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# Stochastic Differential Equation Under ExternalColored Noise

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**Abstract**

In this paper we study the second moment in stochastic differential equation when this equation contains colored noise and white noise . In order to solve this equation we use several steps and derivations when we get the results, we will have several equations that we solved in numerical ways through MATLAB, then we give assumed values for the parameters and we make a table for the second moment with the proposed values, In the second part we find a probability density function (pdf), and we take a maximum likelihood to this pdf and find result the posterior to colored noise from proposed function.

**Keywords**: Velocity, Position, mass, colored and white noise, Euler's method, maximum likelihood .

## 1 Introduction

The stochastic equation is used in many scientific applications as: Engineering Vehicles , biology [1,2] . A stochastic differential equation is essentially**,** classical differential equation which is perturbed by a random white noise and colored . And to solve these random equations that contain a perturbation factor such as colored and white noise we use the methods of solving equations, so we have several systems that we solve using MATLAB which solving them numerically[3]. As well as effect of noise factors, as there are many uses of noise factories [4]. we also saw what speed limits we used in several disciplines, including arithmetic [5,6]. Then we adopted a function [7], then we get a probability density function pdf contain a change of velocity then It takes a maximum likelihood to the parameter a, b in pdf. It is the numerical method that begins with $(x_0, y_0)$, and we find a value of posterior [8,9 ,10] to the colored noise from a proposed function [7]. Finally we give our conclusions.

Our system is define as :

$\frac{d}{dt}h_t = -\frac{-2a}{r}h_t - \frac{b^2}{r}q_t - \frac{b^2}{r}q_t z_t + \frac{\alpha}{r}w_t$

$\frac{d}{dt}q_t = h_t$

When $q_t = Q$ is a position $h_t = H$ is velocity

a ,b are parameters , r is a mass , $z_t = Z$ is colored noise , $w_t = W$ is white noise .

## 2 Second Moment

In this section we found the value of the second moment from the proposed equation depending on source [3], so we perform several steps and simplifications for this equation, then we have a set of systems of equations, also these equations are solved by numerical methods through a MATLAB, then we draw these moment and taking different values of the proposed constants, we notice the change in the value of the second moment through the table .

$$\frac{d}{dt}Q = H \quad ......(1)$$

$$\frac{d}{dt}H = \frac{-2a}{r}H - \frac{b^2}{r}Q - \frac{b^2}{r}QZ + \frac{\alpha}{r}W \quad .....(2)$$

Now multiply (1) by )$Q$) and (2) by $(H)$ then we get that :

$$\frac{d}{dt}Q^2 = 2QH. \quad .....(3)$$

$$\frac{d}{dt}H^2 = \frac{-4a}{r}H^2 - \frac{2b^2}{r}QH - \frac{2b^2}{r}ZQH + \frac{2\alpha}{r}WH. \quad ......(4)$$

By taking the average to above equations then the system becomes :

$$\frac{d}{dt} < Q^2 >= 2 < QH > \quad .....(5)$$

$$\frac{d}{dt} < H^2 >= \frac{-4a}{r} < H^2 > - \frac{2b^2}{r} < QH > - \frac{2b^2}{r} < ZQH > + \frac{2\alpha}{r} < WH >. \quad ........(6)$$

Where the average $\langle \cdots \rangle$ the mathematical system

From equation (1) we know that $\frac{d}{dt}Q = H$ when we put this value in $<WH>$ we have a following value :

$$< W\frac{d}{dt}Q >= (\frac{d}{dt}+\gamma) < WQ >$$

Depending on [3] we conclude that the amount becomes :

$$)\frac{d}{dt}+\gamma) < WQ >= (\frac{d}{dt}+\gamma)D$$

When we put above value in equation 6 become as :

$$\frac{d}{dt} < H^2 >= \frac{-4a}{r} < H^2 > -\frac{2b^2}{r} < QH > -\frac{2b^2}{r}Z < QH > +\frac{2\alpha}{r}(\frac{d}{dt}+\gamma)D)$$

To multiply (3) by $W$ and (4) by $W$ then we get,

$$W\frac{d}{dt}Q^2 = 2WQH \quad \text{.....(7)}$$

$$W\frac{d}{dt}H^2 = \frac{-4a}{r}WH^2 - \frac{2b^2}{r}WQH - \frac{2b^2}{r}WQZH + \frac{2\alpha}{r}W^2H \quad \text{....(8)}$$

Average to (7) and (8) then the output will be as follows :

$$)\frac{d}{dt}+\gamma) < WQ^2 >= 2 < WQH > \quad \text{......(9)}$$

$$(\frac{d}{dt}+\gamma) < WH^2 >= \frac{-4a}{r} < WH^2 > -\frac{2b^2}{r} < WQH > +\frac{2\alpha}{r} < W^2H > \quad \text{.....(10)}$$

Since $Z \amalg W$ then $< WZ >= 0$

$$\langle W^2 \rangle = (\frac{d}{dt}+\gamma) < WQ >$$

When we put above value in equation (10) we have that :

$$(\frac{d}{dt}+\gamma) < WH^2 >= \frac{-4a}{r} < WH^2 > -\frac{2b^2}{r} < WQH > +(\frac{2\alpha}{r}\gamma + \frac{2\alpha}{r}\frac{d}{dt}) < WQH > \quad \text{...(11)}$$

From equation (9) then :

$$\frac{d}{dt} < WQ^2 >= 2 < WQH > -\gamma < WQ^2 > \qquad \text{.....(12)}$$

Multiply (1) by (H) and (2) by ( $Q$ ) then we get :

$$\frac{d}{dt} QH = H^2 \qquad \text{......(13)}$$

$$\frac{d}{dt} QH = \frac{-2a}{r} QH - \frac{b^2}{r} Q^2 - \frac{b^2}{r} ZQ + \frac{\alpha}{r} WQ.. \qquad \text{.....(14)}$$

Average to (13) and (14) we conclude that :

$$\frac{d}{dt} < QH >=< H^2 . > . \qquad \text{......(15)}$$

$$\frac{d}{dt} < QH >= \frac{-2a}{r} < QH > - \frac{b}{r} < Q^2 > - \frac{b^2}{r} Z < Q^2 > + \frac{\alpha}{r} D \quad when \quad < \quad WQ >= \quad D) \qquad \text{.....(16)}$$

Multiply 13 and 14 by $W_t$ we have :

$$W \frac{d}{dt} QH = WH^2 \qquad \text{......(17)}$$

$$W \frac{d}{dt} QH = \frac{-2a}{r} wqH - \frac{b^2}{r} wQ^2 - \frac{b^2}{r} WZQ^2 + \frac{\alpha}{r} W^2 Q. \qquad \text{.....(18)}$$

Average to (17), (18) then :

$$)\frac{d}{dt} + \gamma) < WQH >=< WH^2 > . \qquad \text{......(19)}$$

$$(\frac{d}{dt} + \gamma) < WQh >= \frac{-2a}{r} < WQH > + (\frac{\alpha}{r}\gamma + \frac{\alpha}{r}\frac{d}{dt}) < WQ^2 > .. \qquad \text{.....(20)}$$

By [3] so that we have (6) *vairables* with (6) equations :

$< H^2 >$, $< Q^2 >$ $< WQ^2 >$ $< WH^2 >$, $< QH >$ $< WQH >$ :

From equation 6 we can extract

$< WH >$ also we known that

$\frac{d}{dt}Q = H$ when we put this value in equation 6 we get that :

$$<WH>=<W\frac{d}{dt}Q>$$

$$<WH>=<\frac{d}{dt}+\gamma)<wQ>=W^2$$

Note that equation 6 become :

$$\frac{d}{dt}<H^2>=\frac{-4a}{r}<H^2>-(\frac{b^2}{r}+\frac{b^2}{r}z)QH>+\frac{2\alpha}{r}W^2 \quad \ldots..(21)$$

When we arrange above systems we have a set of equations as follows :

$$\frac{d}{dt}<H^2>=\frac{-4a}{r}<H^2>-<QH>-\frac{2b^2}{r}z<QH>+\frac{2\alpha}{r}\langle w^2\rangle) \quad \ldots(22)$$

$$\frac{d}{dt}<Q^2>=2<QH> \quad \ldots..(23)$$

$$\frac{d}{dt}<wQ^2>=2<wQH>-\gamma<wQ^2>. \quad \ldots..(24)$$

From equations 12 we get :

$$\frac{d}{dt}<WH^2>=\frac{-4a}{r}<WH^2>-<WQH>-\frac{2b^2}{r}Z<WQH>+\frac{2\alpha}{r}<WQ^2>-\gamma<WH^2> \quad \ldots..(25)$$

And also the follow equation :

$$\frac{d}{dt}<QH>=\frac{-2a}{r}<QH>-\frac{b^2}{r}<Q^2>-\frac{b^2}{r}Z<Q>+\frac{2\alpha}{r}D \quad \ldots..(26)$$

And equation :

$$\frac{d}{dt}<WQH>=2<WH^2>-\gamma<WQH> \quad \ldots\ldots(27)$$

Now we will set the variables to one side and the reward to the other side as:

$$(\frac{d}{dt}-\frac{4a}{r})<H^2>+(\frac{2b^2}{r}+\frac{2b^2}{r}Z)<QH>=\frac{2\alpha}{r}(W^2)$$
$$\frac{d}{dt}<Q^2>-2<QH>=0$$
$$(\frac{d}{dt}+\gamma)<WQ^2>-2\langle WQH\rangle=0$$
$$(\frac{d}{dt}+\gamma+\frac{4a}{r})<WH^2>+\frac{2b^2}{r}<WHQ>=\frac{2\alpha}{r}D<WQ^2>$$
$$(\frac{d}{dt}+\frac{2a}{r})<QH>+(\frac{b^2}{r}+Z)<Q^2>=\frac{\alpha}{r}D$$
$$(\frac{d}{dt}+\gamma)<WQH>-<WH^2>=0$$

Then we use a MATLAB to solve the above equations and takes the initial values as t=0 then the variables becomes :

$$H^2=0\ ,\quad Q^2=0\quad WQ^2=0\quad WH^2=0\quad QH=0\quad WQH=0$$

We will take suggested values for some of the constants in the equations, for example the following values,

$$D=1\quad Z=1\quad w^2=1$$

*if* $\gamma=1\quad a=1\quad b^2=1\quad r=1\quad \alpha=1$ then :

$$<H^2>=\frac{3+3\sqrt{3}i}{4}e^{-4t}-e^{(\frac{2\sqrt{3}i-2t}{4})}+e^{-(\frac{2+t}{4})}-e^{-2t}$$

$$<Q^2>=(\frac{-3\sqrt{3}+(3\sqrt{3}+2)i}{12})-e^{(-\frac{3}{4}+\frac{\sqrt{3}i}{4})}+(-\sqrt{3}+\sqrt{3}i)e^{(\frac{\sqrt{3}}{6}-\frac{i}{6})t}$$

$$<WQ^2>=0$$
$$<WH^2>=0$$

$$<QH>=-\sqrt{3}e^{-t}+\sqrt{3}i\,e^{(\frac{-2\sqrt{3}-i)t}{3}}$$

$$<WQH>=0$$

Now if :

$$r=0.3\quad a=0.5\quad \gamma=1\quad \alpha=0.5,\quad b^2=0.2$$

$$< H^2 >= \frac{60}{527} e^{\frac{\sqrt{355}t-25}{15}} - \frac{767}{1054} e^{\frac{(-20t)}{3}t} + \frac{60}{527} e^{\frac{(-\sqrt{355}t+25)}{15}t} - \frac{900\sqrt{355}}{3747} e^{(\frac{\sqrt{355}}{15}t-\frac{5}{15})}$$

$$< Q^2 >= \frac{1775}{639} e^{\frac{(-\sqrt{355}t+25)}{15}t} - \frac{1775}{639} e^{\frac{(-\sqrt{355}t-5)}{15}}$$

$$< wQ^2 >= 0.$$

$$< WH^2 >= 0$$

$$< QH >= \frac{5\sqrt{355}}{71} e^{\frac{\sqrt{355}t+40}{15}}$$

Also $<WQH >= 0$

Now if r=1.2 a=1.5 $\gamma = 1.5$ $\alpha = 1.1$ $b = 1.3$

and when we use the MATLAP to solve those equations we get that

$$< H^2 >= \frac{2\sqrt{1145}}{4352} e^{\frac{-150t+2i\sqrt{14655}}{60}} + \frac{14655}{4251904} e^{\frac{-75t+\sqrt{14655}}{135950}} - \frac{2233}{2176} e^{-5t} + \dots \dots\dots$$

$$< Q^2 >= \frac{\sqrt{14655}}{165113} e^{\frac{-5t}{4}+-\frac{-5i+\sqrt{14655}}{60}} - \dots\dots\dots\dots$$

$$< WQ^2 >= 0$$

$$< WH^2 >= 0$$

$$< QH >= -5\sqrt{14655}\, e^{-75t} + \frac{\sqrt{14655}i}{60} e^{\frac{-30\sqrt{14655}-i}{977}}$$

$$< WQH >= 0$$

**we take assume some values for (t) by creating tables containing a set of approximate values**

Table (1) for $H^2$: if a=1 r=1 $b^2 = 1$ $\gamma = 0.5$ $\alpha = 1$

| t | $< H^2 >$ | t | $< H^2 >$ | t | $< H^2 >$ | t | $< H^2 >$ | t | $< H^2 >$ |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 1.0101 | 0.13967 | 2.0202 | 0.47776 | 3.0303 | 0.54829 | 4.0404 | 0.49665 |
| 0.20202 | 0.19549 | 1.21212 | 0.1946 | 2.22222 | 0.51967 | 3.23232 | 0.5369 | 4.24242 | 0.49259 |
| 0.25253 | 0.20271 | 1.26263 | 0.21189 | 2.27273 | 0.5275 | 3.28283 | 0.53378 | 4.29293 | 0.49196 |
| 0.35354 | 0.19463 | 1.36364 | 0.24909 | 2.37374 | 0.54012 | 3.38384 | 0.5275 | 4.39394 | 0.49111 |
| 0.40404 | 0.1843 | 1.41414 | 0.26853 | 2.42424 | 0.54498 | 3.43434 | 0.5244 | 4.44444 | 0.49087 |
| 0.50505 | 0.15978 | 1.51515 | 0.30804 | 2.52525 | 0.55205 | 3.53535 | 0.5184 | 4.54545 | 0.49071 |
| 0.60606 | 0.13751 | 1.61616 | 0.34713 | 2.62626 | 0.5559 | 3.63636 | 0.51283 | 4.64646 | 0.49092 |
| 0.65657 | 0.12901 | 1.66667 | 0.3661 | 2.67677 | 0.55676 | 3.68687 | 0.51024 | 4.69697 | 0.49114 |
| 0.70707 | 0.12282 | 1.71717 | 0.38452 | 2.72727 | 0.55699 | 3.73737 | 0.5078 | 4.74747 | 0.49143 |
| 0.80808 | 0.11813 | 1.81818 | 0.41921 | 2.82828 | 0.55577 | 3.83838 | 0.5034 | 4.84848 | 0.49216 |
| 0.85859 | 0.11976 | 1.86869 | 0.4353 | 2.87879 | 0.55445 | 3.88889 | 0.50145 | 4.89899 | 0.49259 |
| 0.90909 | 0.12398 | 1.91919 | 0.45046 | 2.92929 | 0.55272 | 3.93939 | 0.49968 | 4.94949 | 0.49305 |
| 0.9596 | 0.13067 | 1.9697 | 0.46463 | 2.9798 | 0.55065 | 3.9899 | 0.49808 | 5 | 0.49353 |

Table(2) for $H^2$: $a = 0.5$ $b^2 = 0.2$ $\alpha = 0.5$ $\gamma = 1$ $r = 0.3$

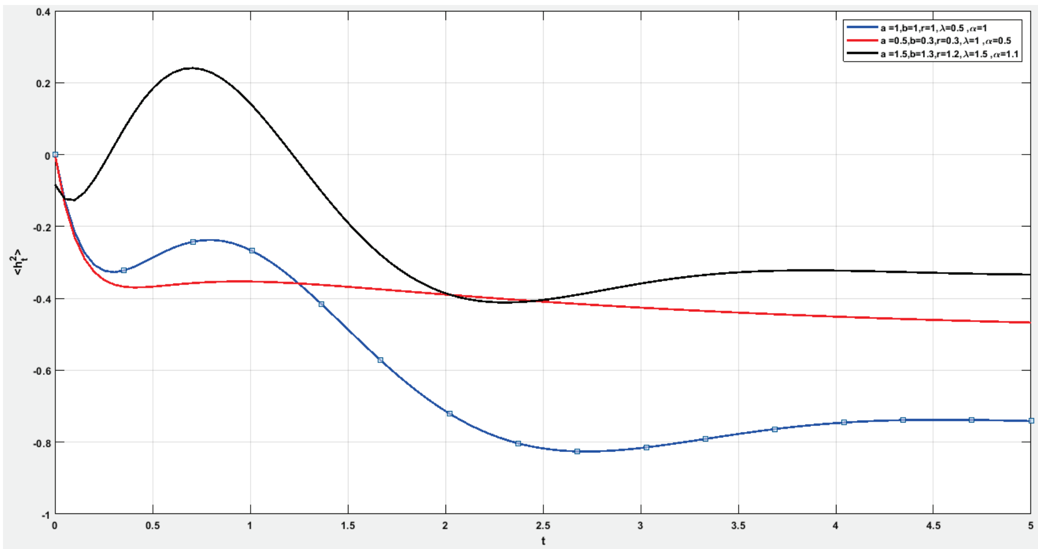
| | $< H^2 >$ | | $< H^2 >$ | | $< H^2 >$ | | $H^2$ |
|---|---|---|---|---|---|---|---|
| | | .0101 | .35349 | .0303 | .40229 | .0404 | .43541 |
| .05051 | .13718 | .06061 | .35632 | .08081 | .40428 | .09091 | .43673 |
| .15152 | .28027 | .16162 | .36187 | .18182 | .40816 | .19192 | .4393 |
| .20202 | .3126 | .21212 | .36459 | .23232 | .41003 | .24242 | .44055 |
| .30303 | .33771 | .31313 | .3699 | .33333 | .41368 | .34343 | .44296 |
| .35354 | .33935 | .36364 | .3725 | .38384 | .41545 | .39394 | .44413 |
| .45455 | .33347 | .46465 | .37756 | .48485 | .41888 | .49495 | .4464 |
| .55556 | .32374 | .56566 | .38245 | .58586 | .42217 | .59596 | .44858 |
| .60606 | .31906 | .61616 | .38482 | .63636 | .42376 | .64646 | .44963 |
| .70707 | .31143 | .71717 | .38945 | .73737 | .42686 | .74747 | .45168 |
| .75758 | .30865 | .76768 | .39169 | .78788 | .42836 | .79798 | .45267 |
| .85859 | .3052 | .86869 | .39605 | .88889 | .43126 | .89899 | .45459 |
| .90909 | .30445 | .91919 | .39817 | .93939 | .43267 | .94949 | .45553 |
| .9596 | .30427 | .9697 | .40025 | .9899 | .43406 | | .45644 |

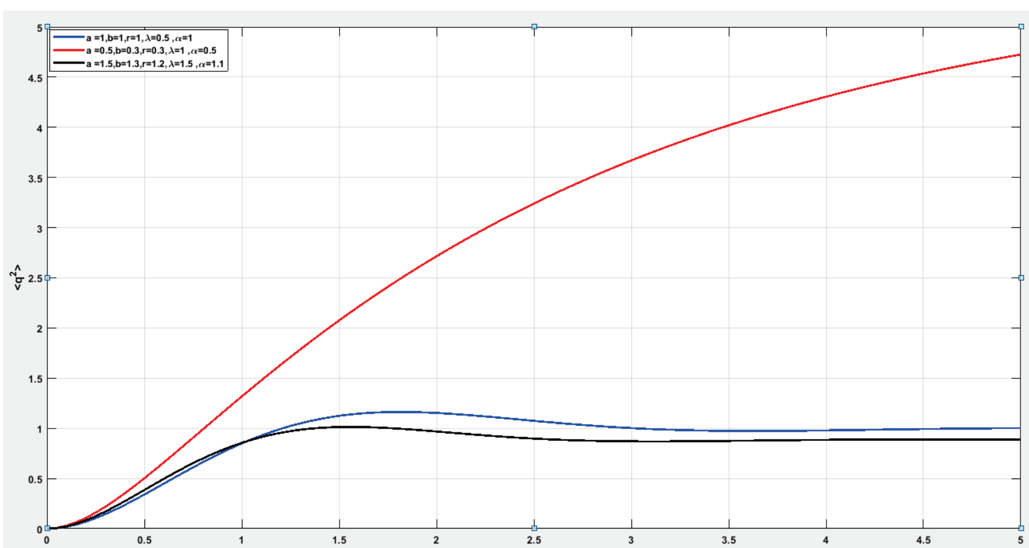
Table(3) for $H^2$ : $a = 1.5$ $b^2 = 1.3$ $r = 1.2$ $\alpha = 1.1$ $\gamma = 1.5$:

| t | $< H^2 >$ | t | $< H^2 >$ | t | $< H^2 >$ | t | $< H^2 >$ | t | $< H^2 >$ |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 1.0101 | 0.1793 | 2.0202 | 0.53788 | 3.0303 | 0.51547 | 4.0404 | 0.4930 |
| 0.05051 | 0.1008 | 1.06061 | 0.2007 | 2.07071 | 0.54307 | 3.08081 | 0.51256 | 4.09091 | 0.4933 |
| 0.10101 | 0.1604 | 1.11111 | 0.2234 | 2.12121 | 0.54705 | 3.13131 | 0.50982 | 4.14141 | 0.4937 |
| 0.20202 | 0.2003 | 1.21212 | 0.2710 | 2.22222 | 0.55179 | 3.23232 | 0.50493 | 4.24242 | 0.4946 |
| 0.30303 | 0.1851 | 1.31313 | 0.3192 | 2.32323 | 0.55285 | 3.33333 | 0.50088 | 4.34343 | 0.4955 |
| 0.40404 | 0.1520 | 1.41414 | 0.3655 | 2.42424 | 0.551 | 3.43434 | 0.4977 | 4.44444 | 0.4965 |
| 0.50505 | 0.1216 | 1.51515 | 0.4081 | 2.52525 | 0.54698 | 3.53535 | 0.49534 | 4.54545 | 0.4974 |
| 0.60606 | 0.1035 | 1.61616 | 0.4458 | 2.62626 | 0.54149 | 3.63636 | 0.49374 | 4.64646 | 0.4982 |
| 0.70707 | 0.1014 | 1.71717 | 0.4777 | 2.72727 | 0.53512 | 3.73737 | 0.49281 | 4.74747 | 0.4990 |
| 0.80808 | 0.115 | 1.81818 | 0.5036 | 2.82828 | 0.5284 | 3.83838 | 0.49246 | 4.84848 | 0.4996 |
| 0.9596 | 0.1596 | 1.9697 | 0.5314 | 2.9798 | 0.51854 | 3.9899 | 0.49275 | 5 | 0.5003 |
| 0 | 0 | 1.0101 | 1.33365 | 2.0202 | 2.73803 | 3.0303 | 3.693 | 4.0404 | 4.32521 |
| 0.05051 | 0.00804 | 1.06061 | 1.41476 | 2.07071 | 2.79557 | 3.08081 | 3.73121 | 4.09091 | 4.35046 |
| 0.10101 | 0.03046 | 1.11111 | 1.49496 | 2.12121 | 2.85196 | 3.13131 | 3.76864 | 4.14141 | 4.37519 |
| 0.15152 | 0.06497 | 1.16162 | 1.57418 | 2.17172 | 2.90723 | 3.18182 | 3.8053 | 4.19192 | 4.39941 |
| 0.20202 | 0.1096 | 1.21212 | 1.65233 | 2.22222 | 2.96139 | 3.23232 | 3.84121 | 4.24242 | 4.42314 |

In above tables we used R program to solve the values that we assumed to find the amount of change every time we change the values of the constants and to note the differences in the results through the tables .

**Draw of the second moment**




**3 posterior of colored noise**

In this section we take a proposed probability density function to which we apply the special law of the posterior, and calculate the $E(q_t = Q, h_t = H) and\ E(q_t = Q, h_t = H)$ in order to find the variance, depending on [7]. Let us conclude that the function :

$$P_s = \frac{2ab(r(1+z_t)^{\frac{1}{2}}}{\pi^2 a^2 D} \exp \frac{arh_t^{\ 2} + ab^2(1+z_t)q_t^{\ 2}}{2D\pi a^2} \qquad \text{……….(28)}$$

Thus the posterior distribution for $z$ is

$$p(q_1, \ldots\ldots, q_{n,} h_1, \ldots\ldots, h_n) = \frac{\pi(z_t)\prod_{i=1}^{n} p_s(q_t, h_t)}{\int_0^\infty \pi(z_t)\prod_{i=1}^{n} p_s(q_t, h_t)dz_t} \qquad \text{… … (29)}$$

It may be noted here that the posterior distribution of $(z_t)$ takes a ratio form that involves integration in the denominator and cannot be reduced to a closed form. Hence, the evaluation of the posterior expectation for obtaining the Bayes estimator of $z$ will be tedious. Among the various methods suggested to approximate the ratio of integrals of the above form, perhaps the simplest one is Lindley's [14] approximation method, which approaches the ratio of the integrals as a whole and produces a single numerical result. Thus, we propose the use of Lindley's approximation [14] for obtaining the Bayes estimator of $z$. In this section we calculate $E(q_t, h_t) and\ E(q_t, h_t)$ in order to find the posterior variance estimates given

$$Var(q_t, h_t) = E(q_t, h_t) - (E(q_t, h_t))^2$$

$$I(z_t) = E(z_t) = \widehat{z_{tB}} = \frac{\int_0^\infty z_t\, e^{loglog\left(\pi(z_t)\right) + L\left(p_s(q_t,h_t)\right)} dz_t}{\int_0^\infty e^{loglog\left(\pi(z_t)\right) + L\left(p_s(q_t,h_t)\right)} dz_t} \text{………(30)}$$

Where $L(p_s(q_t, h_t))$ is log likelihood $p_s(q_t, h_t)$

Therefore

$$\widehat{z_B} = \frac{\int_0^\infty z_t\, e^{log(\frac{e^{-\frac{z_t^2}{2t^2}}}{\sqrt{2\pi}\,t})} e^{log(\frac{2^n a^n b^n (r(1+z_t))^{\frac{n}{2}}}{\pi^{2n}\alpha^{2n}D^n} e^{-\frac{(ar\sum_{i=1}^{n} h_t^2 + ab^2(1+z_t))\sum_{i=1}^{n} q_t^2)}{2D\alpha^2\pi}})} dz_t}{\int_0^\infty e^{log(\frac{e^{-\frac{z^2}{2t^2}}}{\sqrt{2\pi}\,t})} e^{log(\frac{2^n a^n b^n (r(1+z_t))^{\frac{n}{2}}}{\pi^{2n}\alpha^{2n}D^n} e^{-\frac{(ar\sum_{i=1}^{n} h_t^2 + ab^2(1+z_t))\sum_{i=1}^{n} q_t^2)}{2D\alpha^2\pi}})} dz_t}$$

…….(31)

Subsequently

$$\widehat{z_B}=\frac{\int_0^{\infty} \quad z_t\, e^{-\frac{z_t^{\,2}}{2t^2}} e^{log((1+z_t)^{\frac{n}{2}} e^{-\frac{z_t}{2\pi D\alpha^2}\sum_1^n \quad q_t^2})} dz_t}{\int_0^{\infty} \quad e^{-\frac{z^2}{2t^2}} e^{log((1+z_t)^{\frac{n}{2}} e^{-\frac{z}{2\pi D\alpha^2}\sum_1^n \quad q_t^2})} dz_t}$$

We use Lindley's approximation for obtaining the Bayes estimator of $z_t$ by evaluated as

$$\widehat{z_B}= \widehat{z_t} + \frac{1}{2}\sum_i \quad \sum_j \quad (u_{ij}+2u_i\rho_j)\sigma_{ij}+\frac{1}{2}\sum_i \quad \sum_j \quad \sum_k \quad \sum_l \quad (L_{ijk}\sigma_{ij}\sigma_{kl}u_1)$$

Where $\widehat{z_t}$ is MLE estimation of $z$ and $i, j, k, l$ equal to 1 then we get

$$\widehat{z_B}= \widehat{z_t} + \frac{1}{2}(u_{11}+2u_1\rho_1)\sigma_{ij}+\frac{1}{2}L_{111}\sigma_{11}\sigma_{11}u_1$$

Where $u=z_t$ therefore $u_1=\frac{\partial u}{\partial z_t}=1$ and $u_{11}=\frac{\partial^2 u}{\partial {z_t}^2}=0$

$\rho = log\; log\left(\frac{e^{-\frac{z^2}{2t^2}}}{\sqrt{2\pi}\, t}\right) = -\frac{{z_t}^2}{2t^2} - log\; log\left(\sqrt{2\pi}\, t\right)$ Then $\rho_1=\frac{\partial \rho}{\partial z_t}=-\frac{z_t}{t^2}$

$$L = nLn2 + nLna + nLnb + Lnr + \frac{n}{2}Ln(1+Z_t)) - 2nLn\pi - 2nLn\alpha - 2LnD$$

$$-\frac{(ar\sum_{i=1}^{n}\hat{h}^2{}_t + ab^2(1+Z_t)\sum_{i=1}^{n}\hat{q}_t{}^2}{2D\alpha^2\pi}$$

$$L_1 == \frac{n}{2(1+z_t)} - \frac{ab^2\sum_1^n \quad q^2}{2\pi D\alpha^2}$$

$L_{11} == -\frac{n}{2(1+z_t)^2}$ , $L_{111} == \frac{n}{(1+z_t)^3}$

$$and \quad \sigma_{ij}=-L_{ij}^{-1}$$

One can obtained

$$\widehat{z_{t_B}} = \widehat{z_t} + \frac{1}{2}\left(0-2\frac{\widehat{z_t}}{t^2}\right)\left(\frac{2(1+\widehat{z_t})^2}{n}\right)+\frac{1}{2}\frac{n}{(1+\widehat{z_t})^3}\frac{4(1+\widehat{z_t})^4}{n^2}$$

…………(32)

Subsequently

$$\widehat{z_{t_B}} = \hat{z}_t - \left(\frac{2\hat{z}_t(1+\hat{z}_t)^2}{nt^2}\right) + \left(\frac{2(1+\hat{z}_t)}{n}\right) \quad \ldots\ldots(33))$$

When we taking values for Z=0.5,1,2….. and n=75,100…. the results are different for the probability density function depending on the values taken each time.

**Conclusion**

In this paper we talk about a proposed system of random equations, then we obtained some results as second moment, posterior for the disturbance factor colored noise. whereas the first section included taking the proposed system, and after several steps, we got several equations , and theses equations cannot be solved in approximate ways, so we used numerical methods to solve them through MATLAB after that they became exponential equations, and then we started taking approximate values for the existing coefficients, and discussed the results through tables. In two sections we find the result of a density function that contained a change in the speed at a certain time as well as at a primitive time . In third section we used a posterior to find the approximate value to colored noise .

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