\times C$, there exists a commutative square.

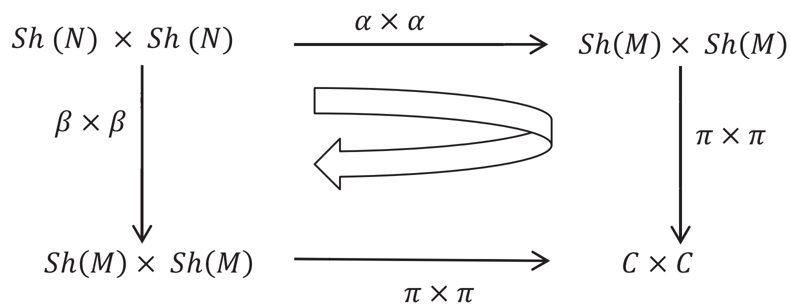


$$D \times D \xrightarrow[\mathrm{p} \times \mathrm{p}]{} C \times C$$

for which a projection formula $(\varphi_1 \times \varphi_1)(\, i_{\mathrm{p}}^* \times i_{\mathrm{p}}^*) = (\mathrm{p}^* \times \mathrm{p}^*)\ (\pi_1 \times \pi_1)$ holds. $D \times D$ =$Sh(\mathrm{p}\alpha)\times$ $Sh(p\alpha)$, while $\varphi \times \varphi$ and $i_{\mathrm{p}} \times i_{\mathrm{p}}$ are induced by the continuous maps of space $\mathrm{p}\alpha \times \mathrm{p}\alpha \xleftarrow{\varphi\times\varphi}$ $E_n(\alpha_p) \times E_n(\alpha_{\mathrm{p}}) \xrightarrow{i_{\mathrm{p}}\times i_{\mathrm{p}}} M \times M$.

Preposition (2.7):

A source and target maps α, β: $N \to M$ fit in the square (1) for topos morphism.



....(1)

which commutes up to the canonical isomorphism

$\grave{\mu} :\ (\alpha^*\ \times\ \alpha^*)\ (\pi^*\ \times\ \pi^*) \cong (\beta^*\ \times\ \beta^*)\ (\pi^*\ \times\ \pi^*)$. A projection formula hold of that square, i. e, an induced natural transformation

$t \times t$:$((\alpha_1\ \times\ \alpha_1)(\beta^*\ \times\ \beta^*)) \to ((\pi^*\ \times\ \pi^*)(\pi \times \pi))$ is an isomorphism.

Proof

Let g equals [$(\grave{\mathrm{p}}, \grave{m}) \times (\grave{\mathrm{p}}, \grave{m}) \overset{\theta}{\to} (q, t) \times (q, t)$]

of $N \times N$ and of all object $F \times F$ of $C \times C$, we have $(\alpha^*\ \times\ \alpha^*)\ (\pi^*\ \times\ \pi^*)\ (F_g \times F_g)\ = F_{\mathrm{p}} \times F_{\mathrm{p}}$, $(\beta^*\ \times$ $\beta^*)\ (\pi^*\ \times\ \pi^*)\ (F_g \times F_g)$ = $(F_q \times F_q)$ and a stalk for an isomorphism $\grave{\mu}_F\ \times \grave{\mu}_F\ :\ (\alpha^*\ \times\ \alpha^*)\ (\pi^*\ \times$ $\pi^*)(F \times F) \longrightarrow\ (\beta^*\ \times\ \beta^*)\ (\pi^*\ \times\ \pi^*)\ (F \times F)$ at a point g be defined to be an isomorphism

$\theta_F \times\ \theta_F : (F_{\mathrm{p}} \times F_{\mathrm{p}}) \to (F_q \times\ F_q)$. Next, we prove of all sheaf $E \times E$ on $M \times M$

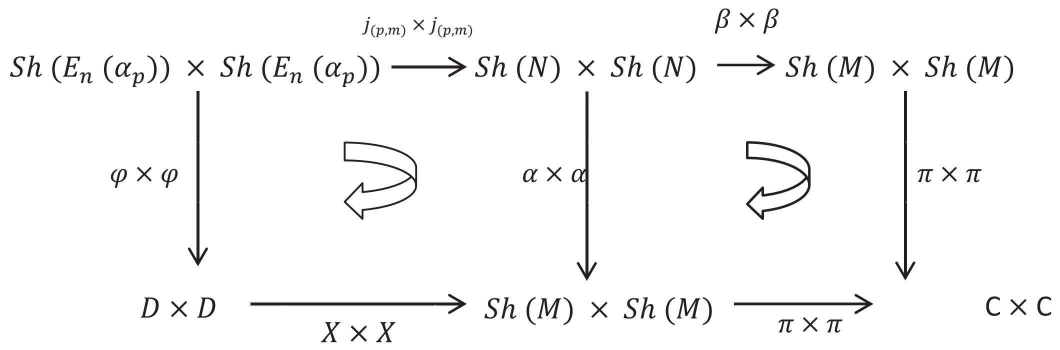
that $((\alpha_1\ \times\ \alpha_1)(\beta^*\ \times\ \beta^*))\ (E \times E) = ((\pi^*\ \times\ \pi^*)(\pi \times \pi))(E \times E)$ (or more

precisely, that a canonical map $((\alpha_1\ \times\ \alpha_1)(\beta^*\ \times\ \beta^*))\ (E \times E) \longrightarrow ((\pi^*\ \times\ \pi^*)(\pi \times \pi))(E \times E)$ be the isomorphism). It suffices into cheak that

$((\alpha_1\ \times\ \alpha_1)(\beta^*\ \times\ \beta^*))\ (E \times E)_x =\ ((\pi^*\ \times\ \pi^*)(\pi \times \pi))(E \times E)_x$

of a stalk at the arbitrary point $x = (\grave{p}, \grave{m}) \times (\grave{p}, \grave{m})$ into $M \times M$.

consider of that a diagram(2)



...(2)

From the topological spaces pullback, a left-hand square is presented here. (in page 5 Proposition (2.4)) and $(j_{(\grave{p},\grave{m})} \times j_{(\grave{p},\grave{m})})$ be as defined into the proof of (in page 6 Proposition (2.4))since $\alpha \times \alpha$ be locally connected by (in page 6 Proposition (2.5)(2)) givers a projection formula $(x^* \times x^*)(\alpha_1 \times \alpha_1) = (\varphi_1 \times \varphi_1)\, j^*{}_{(\grave{p},\grave{m})} \times j^*{}_{(\grave{p},\grave{m})}$ ...(3)

of a left- hand square Moreover, since (β×β) $o\ (j_{(\grave{p},\grave{m})} \times j_{(\grave{p},\grave{m})}) = i_p \times i_p$ and $(\pi \times \pi) o\ (x \times x) = \grave{p} \times \grave{p}$ (in page 6 Proposition (2.6)) gives that $((\pi o x)^* \times (\pi o x)^*)\ (\pi \times \pi) = (\varphi_1 \times \varphi_1)((\beta \times \beta) o\ (j_{(\grave{p},\grave{m})} \times j_{(\grave{p},\grave{m})})^*)$ ...(4)

for the composed rectangle. Thus

$(\alpha_1 \times \alpha_1)(\beta^* \times \beta^*)(E_x, E_x) = (x^* \times x^*)(\alpha_1 \times \alpha_1)(\beta^* \times \beta^*)\ (E \times E) = (\varphi_1 \times \varphi_1)\ (j^*{}_{(p,m)} \times j^*{}_{(p,m)})\ (\beta^*(E) \times \beta^*\ (E))$ by...(3)

$= (x^* \times x^*)\ (\pi^* \times \pi^*).(\pi_1(E) \times \pi_1(E))$ by... (4)

$= (\pi^* \times \pi^*).(\varphi_1(E)_x \times \varphi_1(E)_x).$

Conclusion:

We have studied representing topoi by topological groupoid, where new parameters have been studied ,including topos , sheaves , functor ,stalk and we have obtained new proposition that serve the field of algebraic topology.

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