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Analytical Solutions for the Nonlinear Homogeneous Fractional Biological Equation using a Local Fractional Operator

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Abstract:

This study uses the Natural Variation decomposition technique, which is a great tool for solving fractional biological population equations. The fractional derivatives are described in terms of the Caputo's operator sense. A series of variation components that converge to the exact solution of the problem are used to describe the outcome of the suggested technique. Examples are provided to demonstrate how the suggested technique can be used.

Keywords: Caputo's fractional derivative, Natural transform, Variation iteration method.

1-Introduction

Modern technology has radically changed the world and our way of life. Numerous engineering fields, such as fluid dynamics, aerodynamics, the sciences of the body, and finance, utilize technology. The modeling of mathematical objects has a profound impact on and shapes the design of technology. Differential equations may be used to represent the modeling in the form of a mathematical model.

Mathematical computations can be used to model many diseases, and data collection and careful analysis can be used to control them [9]. Using differential equations, biology and mathematics have a strong and fascinating connection. Fractional order differential equations (FDEs) are the name given to the non-integer order differential equations [1,7, 8]. Fractional calculus is the area of mathematics concerned with FDEs [17].

The operators for fractional derivatives have been provided by numerous academics in great number. The fractional derivative operator by Caputo [10] is the most well-known. The fractional order integral operator was developed by Li et al. to handle differential equations [15]. Fractional order differential equations have recently been solved using a variety of transformations. Laplace, Sumudu, and Elzaki transforms are a few of them [4,5,6, 11,12,13,16].

In this work, we will deal with the natural transform iterative method (NTIM), a combination of the natural transform and the new iterative method which is variation iteration method (VIM).

$$D_{\zeta}^{\gamma}\nu = \frac{\partial^2 \nu^2}{\partial \ell^2} + \frac{\partial^2 \nu^2}{\partial \vartheta^2} + \mathrm{h}\nu^{\mathrm{a}}\left(1 - \mathrm{r}\nu^{\mathrm{b}}\right) \tag{1.1}$$

where, $\nu = \nu\,(\ell, \vartheta, \zeta)$is the population density and f are the supply of population due to births and deaths. The h and r are the real numbers and $\mathrm{f}(\ell, \vartheta)$ is the initial condition. The FBPM is a mathematical model of biology, and we are striving to present the $\mathbb{FNVIM}$, a coupling method of the FVIM and NT, and use it to resolve it. The remainder of this work consists of the portions listed below: In Section 2, there are some definitions for fractional calculus. The definition of a natural transform is covered in Section 3 in detail. Section 4 carries out the $\mathbb{FNVIM}$ with CFO analysis. Examples of $\mathbb{FNVIM}$ applications are shown in Section 5. In Section 6, there is a conclusion to the study.

2- Preliminaries

This section goes over several fractional calculus principles and symbols that will come in handy during this inquiry [2, 3, 14].

Definition 2.1. Suppose $\nu(\zeta) \in R, \zeta > 0$, which is in the space $C_m, m \in R$ if there exists

$$\{\,\rho, (\rho > m), s.t. \nu(\zeta) = \zeta^{\rho}\nu_1(\zeta), where\ \nu_1(\zeta) \in C[0,8)\,\}$$

and $\nu(\zeta)$ is known as in the space $C_m^n\ when\ \nu^{\,n} \in C_m, m \in N$.

Definition 2.2. The fractional integral operator of order $\gamma \geq 0$ for Riemann Liouville of $\nu(\zeta) \in C_m, m \geq -1$ is given by the form

$$\mathrm{I}^{\gamma}\,\nu(\zeta) = \begin{cases} \frac{1}{\Gamma(\gamma)}\int_0^{\zeta} (\zeta - \xi)^{\gamma-1}\,\nu(\xi)\,d\xi, & \gamma > 0, \zeta > 0 \\ \mathrm{I}^{0}\nu(\zeta) = \nu(\zeta), & \gamma = 0 \end{cases} \tag{2.1}$$

where $\Gamma(\cdot)$ is the recognizable Gamma function. The following are the characteristics of the operator I^{γ} : $For\ \nu \in Cm, m \geq -1, \gamma, \sigma \geq 0, then$

1. $\mathrm{I}^{\gamma}\mathrm{I}^{\sigma}\, \nu(\zeta) = \mathrm{I}^{\gamma+\sigma}\nu(\zeta)$
2. $\mathrm{I}^{\gamma}\mathrm{I}^{\sigma}\, \nu(\zeta) = \mathrm{I}^{\sigma}\mathrm{I}^{\gamma}\, \nu(\zeta)$

Definition 2.3. In the understanding of Caputo, $\nu(\zeta)$'s fractional derivative is as follows:

$$D^{\gamma}\, \nu(\zeta) = \mathrm{I}^{n-\gamma}D^{n}\, \nu(\zeta) = \frac{1}{\Gamma(n-\gamma)}\int_0^{\zeta}(\zeta-\xi)^{n-\gamma-1}\,\nu^{(n)}(\xi)\,d\xi\,, \qquad (2.2)$$

$such\ that\ n-1 < \gamma \leq n, n \in N, \zeta > 0\ and\ \nu \in C_{-1}^{n}$

Definition 2.4. The following formula gives the Mittag-Leffler function E_{γ} if it satisfies the following:

$$E_{\gamma}\,(z) = \sum_{n=0}^{\infty}\frac{z^{\gamma}}{\Gamma(n\gamma+1)} \qquad \text{, for each } \gamma > 0 \qquad (2.3)$$

3- Natural Transform definition

We present some context for the natural transform approach [3] in this section.

Definition 3.1. The function $\nu(\zeta)$ for $\zeta \in R$ has a natural transform defined by

$$\mathrm{N}[\nu(\zeta)] = \mathrm{R}(\mathrm{S},\mathrm{U}) = \int_{-\infty}^{\infty} e^{-S\zeta}\ \nu(U\zeta)d\zeta, \qquad S,U \in (-\infty,\infty) \qquad (3.1)$$

We denote that the Natural transform of the time function $\nu(\zeta)$ is $N[\nu(\zeta)]$, and the variables η and ℓ are the Natural transform elements. Furthermore, define $\nu(\zeta)H(\zeta)$ as on the axis of positive real, if $H(\zeta)$ is Heaviside function, and $\zeta \in (0,\infty)$. Consider

$$A = \{\nu(\zeta): \exists M, t_1, t_2 > 0, with\ |\nu(\zeta)| \leq Me^{\frac{|\zeta|}{tj}}, for\ \zeta \in (-1)^{j} \times [0,\infty), j \in Z^{+}\}$$

The natural transform, often known as the NT, is defined as follows:

$$N[\nu(\zeta)H(\zeta)] = N^{+}[\nu(\zeta)] = R^{+}\,(S,U) = \int_0^{\infty} e^{-S\zeta}\,\nu(U\zeta)d\zeta\ , S,U \in (-\infty,\infty) \qquad (3.2)$$

4- Fractional Biological Population Equation (𝔽𝔹ℙ𝔼)

Let us consider a generalized non-linear Biological Iteration equation of the form:

$$D_{\zeta}^{\gamma}\nu(\ell,\vartheta,\zeta) = \frac{\partial^2\nu^2}{\partial\ell^2} + \frac{\partial^2\nu^2}{\partial\vartheta^2} + \mathrm{h}\nu^{\mathrm{a}}\,(1-\mathrm{r}\nu^{\mathrm{b}}) \qquad (4.1)$$

with the initial condition:

$$\mathrm{v}(\ell,\vartheta,0) = \mathrm{f}(\ell,\vartheta) \tag{4.2}$$

Applying NT to each side of (4.1), and by using the differential property of FNVIM, we have

$$N\left[D_{\zeta}^{\gamma}\nu(\ell,\vartheta,\zeta)\right] = N\left[\frac{\partial^2\nu^2}{\partial\ell^2} + \frac{\partial^2\nu^2}{\partial\vartheta^2} + \mathrm{h}\nu^{\mathrm{a}}\,(1 - \mathrm{r}\nu^{\mathrm{b}}\,)\right]$$

$$\frac{S^{\gamma}}{U^{\gamma}}\nu(\ell,\vartheta,\zeta) - \frac{S^{\gamma-1}}{U^{\gamma}}\,\mathrm{v}(\ell,\vartheta,0) = N\left[\frac{\partial^2\nu^2}{\partial\ell^2} + \frac{\partial^2\nu^2}{\partial\vartheta^2} + \mathrm{h}\nu^{\mathrm{a}}\,(1 - \mathrm{r}\nu^{\mathrm{b}}\,)\right] \tag{4.3}$$

$$\nu_{n+1}(\ell,\vartheta,\zeta) = \nu_n + \lambda(\xi)\left[\frac{S^{\gamma}}{U^{\gamma}}\nu_n - \frac{S^{\gamma-1}}{U^{\gamma}}\,\mathrm{v}(\ell,\vartheta,0) - N\left[\frac{\partial^2{\nu_n}^2}{\partial\ell^2} + \frac{\partial^2{\nu_n}^2}{\partial\vartheta^2} + \mathrm{h}{\nu_n}^{\mathrm{a}}\,(1 - \mathrm{r}{\nu_n}^{\mathrm{b}}\,)\right]\right]$$

Take the variation

$$\delta[\nu_{n+1}] = \delta[\nu_n] + \lambda(\xi)\delta\left[\frac{S^{\gamma}}{U^{\gamma}}\nu_n - \frac{S^{\gamma-1}}{U^{\gamma}}\,\mathrm{v}(\ell,\vartheta,0) - N\left[\frac{\partial^2{\nu_n}^2}{\partial\ell^2} + \frac{\partial^2{\nu_n}^2}{\partial\vartheta^2} + \mathrm{h}{\nu_n}^{\mathrm{a}}\,(1 - \mathrm{r}{\nu_n}^{\mathrm{b}}\,)\right]\right] \tag{4.4}$$

$$\delta[\nu_n]\left[1 + \lambda(\xi)\frac{S^{\gamma}}{U^{\gamma}}\right] = 0$$

$$\lambda(\xi) = \frac{-U^{\gamma}}{S^{\gamma}} \tag{4.5}$$

$$\nu_{n+1}(\ell,\vartheta,\zeta) = \nu_n - \frac{U^{\gamma}}{S^{\gamma}}\left[\frac{S^{\gamma}}{U^{\gamma}}\nu_n - \frac{S^{\gamma-1}}{U^{\gamma}}\,\mathrm{v}(\ell,\vartheta,0) - N\left[\frac{\partial^2{\nu_n}^2}{\partial\ell^2} + \frac{\partial^2{\nu_n}^2}{\partial\vartheta^2} + \mathrm{h}{\nu_n}^{\mathrm{a}}\,(1 - \mathrm{r}{\nu_n}^{\mathrm{b}}\,)\right]\right] \tag{4.6}$$

By applying Natural inverse to (4.6)

$$\nu_{n+1}(\ell,\vartheta,\zeta) = N^{-1}\left[\frac{1}{S}\mathrm{v}(\ell,\vartheta,0) + \frac{U^{\gamma}}{S^{\gamma}}\,N\left[\frac{\partial^2{\nu_n}^2}{\partial\ell^2} + \frac{\partial^2{\nu_n}^2}{\partial\vartheta^2} + \mathrm{h}{\nu_n}^{\mathrm{a}}\,(1 - \mathrm{r}{\nu_n}^{\mathrm{b}}\,)\right]\right] \tag{4.7}$$

$$\nu_{n+1}(\ell,\vartheta,\zeta) = \mathrm{v}(\ell,\vartheta,0) + N^{-1}\left[\frac{U^{\gamma}}{S^{\gamma}}\,N\left[\frac{\partial^2{\nu_n}^2}{\partial\ell^2} + \frac{\partial^2{\nu_n}^2}{\partial\vartheta^2} + \mathrm{h}{\nu_n}^{\mathrm{a}}\left(1 - \mathrm{r}{\nu_n}^{\mathrm{b}}\,\right)\right]\right]$$

Assume that

$${\nu_n}^2 = \sum_{n=0}^{\infty} A_n \tag{4.8}$$

$$\nu_{n+1}(\ell,\vartheta,\zeta) = \mathrm{v}(\ell,\vartheta,0) + N^{-1}\left[\frac{U^{\gamma}}{S^{\gamma}}\,N\left[\frac{\partial^2 A_n}{\partial\ell^2} + \frac{\partial^2 A_n}{\partial\vartheta^2} + \mathrm{h}{\nu_n}^{\mathrm{a}}\,(1 - \mathrm{r}{\nu_n}^{\mathrm{b}}\,)\right]\right] \tag{4.9}$$

5- Applications

Example 5.1

Consider the nonlinear fractional biological Iteration equation is given as the following:

$$\mathcal{D}_{\zeta}^{\gamma} \nu(\ell,\vartheta,\zeta) - \nu_{\ell\ell}^2 - \nu_{\vartheta\vartheta}^2 - h\nu = 0 \qquad (5.1)$$

w.r.t initial condition

$$\nu(\ell,\vartheta,\zeta) = \sqrt{\ell\vartheta} \qquad (5.2)$$

$$\mathbb{N}\left[\mathcal{D}_{\zeta}^{\gamma} \nu(\ell,\vartheta,\zeta)\right] - \mathbb{N}[\nu_{\ell\ell}^2 + \nu_{\vartheta\vartheta}^2 + h\nu] = 0 \qquad (5.3)$$

$$\frac{S^{\gamma}}{U^{\gamma}} \nu(\ell,\vartheta,\zeta) - \frac{S^{\gamma-1}}{U^{\gamma}} \nu(\ell,\vartheta,0) - \mathbb{N}\left[\frac{\partial^2 \nu^2}{\partial \ell^2} + \frac{\partial^2 \nu^2}{\partial \vartheta^2} + h\nu\right] = 0$$

$$\nu_{n+1}(\ell,\vartheta,\zeta) = \nu_n(\ell,\vartheta,\zeta) + \lambda(\xi)\left[\frac{S^{\gamma}}{U^{\gamma}} \nu_n - \frac{S^{\gamma-1}}{U^{\gamma}} \nu(\ell,\vartheta,0) - \mathbb{N}\left[\frac{\partial^2 {\nu_n}^2}{\partial \ell^2} + \frac{\partial^2 {\nu_n}^2}{\partial \vartheta^2} + h\nu_n\right]\right]$$

By using (4.5), we get

$$\lambda\nu_{n+1}(\ell,\vartheta,\zeta) = \nu_n(\ell,\vartheta,\zeta) - \frac{S^{\gamma}}{U^{\gamma}}\left[\frac{S^{\gamma}}{U^{\gamma}} \nu_n - \frac{S^{\gamma-1}}{U^{\gamma}} \nu(\ell,\vartheta,0) - \mathbb{N}\left[\frac{\partial^2 {\nu_n}^2}{\partial \ell^2} + \frac{\partial^2 {\nu_n}^2}{\partial \vartheta^2} + h\nu_n\right]\right]$$

Taking Natural inverse.

$$\nu_{n+1}(\ell,\vartheta,\zeta) = \nu(\ell,\vartheta,0) + \mathbb{N}^{-1}\left[\frac{S^{\gamma}}{U^{\gamma}} \mathbb{N}\left[\frac{\partial^2 {\nu_n}^2}{\partial \ell^2} + \frac{\partial^2 {\nu_n}^2}{\partial \vartheta^2} + h\nu_n\right]\right] \qquad (5.4)$$

$$\nu_n^2 = \sum_{n=0}^{\infty} A_n$$

$$\nu_{n+1}(\ell,\vartheta,\zeta) = \nu(\ell,\vartheta,0) + \mathbb{N}^{-1}\left[\frac{S^{\gamma}}{U^{\gamma}} \mathbb{N}\left[\frac{\partial^2 A_n}{\partial \ell^2} + \frac{\partial^2 A_n}{\partial \vartheta^2} + h\nu_n\right]\right] \qquad (5.6)$$

$$\nu_0 = \nu(\ell,\vartheta,0) = \sqrt{\ell\vartheta}$$

$$\nu_1 = \nu(\ell,\vartheta,0) + \mathbb{N}^{-1}\left[\frac{S^{\gamma}}{U^{\gamma}} \mathbb{N}\left[\frac{\partial^2 A_0}{\partial \ell^2} + \frac{\partial^2 A_0}{\partial \vartheta^2} + h\nu_0\right]\right] \qquad (5.7)$$

$$A_0 = \nu_0.\nu_0 = \sqrt{\ell\vartheta}.\sqrt{\ell\vartheta} = \ell\vartheta$$

$$\frac{\partial^2 A_0}{\partial \ell^2} = 0\,, \frac{\partial^2 A_0}{\partial \vartheta^2} = 0$$

$$\nu_1 = \sqrt{\ell\vartheta} + \mathbb{N}^{-1}\left[\frac{S^\gamma}{U^\gamma}\mathbb{N}\left[0 + 0 + h\sqrt{\ell\vartheta}\,\right]\right]$$

$$= \sqrt{\ell\vartheta} + \frac{\zeta^\gamma}{\Gamma(\gamma+1)}\, h\sqrt{\ell\vartheta} \qquad (5.8)$$

$$\nu_2 = \nu(\ell,\vartheta,0) + \mathbb{N}^{-1}\left[\frac{S^\gamma}{U^\gamma}\,\mathbb{N}\left[\frac{\partial^2 A_1}{\partial \ell^2} + \frac{\partial^2 A_1}{\partial \vartheta^2} + h\nu_1\right]\right] \qquad (5.9)$$

$$A_1 = \nu_0.\nu_1 + \nu_1.\nu_0$$

$$= \sqrt{\ell\vartheta}\,.\left[\sqrt{\ell\vartheta} + \frac{\zeta^\gamma}{\Gamma(\gamma+1)}\, h\sqrt{\ell\vartheta}\,\right] + \left[\sqrt{\ell\vartheta} + \frac{\zeta^\gamma}{\Gamma(\gamma+1)}\, h\sqrt{\ell\vartheta}\,\right].\sqrt{\ell\vartheta}$$

$$= 2\left(1 + \frac{\zeta^\gamma}{\Gamma(\gamma+1)}\, h\right)\, \ell\vartheta \qquad (2.10)$$

$$\frac{\partial^2 A_1}{\partial \ell^2} = 0\,, \frac{\partial^2 A_1}{\partial \vartheta^2} = 0$$

$$\nu_2 = \sqrt{\ell\vartheta} + \mathbb{N}^{-1}\left[\frac{U^\gamma}{S^\gamma}\;\mathbb{N}\left[0 + 0 + h\sqrt{\ell\vartheta} + \frac{\zeta^\gamma}{\Gamma(\gamma+1)}\, h^2\sqrt{\ell\vartheta}\right]\right]$$

$$= \sqrt{\ell\vartheta} + \frac{\zeta^\gamma}{\Gamma(\gamma+1)}\, h\sqrt{\ell\vartheta} + \frac{\zeta^{2\gamma}}{\Gamma(2\gamma+1)}\, h^2\sqrt{\ell\vartheta}$$

:

$$\nu_n = \sqrt{\ell\vartheta} + \frac{\zeta^\gamma}{\Gamma(\gamma+1)}\, h\sqrt{\ell\vartheta} + \frac{\zeta^{2\gamma}}{\Gamma(2\gamma+1)}\, h^2\sqrt{\ell\vartheta} + \cdots + \frac{\zeta^{n\gamma}}{\Gamma(n\gamma+1)}\, h^n\sqrt{\ell\vartheta}$$

$$\nu(\ell,\vartheta,\zeta) = \lim_{n\to\infty} \nu_n = \sqrt{\ell\vartheta}[1 + \frac{\zeta^\gamma}{\Gamma(\gamma+1)}\, h + \frac{\zeta^{2\gamma}}{\Gamma(2\gamma+1)}\, h^2 + \cdots + \frac{\zeta^{n\gamma}}{\Gamma(n\gamma+1)}\, h^n$$

$$= E_\gamma(h\zeta^\gamma)\sqrt{\ell\vartheta} \qquad (2.11)$$

When $\gamma = 1$

$$\nu(\ell,\vartheta,\zeta) = \sqrt{\ell\vartheta}\,[1 + h\zeta + \frac{h^2\zeta^2}{2!} + \cdots]$$

$$= e^{h\zeta}\sqrt{\ell\vartheta} \qquad (5.12)$$

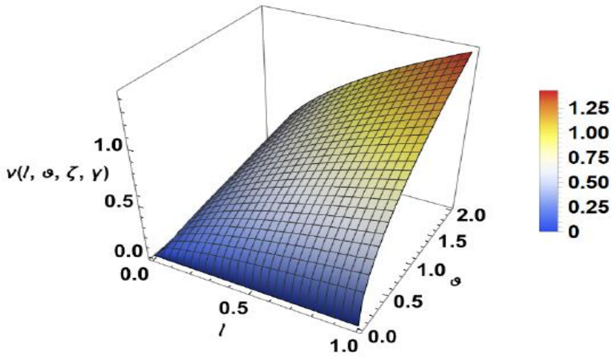

Figure 1: The surface graph of the approximate solution $\nu(\ell,\vartheta,\zeta)$ of (5.1) when $\gamma=1$.

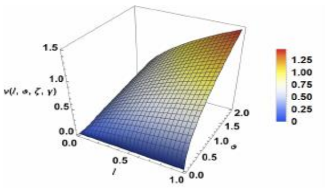

Figure 2: The surface graph of the approximate solution $\nu(\ell,\vartheta,\zeta)$ of (5.1) when $\gamma=0.4$.

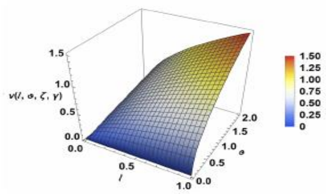

Figure 3: The surface graph of the approximate solution $\nu(\ell,\vartheta,\zeta)$ of (5.1) when $\gamma=0.6$.

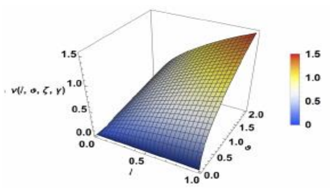

Figure 4: The surface graph of the approximate solution $\nu(\ell,\vartheta,\zeta)$ of (5.1) when $\gamma=0.8$.

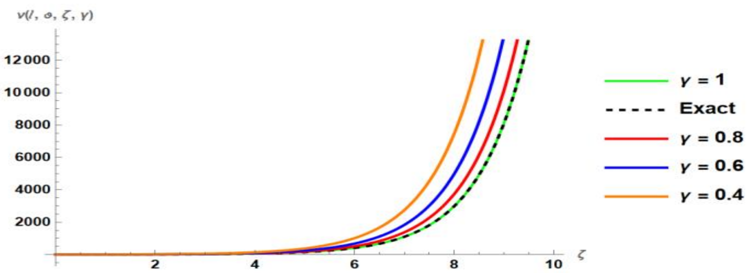

Figure 5: The surface graph of the approximate solution $\nu(\ell,\vartheta,\zeta)$ of (5.1) when $\gamma=1$.

Example 5.2

Consider the nonlinear fractional biological population is given as the following:

$$\mathcal{D}_\zeta^\gamma \nu(\ell,\vartheta,\zeta) - \nu_{\ell\ell}^2 - \nu_{\vartheta\vartheta}^2 - \nu = 0 \qquad (5.13)$$

w.r.t initial condition

$$\nu(\ell,\vartheta,0) = \sqrt{\sin(\ell)\sinh(\vartheta)} \qquad (5.14)$$

$$\mathbb{N}\left[\mathcal{D}_\zeta^\gamma \nu(\ell,\vartheta,\zeta)\right] - \mathbb{N}[\nu_{\ell\ell}^2 - \nu_{\vartheta\vartheta}^2 - \nu] = 0 \qquad (5.15)$$

$$\frac{S^\gamma}{U^\gamma}\nu - \frac{S^{\gamma-1}}{U^\gamma}\nu(\ell,\vartheta,0) - \mathbb{N}\left[\frac{\partial^2\nu^2}{\partial\ell^2} + \frac{\partial^2\nu^2}{\partial\vartheta^2} + \nu\right] = 0 \qquad (5.16)$$

$$v_{n+1}(\ell,\vartheta,\zeta) = v_n(\ell,\vartheta,\zeta) + \lambda(\xi)[\frac{S^\gamma}{U^\gamma}v_n - \frac{S^{\gamma-1}}{U^\gamma}\ v(\ell,\vartheta,0) - \ \mathbb{N}\left[\frac{\partial^2 {v_n}^2}{\partial \ell^2} + \frac{\partial^2 {v_n}^2}{\partial \vartheta^2} + v_n\right]$$

By using (4.5), we get

$$v_{n+1}(\ell,\vartheta,\zeta) = v_n(\ell,\vartheta,\zeta) - \frac{U^\gamma}{S^\gamma}\left[\frac{S^\gamma}{U^\gamma}v_n - \frac{S^{\gamma-1}}{U^\gamma}\ v(\ell,\vartheta,0) - \ \mathbb{N}\left[\frac{\partial^2 {v_n}^2}{\partial \ell^2} + \frac{\partial^2 {v_n}^2}{\partial \vartheta^2} + v_n\right]\right]$$

Taking Natural inverse

$$v_{n+1}(\ell,\vartheta,\zeta) = v(\ell,\vartheta,0) + \mathbb{N}^{-1}\left[\frac{U^\gamma}{S^\gamma}\ \mathbb{N}\left[\frac{\partial^2 {v_n}^2}{\partial \ell^2} + \frac{\partial^2 {v_n}^2}{\partial \vartheta^2} + v_n\right]\right] \qquad (5.17)$$

$$v_n^2 = \sum_{n=0}^{\infty} A_n$$

$$v_{n+1}(\ell,\vartheta,\zeta) = v(\ell,\vartheta,0) + \mathbb{N}^{-1}\left[\frac{U^\gamma}{S^\gamma}\mathbb{N}\left[\frac{\partial^2 A_n}{\partial \ell^2} + \frac{\partial^2 A_n}{\partial \vartheta^2} + v_n\right]\right] \qquad (5.18)$$

$$v_0 = v(\ell,\vartheta,0) = \sqrt{\sin(\ell)\sinh(\vartheta)} \qquad (5.19)$$

$$v_1 = v(\ell,\vartheta,0) + \mathbb{N}^{-1}\left[\frac{U^\gamma}{S^\gamma}\mathbb{N}\left[\frac{\partial^2 A_0}{\partial \ell^2} + \frac{\partial^2 A_0}{\partial \vartheta^2} + v_0\right]\right] \qquad (5.20)$$

$$A_0 = v_0.v_0 = \left(\sqrt{\sin(\ell)\sinh(\vartheta)}\right).\left(\sqrt{sin(\ell)\,sinh(\vartheta)}\right) = sin(\ell)\,sinh(\vartheta)$$

$$\frac{\partial^2 A_0}{\partial \ell^2} = \ -sin(\ell)\,sinh(\vartheta)$$

$$\frac{\partial^2 A_0}{\partial \vartheta^2} = sin(\ell)\,sinh(\vartheta)$$

$$v_1 = \sqrt{sin(\ell)\,sinh(\vartheta)} + \mathbb{N}^{-1}\left[\frac{U^\gamma}{S^\gamma}\mathbb{N}\left[-sin(\ell)\,sinh(\vartheta) + sin(\ell)\,sinh(\vartheta) + \sqrt{sin(\ell)\,sinh(\vartheta)}\right]\right]$$

$$= \left(1 + \frac{\zeta^\gamma}{\Gamma(\gamma+1)}\right)\sqrt{sin(\ell)\,sinh(\vartheta)} \qquad (5.21)$$

$$v_2 = v(\ell,\vartheta,0) + \mathbb{N}^{-1}\left[\frac{U^\gamma}{S^\gamma}\mathbb{N}\left[\frac{\partial^2 A_1}{\partial \ell^2} + \frac{\partial^2 A_1}{\partial \vartheta^2} + v_1\right]\right] \qquad (5.22)$$

$$A_1 = v_0.v_1 + v_1.v_0$$

$$= \sqrt{sin(\ell)\, sinh(\vartheta)} . \left[\left(1 + \frac{\zeta^{\gamma}}{\Gamma(\gamma+1)}\right) \sqrt{sin(\ell)\, sinh(\vartheta)} \right]$$

$$+ \left[\left(1 + \frac{\zeta^{\gamma}}{\Gamma(\gamma+1)}\right) \sqrt{sin(\ell)\, sinh(\vartheta)} \right] . \sqrt{sin(\ell)\, sinh(\vartheta)}$$

$$= \left(1 + \frac{\zeta^{\gamma}}{\Gamma(\gamma+1)}\right) sin(\ell)\, sinh(\vartheta) + \left(1 + \frac{\zeta^{\gamma}}{\Gamma(\gamma+1)}\right) sin(\ell)\, sinh(\vartheta)$$

$$= 2\left(1 + \frac{\zeta^{\gamma}}{\Gamma(\gamma+1)}\right) sin(\ell)\, sinh(\vartheta)$$

$$\frac{\partial^2 A_1}{\partial \ell^2} = -2\left(1 + \frac{\zeta^{\gamma}}{\Gamma(\gamma+1)}\right) sin(\ell)\, sinh(\vartheta) \qquad (5.23)$$

$$\frac{\partial^2 A_1}{\partial \vartheta^2} = 2\left(1 + \frac{\zeta^{\gamma}}{\Gamma(\gamma+1)}\right) sin(\ell)\, sinh(\vartheta)$$

$$v_2 = \sqrt{sin(\ell)\, sinh(\vartheta)} + \mathbb{N}^{-1}[\frac{U^{\gamma}}{S^{\gamma}} \mathbb{N}[-2\left(1 + \frac{\zeta^{\gamma}}{\Gamma(\gamma+1)}\right) sin(\ell)\, sinh(\vartheta) +$$

$$2\left(1 + \frac{\zeta^{\gamma}}{\Gamma(\gamma+1)}\right) sin(\ell)\, sinh(\vartheta) + \left(1 + \frac{\zeta^{\gamma}}{\Gamma(\gamma+1)}\right) \sqrt{sin(\ell)\, sinh(\vartheta)}]\,]$$

$$= \sqrt{sin(\ell)\, sinh(\vartheta)} + \frac{\zeta^{\gamma}}{\Gamma(\gamma+1)} \sqrt{sin(\ell)\, sinh(\vartheta)} + \frac{\zeta^{2\gamma}}{\Gamma(2\gamma+1)} \sqrt{sin(\ell)\, sinh(\vartheta)}$$

$$\vdots$$

$$v_n = \frac{\zeta^{n\gamma}}{\Gamma(n\gamma+1)} \sqrt{sin(\ell)\, sinh(\vartheta)} \qquad (5.24)$$

$$v(\ell, \vartheta, \zeta) = \lim_{n \to \infty} v_n = \sqrt{sin(\ell)\, sinh(\vartheta)} \left[1 + \frac{\zeta^{\gamma}}{\Gamma(\gamma+1)} + \cdots \frac{\zeta^{n\gamma}}{\Gamma(n\gamma+1)} \right]$$

$$= E_{\gamma}(\zeta^{\gamma}) \sqrt{sin(\ell)\, sinh(\vartheta)} \qquad (5.25)$$

When $\gamma = 1$

$$v(\ell, \vartheta, \zeta) = e^{\zeta} \sqrt{sin(\ell)\, sinh(\vartheta)} \qquad (5.26)$$

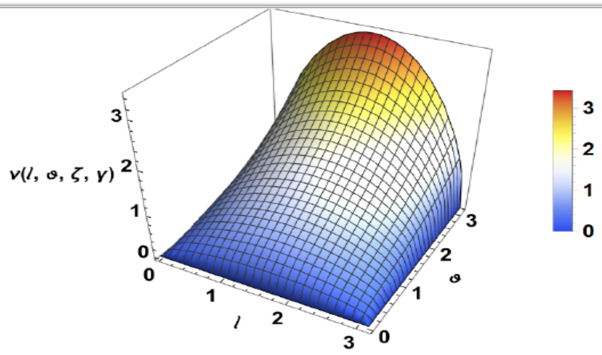

Figure 6: The surface graph of the approximate solution $\nu(\ell,\vartheta,\zeta)$ of (5.13) when $\gamma = 1$.

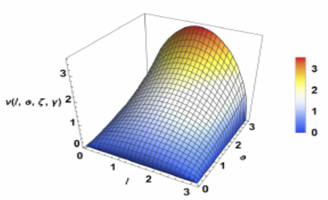

Figure 7: The surface graph of the approximate solution $\nu(\ell,\vartheta,\zeta)$ of (5.13) when $\gamma = 0.4$.

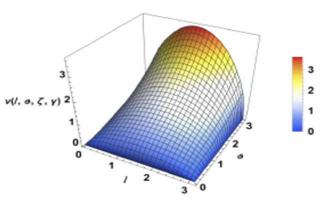

Figure 8: The surface graph of the approximate solution $\nu(\ell,\vartheta,\zeta)$ of (5.13) when $\gamma = 0.6$.

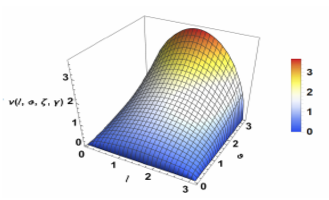

Figure 9: The surface graph of the approximate solution $\nu(\ell,\vartheta,\zeta)$ of (5.13) when $\gamma = 0.8$.

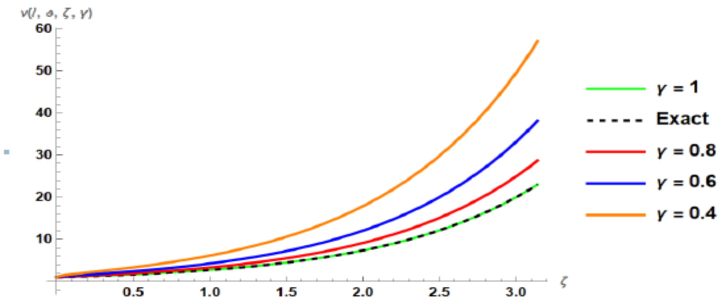

Figure 10: The surface graph of the approximate solution $\nu(\ell,\vartheta,\zeta)$ of (5.13) when $\gamma = 1$.

Example 5.3

Consider the nonlinear fractional biological population equation is given as the following

$$D_{\zeta}^{\gamma}\nu(\ell,\vartheta,\zeta) - {\nu^2}_{\ell\ell} - {\nu^2}_{\vartheta\vartheta} + \nu\left(\frac{8}{9}\nu + 1\right) = 0 \qquad (5.27)$$

w.r.t initial condition

$$\nu(\ell,\vartheta,0) = e^{\frac{1}{3}(\ell+\vartheta)} \qquad (5.28)$$

$$N\left[D_{\zeta}^{\gamma}\nu(\ell,\vartheta,\zeta)\right] - N\left[{\nu^2}_{\ell\ell} + {\nu^2}_{\vartheta\vartheta} - \nu\left(\frac{8}{9}\nu + 1\right)\right] = 0 \qquad (5.29)$$

$$\frac{S^{\gamma}}{U^{\gamma}}\nu(\ell,\vartheta,\zeta)-\frac{S^{\gamma-1}}{U^{\gamma}}\,\nu(\ell,\vartheta,0)-N\left[\frac{\partial^2\nu^2}{\partial\ell^2}+\frac{\partial^2\nu^2}{\partial\vartheta^2}-\nu\left(\frac{8}{9}\nu+1\right)\right]=0$$

$$\nu_{n+1}(\ell,\vartheta,\zeta)=\nu_n+\lambda(\xi)\left[\frac{S^{\gamma}}{U^{\gamma}}\nu_n-\frac{S^{\gamma-1}}{U^{\gamma}}\,\nu(\ell,\vartheta,0)-N\left[\frac{\partial^2{\nu_n}^2}{\partial\ell^2}+\frac{\partial^2{\nu_n}^2}{\partial\vartheta^2}-\nu_n\left(\frac{8}{9}\nu_n+1\right)\right]\right]$$

$$\lambda(\xi)=\frac{-U^{\gamma}}{S^{\gamma}} \qquad (5.30)$$

$$\nu_{n+1}(\ell,\vartheta,\zeta)=\nu_n-\frac{U^{\gamma}}{S^{\gamma}}\left[\frac{S^{\gamma}}{U^{\gamma}}\nu_n-\frac{S^{\gamma-1}}{U^{\gamma}}\,\nu(\ell,\vartheta,0)-N\left[\frac{\partial^2{\nu_n}^2}{\partial\ell^2}+\frac{\partial^2{\nu_n}^2}{\partial\vartheta^2}-\nu_n\left(\frac{8}{9}\nu_n+1\right)\right]\right]$$

Taking Natural inverse

$$\nu_{n+1}(\ell,\vartheta,\zeta)=\nu(\ell,\vartheta,0)+N^{-1}\left[\frac{U^{\gamma}}{S^{\gamma}}N\left[\frac{\partial^2{\nu_n}^2}{\partial\ell^2}+\frac{\partial^2{\nu_n}^2}{\partial\vartheta^2}-\nu_n\left(\frac{8}{9}\nu_n+1\right)\right]\right]$$

$$\nu_{n+1}(\ell,\vartheta,\zeta)=\nu(\ell,\vartheta,0)+N^{-1}\left[\frac{U^{\gamma}}{S^{\gamma}}N\left[\frac{\partial^2{\nu_n}^2}{\partial\ell^2}+\frac{\partial^2{\nu_n}^2}{\partial\vartheta^2}-\frac{8}{9}{\nu_n}^2-\nu_n\right]\right] \qquad (5.31)$$

$${\nu_n}^2=\sum_{n=0}^{\infty}A_n$$

$$\nu_{n+1}(\ell,\vartheta,\zeta)=\nu(\ell,\vartheta,0)+N^{-1}\left[\frac{U^{\gamma}}{S^{\gamma}}N\left[\frac{\partial^2A_n}{\partial\ell^2}+\frac{\partial^2A_n}{\partial\vartheta^2}-\frac{8}{9}A_n-\nu_n\right]\right] (5.32)$$

$$\nu_0=\nu(\ell,\vartheta,0)=e^{\frac{1}{3}(\ell+\vartheta)} \qquad (5.33)$$

$$\nu_1(\ell,\vartheta,\zeta)=\nu(\ell,\vartheta,0)+N^{-1}\left[\frac{U^{\gamma}}{S^{\gamma}}N\left[\frac{\partial^2A_0}{\partial\ell^2}+\frac{\partial^2A_0}{\partial\vartheta^2}-\frac{8}{9}A_0-\nu_0\right]\right] \qquad (5.34)$$

$$A_0=\nu_0.\nu_0=\left(e^{\frac{1}{3}(\ell+\vartheta)}\right).\left(e^{\frac{1}{3}(\ell+\vartheta)}\right)$$

$$=\frac{4}{9}e^{\frac{2}{3}(\ell+\vartheta)}$$

$$\frac{\partial^2A_0}{\partial\ell^2}=\frac{4}{9}e^{\frac{2}{3}(\ell+\vartheta)}\ ,\ \frac{\partial^2A_0}{\partial\vartheta^2}=\frac{4}{9}e^{\frac{2}{3}(\ell+\vartheta)}$$

$$\nu_1=e^{\frac{1}{3}(\ell+\vartheta)}+N^{-1}\left[\frac{U^{\gamma}}{S^{\gamma}}N\left[\frac{4}{9}e^{\frac{2}{3}(\ell+\vartheta)}+\frac{4}{9}e^{\frac{2}{3}(\ell+\vartheta)}-\frac{8}{9}e^{\frac{2}{3}(\ell+\vartheta)}-e^{\frac{1}{3}(\ell+\vartheta)}\right]\right] \qquad (5.35)$$

$$=\left(1-\frac{\zeta^{\gamma}}{\Gamma(\gamma+1)}\right)e^{\frac{1}{3}(\ell+\vartheta)}$$

$$\nu_2(\ell,\vartheta,\zeta)=\nu(\ell,\vartheta,0)+N^{-1}\left[\frac{U^{\gamma}}{S^{\gamma}}N\left[\frac{\partial^2A_1}{\partial\ell^2}+\frac{\partial^2A_1}{\partial\vartheta^2}-\frac{8}{9}A_1-\nu_1\right]\right] \qquad (5.36)$$

$$A_1=\nu_0.\nu_1+\nu_1.\nu_0$$

$$=e^{\frac{1}{3}(\ell+\vartheta)}.\left[\left(1-\frac{\zeta^{\gamma}}{\Gamma(\gamma+1)}\right)e^{\frac{1}{3}(\ell+\vartheta)}\right]+\left[\left(1-\frac{\zeta^{\gamma}}{\Gamma(\gamma+1)}\right)e^{\frac{1}{3}(\ell+\vartheta)}\right].e^{\frac{1}{3}(\ell+\vartheta)}$$

$$= \left(1-\frac{\zeta^{\gamma}}{\Gamma(\gamma+1)}\right)e^{\frac{2}{3}(\ell+\vartheta)}+\left(1-\frac{\zeta^{\gamma}}{\Gamma(\gamma+1)}\right)e^{\frac{2}{3}(\ell+\vartheta)}$$

$$= 2\left(1-\frac{\zeta^{\gamma}}{\Gamma(\gamma+1)}\right)e^{\frac{2}{3}(\ell+\vartheta)}$$

$$\frac{\partial^2 A_1}{\partial \ell^2} = \frac{8}{9}\left(1-\frac{\zeta^{\gamma}}{\Gamma(\gamma+1)}\right)e^{\frac{2}{3}(\ell+\vartheta)} \quad , \quad \frac{\partial^2 A_1}{\partial \vartheta^2} = \frac{8}{9}\left(1-\frac{\zeta^{\gamma}}{\Gamma(\gamma+1)}\right)e^{\frac{2}{3}(\ell+\vartheta)}$$

$$\nu_2 = e^{\frac{1}{3}(\ell+\vartheta)} + N^{-1}\left[\frac{U^{\gamma}}{S^{\gamma}}N\left[\frac{8}{9}\left(1-\frac{\zeta^{\gamma}}{\Gamma(\gamma+1)}\right)e^{\frac{2}{3}(\ell+\vartheta)}+\frac{8}{9}\left(1-\frac{\zeta^{\gamma}}{\Gamma(\gamma+1)}\right)e^{\frac{2}{3}(\ell+\vartheta)}\right.\right.$$

$$\left.\left.-\frac{16}{9}\left(1-\frac{\zeta^{\gamma}}{\Gamma(\gamma+1)}\right)e^{\frac{2}{3}(\ell+\vartheta)} - e^{\frac{1}{3}(\ell+\vartheta)} + \frac{\zeta^{\gamma}}{\Gamma(\gamma+1)}e^{\frac{1}{3}(\ell+\vartheta)}\right]\right]$$

$$= e^{\frac{1}{3}(\ell+\vartheta)} - \frac{\zeta^{\gamma}}{\Gamma(\gamma+1)}e^{\frac{1}{3}(\ell+\vartheta)} + \frac{\zeta^{2\gamma}}{\Gamma(2\gamma+1)}e^{\frac{1}{3}(\ell+\vartheta)}$$

$$\vdots$$

$$\nu_n = (-1)^n\frac{\zeta^{n\gamma}}{\Gamma(n\gamma+1)}e^{\frac{1}{3}(\ell+\vartheta)} \qquad (5.37)$$

$$\nu(\ell,\vartheta,\zeta) = \lim_{n\to\infty}\nu_n = e^{\frac{1}{3}(\ell+\vartheta)}\left[1-\frac{\zeta^{\gamma}}{\Gamma(\gamma+1)}+\cdots\frac{\zeta^{n\gamma}}{\Gamma(n\gamma+1)}\right]$$

$$= E_{\gamma}(-\zeta^{\gamma})e^{\frac{1}{3}(\ell+\vartheta)} \qquad (5.38)$$

When $\gamma = 1$

$$\nu(\ell,\vartheta,\zeta) = e^{\frac{1}{3}(\ell+\vartheta)}\left[1-\zeta+\frac{\zeta^2}{2!}-\cdots\right]$$

$$\nu(\ell,\vartheta,\zeta) = e^{\frac{1}{3}(\ell+\vartheta)-\zeta} \qquad (5.39)$$

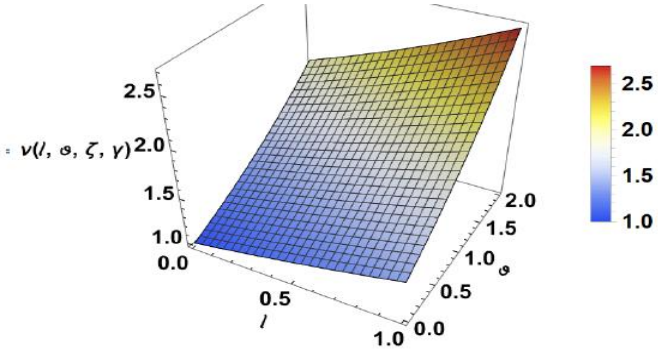

Figure 11: The surface graph of the approximate solution $\nu(\ell, \vartheta, \zeta)$ of (5.27) when $\gamma = 1$.

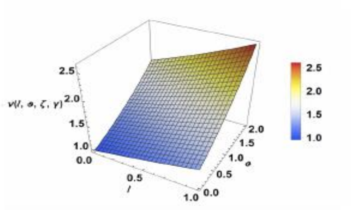

Figure 12: The surface graph of the approximate solution $\nu(\ell, \vartheta, \zeta)$ of (5.27) when $\gamma = 0.4$.

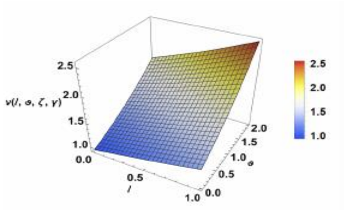

Figure 13: The surface graph of the approximate solution $\nu(\ell, \vartheta, \zeta)$ of (5.27) when $\gamma = 0.6$.

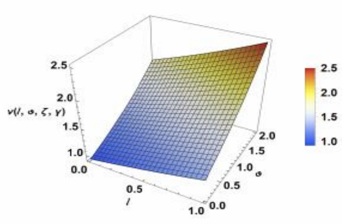

Figure 14: The surface graph of the approximate solution $\nu(\ell, \vartheta, \zeta)$ of (5.27) when $\gamma = 0.8$.

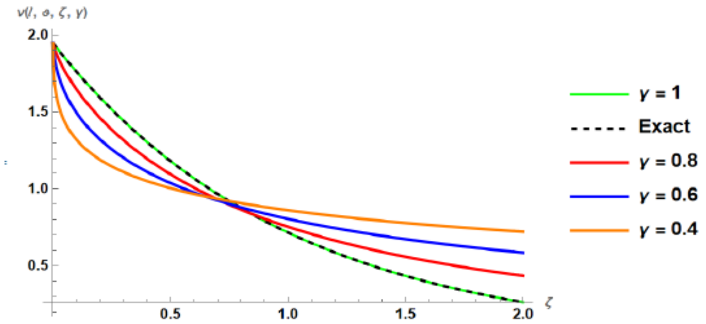

Figure 15: The surface graph of the approximate solution $\nu(\ell, \vartheta, \zeta)$ of (5.27) when $\gamma = 1$.

6- Conclusion

Using the NVIM, we examined the time-fractional biological equation's approximate solution. Because of the minimal number of computations, the suggested method's technique is determined to be more dependable than other analytical methods. Because it comprises of applying the NT directly to the provided issue and then applying the VIM, the approach is easily understood by the readers. As a result, the proposed approach is regarded to be a useful analytical tool for solving fractional PDEs.

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