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A Review of Continuous and Discontinuous Galerkin Finite Element Methodfor Differential Equations

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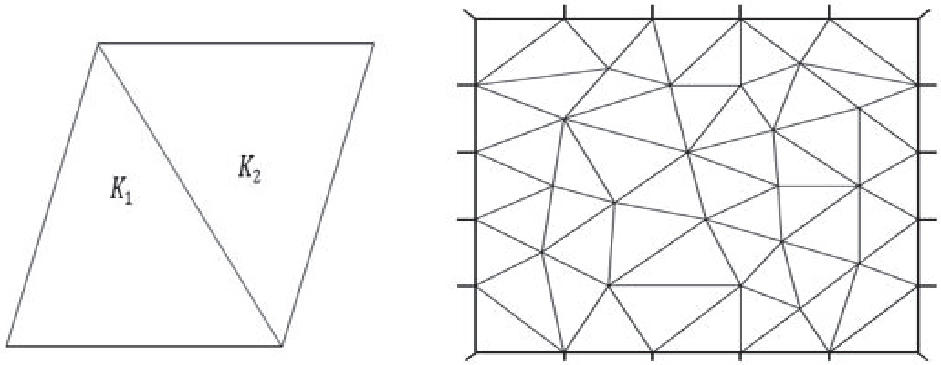
Abstract:

This study is a review of both the continuous Galerkin finite element method (CGFEM)and the discontinuous Galerkin finite element method (DGFEM).A group of the most important research over the last 13 years has been compared to both methods and their historical development, as well as comparing the advantages and disadvantages of both methods.This review is aimed at determining the best and most efficient method to solve complex problems in the future.

Keywords: continuous Galerkin finite element method, discontinuous Galerkin finite element method, differential equations.

1. Introduction

The finite element method (FEM) is a computational tool for solving physics and engineering differential equations. The approach was initially introduced in an article by Turner, Clough, and Martin [48] in the aerospace sector in the early 1950s. Clough [15] (1960) used the phrase CGFEM in a work about planar elasticity applications. Argyris [9] (1960) proposed a straightforward technique based on the virtual work idea. He created this work to use computational tools to address extremely difficult situations. Researchers in the early 1960s quickly shifted their focus to the solution of nonlinear issues. The CGFEMapproachwas originally used for conduction heat transport and fluid dynamics by Zienkiewicz and Cheung (1965)[56], and Wilson and Nickell (1966)[50]. Other researchers, Szabo and Lee (1969)[47], and Zienkiewicz (1971)[55], demonstrated that the element equations for structural mechanics, heat transfer, and fluid mechanics may be found using a weighted residual strategy such as Galerkin's method or the least-squares approach. This understanding is crucial to the theory since it enables the application of the FEM to any differential problem. The FEM has advanced from a structural numerical technique to a generic numerical technique for solving a differential equation or a collection of differential equations. Figure 1.1 clarifies the mesh of CGFEM.



(a)Two adjacent triangles intersected by a side | (b)Cut the space in the form of a grid of triangles

Figure 1.1: The discretization of CGFEM.

Discontinuous Galerkin finite element methods(DGFEMs) are a type of numerical method used to solve differential equations. They integrate elements of the finite volume (FV) and finite element (FE) frameworks and have been effectively used forelliptic, hyperbolic, parabolic,and issues with mixed forms arising from a range of applications. For problems with a dominating first-order portion, such as fluid mechanics, electrodynamics, and plasma physics, DGEM approaches have attracted a lot of attention. The method consists of three main cases: if $\varepsilon = -1$, the method is called symmetric interior penalty Galerkin (SIPG), if$\varepsilon = 1$, the method is called nonsymmetric interior penalty Galerkin (NIPG), or if $\varepsilon = 0$,the method is called inconsistent penalty Galerkin (IIPG), when ε is called the average termparameter[16],[27]. In the early 1970s, DGEMs were developed and studied as a method for numerically solving partial differential equations. Reed and Hill [40] proposed the DGEM approach for solving the hyperbolic neutron transport problem in 1973. In 1977, Baker [11] published a study on DGEM techniques for elliptic problems in the setting of 4th-order equations. Arnold et al.[10] published a book that gives a more detailed overview of the historical evolution as well as an introduction to DGEM approaches for elliptic problems. The proceedings book published by Cockburn et al.[11] contains several research objectives and problems on DGEM approaches Figure 1.2 clarifies the mesh of DGFEM.

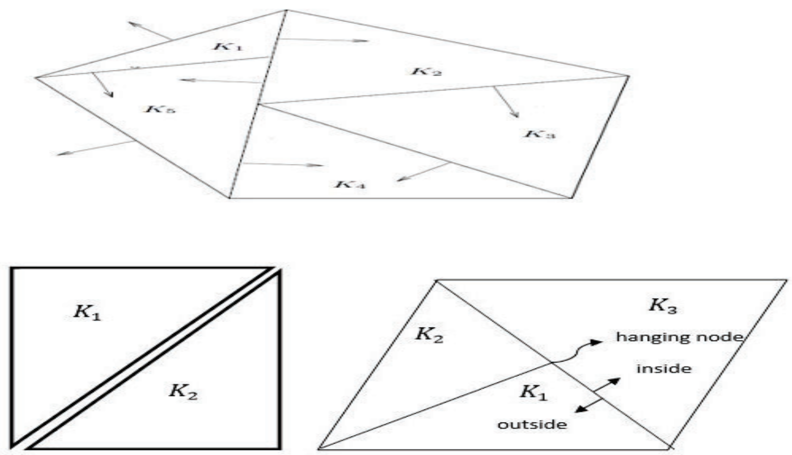


Figure 1.2: The discretization of DCGFEM

These numerical approaches help to solve the equations that have applications in different sciences, one of them is image processing applications such as image inpainting [6], [4] image denoising [3], image segmentation [23], and image classification [5].

The rest of the paper is organized as follows. Section 2 introduces a review of the continuous Galerkin finite element methods. We present a review ofdiscontinuous Galerkin finite element methods in Section 3. The comparison of continuous and discontinuous Galerkin finite element methods illustrates in section 4. Finally, in section 5, we finish the paper by giving the conclusion and future directions.

2. Continuous Galerkin Finite Element Method

In 2010, Pironneau and Tabata [39] proposed a Galerkin-characteristics lumped mass FEM for convection-diffusion problems. The scheme showed stability and convergence at the L^∞ −norm under the hypothesis of acute weak triangulation.

In 2011, Anca et al.[8] published a paper on utilizing FEM to simulate fusion welding. A moving heat source, the temperature dependency of thermo-physical characteristics, and elastoplasticity are all included in the models. The 3D stress condition of a butt-welded joint is derived as an example of an application.

In 2012, Galvin et al. [22] investigated the issue of poor conservation of mass in mixed FEM for flow issues with significant rotation-free forcing inside the numerical solution. For a benchmark natural convection issue, the theory is proved via numerical experiments.

In 2013, Dziuk and Elliott [18] considered FEMs for approximating PDEs on surfaces. They concentrated on surface FEs on triangulated surfaces as well as implicit surface approaches based on level-set surface descriptions. Some concepts of numerical analysis are studied with numerical examples to illustrate them.

In 2014, Yi [52] proposed a mixed FEM modulus for the 2D Biot uniformity model of porosity elasticity. Key unknown variables in the new mixed formula include the total stress tensor and fluid flow, as well as displacement and pore pressure. Optimal a priori error estimates have been demonstrated for bothfully discrete and semi-discrete issues.

In 2015, Roos and Schopf[45] showed that scaling the H^1 semi-norm differently results in a balanced norm that appropriately captures layer behavior. They demonstrated error estimates in balanced norms, explored stability issues, and suggested a novel $C0$ interior penalty approach that outperforms the FEM in terms of stability.

In 2016, authors [20] studied two different types of space-time fractional diffusion equations over a finite region. They used the Crank-Nicolson approach to calculate the time derivative and find the best convergence rate $(\tau^2 + h^2)$. In a few numerical instances, the numerical results agree well with the theoretical approach.

In 2017, John et al. [29] difference constraint for the incompressible Navier-Stokes equations

$$\frac{\partial u}{\partial t} - \nu\Delta\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p = \boldsymbol{f}, \qquad \nabla\cdot\boldsymbol{u} = 0, \tag{2.1}$$

has been re-examined in a mixed FE framework. Several techniques for improving discrete mass balancing or perhaps even calculating difference-free solutions are explored.

In 2018, the authors in [21] studied the convergence of a mixed FE–FVFE–FV numerical technique for the isentropic Navier–Stokes system.As long as the latter persists, numerical solutions of the limit system converge strongly to strong solutions of the limit system, according to the recently established weak–strong uniqueness principle in the class of dissipative measure-valued solutions.

Also in 2018, Furihata et al. [19] investigated the stochastic Cahn-Hilliard equation in a convex domain with a polygonal border of dimension driven by additive Gaussian noise. They used a typical FEM in space and a completely implicit backward Euler approach in time to discretize the problem. The numerical solution converges strongly with the solution by providing optimal error estimates on arbitrarily high probability subsets of the probability space and uniform-in-time moment constraints.

In 2019, Gong et al. [24] describe an adaptive FEM for control concerns with Dirichlet elliptical limits in energy space. By adding auxiliary issues and establishing specific standard equations, pre-error estimates were easily produced, and it was shown that adaptively constructed sequences of discrete solutions converge to the real ones.

The authors in [31] presented a demonstration of convergence for complete and semi-discretizations of the closed flow's mean curvature of 2-D surfaces. In the situation of FE with at least two polynomial degrees, this numerical technique allows for a convergence analysis. To get optimal order H^1-norm error bounds, the error analysis includes consistency and stability calculations.

In 2020, Chen et al. [14] looked at using FEM to identify defects in 2-D engineering materials using ultrasonic time-of-flight diffraction (TOFD) phased array technology. The feasibility of FEM detection was determined by comparing the error ratios of the simulation and experiment.

The authors in [28] have addressed the issue of temporal Allen-Cahn fractions, in which the spatial domain Ω is a finite subset of $\mathbb{R}^d$for some $d \in \{1,2,3\}$. For the given derivatives of the solution, new limits were obtained, which were used in the analysis of the numerical method. For the Sobolev H^1 standard, the calculated solution resulted in the best affinity rate.

In 2021, Metzger [35] rethought a unified approach to solving nonlinear equations in Hilbert spaces. It was determined that iterative linear finite element techniques (AILFEMs) achieve complete linear convergence. In terms of degrees of freedom and overall computational time, optimal algebraic rates were found.

Also, the authors in [53] discovered that affinity stalls are dependent on the Bakhvalov-Shishkin mesh in the case of $N^{-1} \leq \varepsilon$ where ε is the singular perturbation parameter and N is the number of mesh intervals. They proposed uniform convergence analysis of finite element techniques on Bakhvalov-type meshes, a popular kind of graded mesh closely linked to Bakhvalov-Shishkin mesh, where $N^{-1} \leq \varepsilon$, to examine these phenomena. This optimal order of convergence was demonstrated and this result was used to improve the Bakhvavers network. Numerical experiments backed up the theoretical findings.

In 2022, Gong et al. [25] achieved the best approximation of the FE approximation of a problem with PDE constraints. The shrinkage property of the adaptive algorithm and the average improvement rate concerning error estimators and solution errors are demonstrated, and extensive numerical experiments were performed to support the theoretical results.

Also, the authors in [32] studied the convergence of a mixed FE and FV plot of the compactable Navier-Stokes equations. They have demonstrated stability standardized estimates and proven solutions. In terms of h spatial estimation, they calculated a strong convergence rate.

3. Discontinuous Galerkin Finite Element Method

In 2010, Wilcox et al. [49] proposed a new, discrete Galerkin approach to issues involving convective interaction and diffusion. In an approximation to the standard variable, the approach demonstrates the features of hyper-convergence. To produce an approximation faster than the initial approximation, post-processing of the approximation was performed for each element separately.

In 2011, Nguyen et al. [36] developed a DGFEM for the incompressible, time-dependent, steady-state Navier-Stokes equations that have implicitly high-order hybridizability. The convergence of globally unknown technology with traceability of velocity and average pressure on element boundaries in the base converges with the order $k + 2$ for $k = 1$ and with the order 1 for $k = 0$. A new approach is provided to impose boundary conditions for stress, viscous pressure, vortex, and pressure independent of the weak formulation of the method.

In 2012, Heath et al. [26] introduced the DGFEM to approximate the Vlasov-Poisson system

$$\begin{aligned} 0 &= \partial_t f + \operatorname{div}(\alpha f) = \partial_t f + v \cdot \nabla_x f - E \cdot \nabla_t f, \text{ in } \Omega \times (0, T] \\ E &= -\nabla_x \Phi; -\Delta_x \Phi = 1 - \int_{\Omega^\varepsilon} f dv, \text{ in } \Omega_x \times (0, T] \end{aligned} \tag{3.1}$$

to the equations characterizing the time evolution of non-colliding plasmas. This approach is collectively conservative and retains the positivity of the electron distribution function when multiple-definition invariant functions are used as a basis. The finer details of the BGK-like state have been available for a long time for the latter. Conservation regulations have been investigated, and many similarities have been made with the theory.

In 2013, Rhebergen et al. [43] introduced the discrete, temporal, and DGFEM of the incompressible Navier-Stokes equations. A numerical investigation was conducted to demonstrate the method's flexibility in field distortion and to evaluate the behavior of solution convergence rates.

In 2014, Xu and Hesthaven[51] proposed a DGFEM for partial convective diffusion equations. The fractional Laplacian was used to obtain the superdiffusion factor, and it was proved that the diffusion problem has a stable and perfect convergence order, and the numerical examples support the theoretical conclusion.

In 2015, Oikawa [37] proposed a hybrid discrete Galerkin (HDGFE) approach with weak Poisson equation stability. To estimate the unknowns between the components, several degrees of separation was used to reduce stability, and the proposed approach was validated using numerical results.

In 2016, Stanglmeier et al. [46] proposed an explicitly hybridized discrete Galerkin (HDGFE) approach to solve the sound wave equation numerically. The approach is easy to use, has high accuracy in both space and time, and is similar to the traditional discrete Galerkin (DGFE) method. For all approximation variables, including the solution gradient, it yields the optimal $k + 1$ convergence order. When the time step technique is on the order of$k + 2$, it has the property of superconvergence, which allows one to obtain better estimates of scalar domain variables.

In 2017, Cesmelioglu et al. [13] presented the first hybridized Galerkin discrete-hybridizable a priori error analysis to approximate the Navier-Stokes equations. This approach is designed to match simple networks to a fixed-degree polynomial approximation for each of the velocity, velocity, and pressure gradient components.

In 2018, Ladonkina et al. [33] investigated the DGFEM's accuracy in computing the discontinuous solutions of a quasi-deterministic system of conservation equations with variable-"speed shock waves". It appears that, in the shock wave field of effect, it reduces the order of convergence to first order, similar to TVD and WENO plots for order of increasing approximation over smooth solutions.

Also, the authors in [41] proposed a non-continuous Galerkin hybridization approach for incompressible Navier-Stokes equations with no point divergence in the estimated velocity range. 2-D and 3-D numerical examples, as well as other orders of polynomial approximation, supported the theoretical conclusions.

In 2019, An and Yu [7] provided a novel strategy for tackling the 2-D fractional diffusion-wave issue. The suggested system is unconditionally stable, with a spatial global convergence order of $O\ (\Delta t\ + h^{k+1}\ )$ and temporal convergence order of P^Kpolynomial in the $L^2 -$ norm.

Also, the authors in [38] proposed a linear, stable DGEM, independent of the entropy of the deterministic conservation laws. It was concluded that the computation takes significantly less time than the usual DGEM approach, and the usefulness of the method in a variety of test conditions is demonstrated by numerical data.

In 2020, Cangiani et al. [12] demonstrated a fundamental error contraction solution for an elliptic interface issue using an adaptive DGFEM. A residual-type a posteriori error estimator with bulk refinement criteria underpins the approach. The same procedure, as a corollary, converges for non-essential boundary conditions posed on generic curved borders.

Also, the authors in [42] proposed a novel DGFEM hybrid embed for the Stokes problem.

$$\begin{aligned} -v\nabla^2 u + \nabla p &= f && \text{in } \Omega, \\ \nabla \cdot u &= 0 && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \\ \int_\Omega p\mathrm{d}x &= 0, \end{aligned} \tag{3.2}$$

Full hybrid features, such as velocity field conformation and point fulfillment of the continuity equation, make this technique appealing. A collection of numerical examples exhibiting convergence rates backed up the findings of the study.

In 2021, the authors in [30]proved the C0-interior penalty techniques and convergence of adaptive DGFEM for Isaacs equations and nonlinear second-order elliptic Hamilton–Jacobi–Bellman. They investigated a variety of strategies for upgrading conforming simplicial models in 2 and 3-D adaptively.

Also, Liu [34] proposed a methodology to analyze direct DGFEMs for multidimensional convective-diffusion equations with various boundary conditions. The optimal drop error in L^2 was achieved using an arbitrary and locally regular section of the field, which is well-defined for a class of diffuse flow parameters.

In 2022, the authors in [2] presented the theory of the DGFEM in the space-time approach to estimate non-stationary linear heat transfer problems. At different time scales, PDGEMFE technology has been implemented separately in space using different space networks. They demonstrate that the approximate solution converges with an order error of $O(h^2\ +\ \tau^3)$, as well as binary form and invariance features.

Also, Zhou and Chia [54] published a Lagrangian-Euler analysis of one-dimensional linear thermal propagation problems using a local stochastic discrete Galerkin approach. They also spoke of three different systems in which the implicit-explicit-Rong-Kota time cycle was used. To illustrate the theoretical results, are illustrated with numerical examples.

4. Comparison of continuous and discontinuous Galerkin Finite Element Methods

From a practical standpoint, we show a comparison of the continuous Galerkin finite element method(CGEM) and discontinuous Galerkin finite element method(DGEM).

a) CGEM solutions are continuous polynomials, whereas DGEM solutions are discontinuous polynomials [44].
b) **Age of the method:** Hundreds of books have been produced about the CGEM, which has been around for more than 70 years. The DGEM has recently emerged and has gained great interest in the scientific community[44].
c) **Size of the problem:** The number of elements in the mesh is proportional to the total number of degrees of freedom for DGEM. The proportionality constant varies with the polynomial degree. In CGEM, the degrees of freedom is proportional to the number of vertices[26].
d) **Hanging nodes:**Mesh vertices in the CGEM correspond to nodes or degrees of freedom." Hanging node" means any mesh vertex on the interior of an edge (or face). The DGEM allows for the use of nonconforming networks, but the CGEM does not. Because there are no restrictions on continuity between the elements, many hanging nodes can be configured for each face in the DGEM [26].
e) **Accuracy:**When the network size or the degree of the polynomial decreases, both techniques converge. In the power criterion, the error ratings of the DGEM are optimal. On the other hand, the error estimates of the L^2parameters are optimal in the CGEM, but they are only optimal in the case of SIPG if the DGEM approach does not use a super penalty ([44],[17]).
f) **Boundary condition:**Dirichlet boundary conditions are often weakly enforced when using DGEM but are used strongly in CGEM. On the other hand, the DGEM allows us to impose strong boundary conditions ([26],[1]).
g) **Mass conservation:** A local mass balance is satisfied using the DGEM.The CGEM can only fulfill a global mass balance over the whole computational domain.([44],[17]).

5. Conclusion and Future Work

The CGFEM and DGFEM are presented in this survey paper. A comparison of CGFEM and DGFEM was provided to evaluate their performance and explain the benefits and drawbacks of each. Through the comparison, it was found that the DGFEM is more general and comprehensive than the CFEM, as it can be implemented on all types of networks, regardless of the network's difficulty and complexity, as it is considered an improvement of the FEM.

In the future, we will suggest the DGFEM solve the problems that contain the diffusion term and the convection term, especially when the diffusion term is less than the convection term. Where oscillation occurs in the CGFEM, the DGFEM has proven its efficiency in reducing and eliminating this oscillation. For situations involving irregular networks, the DGFEM is also suggested.

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