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NEW FINDINGS RELATED TO GP- METRIC SPACES

Rihab T. Muhammed, **Ayed E. Hashoosh**
rihab.taima.math@utq.edu.iq **, Ayed197991@yahoo.com**

University of Thi-Qar-College of Education for Pure Sciences

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Abstract

In this study, several conclusions of fixed point theorems for GP-metric spaces are developed using lower semi-continuous mappings. We also extend Karapinar's findings that depend on partial metric spaces.

Keywords: Fixed point theorem, Lower semi-continuous, Partial metric space, G-mertic space, G_p- metric space.

1 INTRODUCTION

In [16], Matthews developed the idea of partial metric space to widen and generalizes the idea of metric spaces in which $d(\varkappa,\varkappa)$ are no longer invariably zero. While researching several computer application-related issues, he came up with the concept of partial metric spaces. In order to resolve some challenging issues in computer sciences, the conventional theory of a fixed point can be used. Furthermore, partial metric spaces have a spacious area of application in the field of fixed point theory and have been utilized in extensively for multiple generalizations of the Banach contraction principle. Also, Matthews in [16] proved the partial metric contraction mapping theorem which is analogous to the Banach contraction mapping principle in metric spaces. After this fabulous contribution, A lot of authors concentrated on the features of partial metric space. In 1996, O'Neill [19] generalized Matthews notion of partial metric, in order to establish connections between these structures and the topological aspects of domain theory. After then, Oltra and Valero [18] obtained the Banach fixed point theorem for complete partial metric spaces in the sense of O'Neill and Valero in [20]. Künzi et al. [15] introduced and investigated the concept of partial quasi-metric and some of its applications. In 2008, Altun and Simsek [1] gave an order relation on dualistic partial metric space and by using this relation they proved some fixed point theorems for single and multi valued mappings on ordered dualistic partial metric space. And Altun et al. [2], Altun and Erduran [3]. In [4],

Abdeljawad et al. introduced a general variant of the weak ϕ-contraction on partial metric space for getting a common fixed point. Ćirić et al. [8] evidenced the common fixed point theorems for four mappings satisfying a generalized non-linear contraction type condition on partial metric space. Haghi et al. [9] proved that some generalizations in fixed point theory are not real generalizations. Han et al. [11] studied topologically the partial metric space which may be regarded as a novel sub-branch of the pure asymmetric topology. Recently, Zand and Nezhad [22] and [24] introduced a new structure of generalized metric spaces called G_p-metric spaces, which are a generalization of the concepts of partial metric spaces, and G-metric spaces, which are symmetric. Yazdy et al. [21] obtained a coupled fixed point theorem in a complete G_p-metric space which can be symmetric or asymmetric. Zand and Yazdy [23] demonstrated a connection between G-metric spaces, partial metric spaces and G_p- metric spaces. The goal of this paper is to construct a new finding of some important theories related to G_p- metric space by using lower semi-continuous mappings.

1.2 Preliminaries

In this section, we will highlight some fundamental notations, concepts, and results that will be employed subsequently.

Throughout this paper, we denote by $\mathbb{R}, \mathbb{R}^+,$ and $\mathbb{N}$ the set of all real numbers, the set of all non negative real numbers, and the set of all non negative integer numbers, respectively.

Theorem 1.2.1. [23] Given that S is a self-mapping of a set $\mathcal{M}$, suppose that $(\mathcal{M}, G_p)$ is a G_p- metric space, $\psi: \mathcal{M}^2 \to \mathbb{R}^+$ be a lower semi continuous function (l.s.c.), and

For each $\varkappa \in \mathcal{M}$ $\quad G_p(\varkappa, S\varkappa, S\varkappa) \leq \psi(\varkappa) - \psi(T\varkappa).$

A fixed point exists in the self-mapping S if $(\mathcal{M}, G_p)$ is a G_p- complete.

Definition 1.2.2. [25] Suppose that $(\mathcal{M}, G_p)$ is a G_p- metric space and that $\psi: \mathcal{M}^2 \to \mathbb{R}^+$is a function. If $S: (\mathcal{M}, G_p) \to (\mathcal{M}, G_p)$ is a G_p-continuous function, $\{\varkappa_n\}_{n\geq 1}$is a G_p-convergent sequence to $x \in \mathcal{M}$ such that the following assumption occurs,

$lim_{n,m\to\infty} G_p(\varkappa_n, \varkappa_m, S\varkappa_m) = G_p(\varkappa, \varkappa, \varkappa)$ then,

$\psi(\varkappa, S\varkappa) \leq lim_{m\to\infty} inf\ \psi(\varkappa_m, S\varkappa_m)\ sup_{n\geq 1} inf_{m\geq n}\ \psi(\varkappa_m, S\varkappa_m),$

The function ψ is thus called (S-l.s.c.) on $\mathcal{M}^2$.

Furthermore, if for every $\varkappa \in \mathcal{M}$ the following inequality is true:

$$G_p(\varkappa, S\varkappa, S^2\varkappa) \leq \psi(\varkappa, S\varkappa) - \psi(S\varkappa, S^2\varkappa),$$

then S is said to be a Caristi map on $(\mathcal{M}, G_p)$.

Theorem 1.2.3. [23] Let $(\mathcal{M}, G_p)$ be a G_p- metric space that is G_p-complete and let $\psi: \mathcal{M}^2 \to \mathbb{R}^+$, $S: \mathcal{M} \to \mathcal{M}$ be two mappings. If ψ is S-l.s.c. map on $\mathcal{M}^2$ and S is a Caristi map on $(\mathcal{M}, G_p)$. As a result, S in $\mathcal{M}$ has a fixed point.

Theorem 1.2.4. [21] Let the space $(\mathcal{M}, G_p)$ be a G_p-metric. The function $G_p(\varkappa, \partial, z)$ becomes jointly continuous in all three variables.

Proposition 1.2.5. [22] Every G_p- metric space $(\mathcal{M}, G_p)$ generates a partial metric

space $(\mathcal{M}, P_{Gp})$, where

$P_{Gp}(\varkappa, \partial) = G_p(\varkappa, \partial, \partial)$ for every $\varkappa, \partial \in \mathcal{M}$.

Corollary 1.2.6. [23] Consider that G_p- metric space $(\mathcal{M}, G_p)$,

(1) In a G_p-metric space $(\mathcal{M}, G_p)$, a sequence $\{\varkappa_n\}_{n\geq 1}$is a G_p-Cauchy sequence if and only if it is a Cauchy sequence in the partial metric space it belongs to $(\mathcal{M}, P_{Gp})$.
(2) The corresponding partial metric space $(\mathcal{M}, P_{Gp})$ is complete, if and only if G_p-metric space $(\mathcal{M}, G_p)$ is G_p-complete.

Theorem 1.2.7. [23] Each G_p-metric space $(\mathcal{M}, G_p)$ generates a G-symmetric space $(\mathcal{M}, G_{Gp})$, such that:

$G_{Gp}(\varkappa, \partial, z) = G_p(\varkappa, \partial, \partial) + G_p(\varkappa, z, z) + G_p(\partial, z, z) - G_p(\varkappa, \varkappa, \varkappa) - G_p(\partial, \partial, \partial) - G_p(z, z, z)$ for any $\varkappa, \partial, z \in \mathcal{M}$.

Corollary 1.2.8. [23] Let $(\mathcal{M}, G_p)$ be a G_p- metric space, $(\mathcal{M}, G_{Gp})$ is the G- metric space that it is associated with. Then the following circumstances exist.

(1) A sequence $\{\varkappa_n\}_{n\geq 1}$ is said to be G-Cauchy in $(\mathcal{M}, G_{Gp})$ if and only if it is G_p- Cauchy in $(\mathcal{M}, G_p)$.
(2) $(\mathcal{M}, G_{Gp})$ is a G-complete if and only if $(\mathcal{M}, G_p)$ is G_p-complete. Furthermore, $lim_{n,m\to\infty} G_{Gp}(\varkappa, \varkappa_n, \varkappa_m) = 0$ if and only if $G_p(\varkappa, \varkappa, \varkappa) = lim_{n\to\infty} G_p(\varkappa, \varkappa_n, \varkappa_n)$
$= lim_{n,m\to\infty} G_p(\varkappa_n, \varkappa_m, \varkappa_m)$

$= lim_{n\to\infty} G_p(\varkappa_n, \varkappa_n, \varkappa_n)$.

1.3 Main results

Theorem 1.3.1. Assume the G_p- metric space $(\mathcal{M}, G_p)$ be complete. And let $\mathcal{H}$ be the family of mappings $h: \mathcal{M} \to \mathcal{M}$. If there is a function $\psi: \mathcal{M}^2 \to \mathbb{R}^+$ which is S-l.s.c. function on $\mathcal{M}^2$, $S: \mathcal{M} \to \mathcal{M}$ is a Caristi map on $(\mathcal{M}, G_p)$ corresponding to,

$G_p\big(\varkappa, h(\varkappa), Sh(\varkappa)\big) \leq \psi(\varkappa, S\varkappa) - \psi\big(h(\varkappa), Sh(\varkappa)\big)$ for any $\varkappa \in \mathcal{M}$ and for any

$h \in \mathcal{H}$. (3.1)

so, for each $\varkappa \in \mathcal{M}$, $\mathcal{H}$ has a common fixed point r such that ,

$G_p(\varkappa, r, Sr) \leq \psi(\varkappa, S\varkappa) - u$, where $u = inf\ \{\psi(\varkappa, S\varkappa),\ \varkappa \in \mathcal{M}\}$

Proof. Let $D(\varkappa), \alpha(\varkappa)$ symbolize the following sets for $\varkappa$ in $\mathcal{M}$:

$D(\varkappa) = \{\varkappa \in \mathcal{M}: G_p(\varkappa, r, Sr) \leq \psi(\varkappa, S\varkappa) - \psi(r, Sr)\}$, (3.2)

$\alpha(\varkappa) = inf\ \{\psi(r, Sr): r \in D(\varkappa)\}$. (3.3)

$D(\varkappa) \neq \emptyset$, and Since $\varkappa \in D(\varkappa)$. Let's say that the expression $r \in D(\varkappa)$ imply that $0 \leq \alpha(\varkappa) \leq \psi(r, Sr) \leq \psi(\varkappa, S\varkappa)$.

In order to construct the sequence $\{\varkappa_n\}_{n\geq 1}$, follow the instructions below:

$$\varkappa_1 := \varkappa,$$

For any $n \in \mathbb{N}$, $\varkappa_{n+1} \in D(\varkappa_n)$ such that $\psi(\varkappa_{n+1}, S\varkappa_{n+1}) \leq \alpha(\varkappa_n) + \frac{1}{n}$. (3.4)

Consequently, one can see that for every n,

(i) $G_p(\varkappa_n, \varkappa_{n+1}, S\varkappa_{n+1}) \leq \psi(\varkappa_n, S\varkappa_n) - \psi(\varkappa_{n+1}, S\varkappa_{n+1})$.

(ii) $\alpha(\varkappa_n) \leq \psi(\varkappa_{n+1}, S\varkappa_{n+1}) \leq \alpha(\varkappa_n) + \frac{1}{n}$.

and so $\{\psi(\varkappa_n, S\varkappa_n)\}$ is a decreasing sequence of non-negative real numbers. Hence, the sequence $\{\psi(\varkappa_n, S\varkappa_n)\}$ is convergent to a real number L, and as a result, via (i), (ii), we obtain

$$L = \lim_{n\to\infty} \psi(\varkappa_n, S\varkappa_n) = \lim_{n\to\infty} \alpha(\varkappa_n). \qquad (3.5)$$

As stated in (3.5), there exists $N_k \in \mathbb{N}$ for every $k \in \mathbb{N}$, such that

$$\psi(\varkappa_n, S\varkappa_n) \leq L + \frac{1}{k}, \text{ for any } n \geq N_k. \qquad (3.6)$$

Since $\{\psi(\varkappa_n, S\varkappa_n)\}$ is decreasing sequence, we obtain

$$L \leq \psi(\varkappa_m, S\varkappa_m) \leq \psi(\varkappa_n, S\varkappa_n) \leq L + \frac{1}{k}, \text{ for all } m \geq n \geq N_k$$

and so

$$\psi(\varkappa_n, S\varkappa_n) - \psi(\varkappa_m, S\varkappa_m) < \frac{1}{k}, \text{ for all } m \geq n \geq N_k. \qquad (3.7)$$

By using (G_{p1}), (G_{p3}) and (i), (ii), the following inequalities are valid.

$$\begin{aligned} G_p(\varkappa_n, \varkappa_{n+2}, S\varkappa_{n+2}) &\leq G_p(\varkappa_n, \varkappa_{n+1}, \varkappa_{n+1}) + G_p(\varkappa_{n+1}, \varkappa_{n+2}, S\varkappa_{n+2}) \\ &\quad - G_p(\varkappa_{n+1}, \varkappa_{n+1}, \varkappa_{n+1}) \\ &\leq G_p(\varkappa_n, \varkappa_{n+1}, S\varkappa_{n+1}) + G_p(\varkappa_{n+1}, \varkappa_{n+2}, S\varkappa_{n+2}) \\ &\leq \psi(\varkappa_n, S\varkappa_n) - \psi(\varkappa_{n+1}, S\varkappa_{n+1}) + \psi(\varkappa_{n+1}, S\varkappa_{n+1}) \\ &\quad - \psi(\varkappa_{n+2}, S\varkappa_{n+2}) \end{aligned}$$

$$\leq \psi(\varkappa_n, S\varkappa_n) - \psi(\varkappa_{n+2}, S\varkappa_{n+2}).$$

The following inequalities are created using the same method as the calculation above.

$$G_p(\varkappa_n, \varkappa_{n+3}, S\varkappa_{n+3}) \leq G_p(\varkappa_n, \varkappa_{n+2}, \varkappa_{n+2}) + G_p(\varkappa_{n+2}, \varkappa_{n+3}, S\varkappa_{n+3})$$

$$- G_p(\varkappa_{n+2}, \varkappa_{n+2}, \varkappa_{n+2})$$

$$\leq G_p(\varkappa_n, \varkappa_{n+2}, S\varkappa_{n+2}) + G_p(\varkappa_{n+2}, \varkappa_{n+3}, S\varkappa_{n+3})$$

$$\leq \psi(\varkappa_n, S\varkappa_n) - \psi(\varkappa_{n+2}, S\varkappa_{n+2}) + \psi(\varkappa_{n+2}, S\varkappa_{n+2})$$

$$-\psi(\varkappa_{n+3}, S\varkappa_{n+3})$$

$$\leq \psi(\varkappa_n, S\varkappa_n) - \psi(\varkappa_{n+3}, S\varkappa_{n+3}).$$

The following inequality arises as a result of a simple induction.

$$G_p(\varkappa_n, \varkappa_m, S\varkappa_m) \leq \psi(\varkappa_n, S\varkappa_n) - \psi(\varkappa_m, S\varkappa_m) \text{ for all } m \geq n, \qquad (3.8)$$

Via (G_{p1}) and Theorem 1.2.7, we obtain:

$$G_{Gp}(\varkappa_n, \varkappa_m, \varkappa_m) \leq 2G_p(\varkappa_n, \varkappa_m, \varkappa_m)$$

$$\leq 2G_p(\varkappa_n, \varkappa_m, S\varkappa_m) \qquad (3.9)$$

The right side of (3.8) tends to be zero. because the sequence $\{\psi(\varkappa_n, S\varkappa_n)\}$ is convergent, and so $G_p(\varkappa_n, \varkappa_m, S\varkappa_m)$ also tends to zero as $n, m \to \infty$. As a result, as per (3.9), $\{\varkappa_n\}_{n\geq 1}$ is G-Cauchy in G-metric space $(\mathcal{M}, G_{Gp})$. Due to Corollary 1.2.8. the related G-metric space $(\mathcal{M}, G_{Gp})$ is G-complete since $(\mathcal{M}, G_p)$ is G_p-complete. Consequently, there exists $r_0 \in \mathcal{M}$ such that the sequence $\{\varkappa_n\}_{n\geq 1}$ is G-convergent to $r_0 \in \mathcal{M}$, as shown by Corollary 1.2.8.

$$G_p(r_0, r_0, r_0) = \lim_{n\to\infty} G_p(r_0, x_n, x_n)$$

$$= \lim_{n,m\to\infty} G_p(\varkappa_n, \varkappa_m, \varkappa_m)$$

$$\leq \lim_{n,m\to\infty} G_p(\varkappa_n, \varkappa_m, S\varkappa_m) = 0. \qquad (3.10)$$

So,$\{\varkappa_n\}_{n\geq 1}$ is a G_p-convergent to r_0, $G_p(r_0, r_0, r_0) = \lim_{n,m\to\infty} G_p(\varkappa_n, \varkappa_m, S\varkappa_m)$ and so by using (3.5) , $\psi(r_0, Sr_0) \leq \lim_{m\to\infty} inf\, \psi(\varkappa_m, S\varkappa_m) = L$ since ψ is S- l.s.c. Hence, As stated in Theorem 1.2.4., the inequality (3.8) implies that

$$\psi(r_0, Sr_0) \leq \lim_{m\to\infty} inf\, \psi(\varkappa_m, S\varkappa_m)$$

$$\leq \lim_{m\to\infty} inf\left[\psi(\varkappa_n, S\varkappa_n) - G_p(\varkappa_n, \varkappa_m, S\varkappa_m)\right]$$

$$= \psi(\varkappa_n, S\varkappa_n) - G_p(\varkappa_n, r_0, Sr_0). \qquad (3.11)$$

Therefore, (3.2) implies that $r_0 \in D(\varkappa_n)$ for all $n \in \mathbb{N}$ and so by (3.5),

$$L = \lim_{n\to\infty} \alpha(\varkappa_n) \le \psi(r_0, Sr_0).$$

Thus $L = \psi(r_0, Sr_0)$.

We are going to demonstrate that for all $h \in \mathcal{H}$, $h(r) = r$. Contrarily, suppose that $h \in \mathcal{H}$ exists such that $h(r) \neq r$. In (3.1) replace $r = \varkappa$ then we have $\psi(h(r), Sh(r)) < \psi(r, Sr) = L$. Thus by (3.5), there are $n \in \mathbb{N}$ such that

$\psi(h(r), Sh(r)) < \alpha(\varkappa_n)$. Since $r \in D(\varkappa_n)$, we have

$G_p(\varkappa_n, h(r), Sh(r)) \le G_p(\varkappa_n, r, r) + G_p(r, h(r), Sh(r)) - G_p(r, r, r)$

$\le G_p(\varkappa_n, r, Sr) + G_p(r, h(r), Sh(r))$

$\le \psi(\varkappa_n, S\varkappa_n) - \psi(r, Sr) + \psi(r, Sr) - \psi(h(r), Sh(r))$

$\le \psi(\varkappa_n, S\varkappa_n) - \psi(h(r), Sh(r))$. (3.12)

Clearly $\psi(h(r), Sh(r)) \le \psi(\varkappa_n, S\varkappa_n)$ for all $n \in \mathbb{N}$ and so $\psi(h(r), Sh(r)) \le L$

by (3.5).

By (3.2) and (3.12) implies that $h(r) \in A(\varkappa_n)$ for all $n \in \mathbb{N}$. Hence, $\alpha(\varkappa_n) \le$

$\psi(h(r), Sh(r))$ which is a contradiction with $\psi\big(h(r), Sh(r)\big) < \alpha(\varkappa_n)$.

Thus $h(r) = r$ for every $x \in \mathcal{M}$ and for all $h \in \mathcal{H}$. Since $r \in A(\varkappa_n)$, we have

$$G_p(\varkappa_n, r, Sr) \le \psi(\varkappa_n, S\varkappa_n) - \psi(r, Sr)$$

$$\le \psi(\varkappa_n, S\varkappa_n) - inf\,\{\psi(y, Sy), y \in \mathcal{M}\}$$

$$\le \psi(\varkappa, S\varkappa) - u\,.$$

Is obtained. □

The Theorem 1.3.1 is the improvement of the result of Karapinar in ([12], Theorem 8). Using the condition bifunction ψ is bounded below, we obtain the following finding.

Corollary 1.3.2. Assume that $S: \mathcal{M} \to \mathcal{M}$ is a Caristi map, and that $\psi: \mathcal{M}^2 \to \mathbb{R}^+$ is S-l.s.c. map on a complete $\boldsymbol{G_p}$- metric space. If $\boldsymbol{\psi}$ is bounded below, then there is $r \in \mathcal{M}$ given:

$\psi(r, Sr) < \psi(\varkappa, S\varkappa) + G_p(r, \varkappa, S\varkappa)$ for every $\varkappa \in \mathcal{M}$ with $\varkappa \neq r$.

Proof. It is sufficient to demonstrate that the point r, as determined by Theorem 1.3.1. satisfies the statement of Theorem 3.3.1. It is necessary to demonstrate that $\boldsymbol{\varkappa} \notin D(r)$ for $\varkappa \neq r$, using the same technique as in the proof of Theorem 1.3.1. Suppose the opposite, which means that we have $t \in D(r)$, for some $t \neq r$.

Then,

$0 < G_p(r,t,St) \leq \psi(r,Sr) - \psi(t,St)$ implies,

$\psi(t,St) < \psi(r,Sr) = L$. By triangular inequality,

$$G_p(\varkappa_n,t,St) \leq G_p(\varkappa_n,r,r) + G_p(r,t,St) - G_p(r,r,r)$$

$$\leq G_p(\varkappa_n,r,Sr) + G_p(r,t,St)$$

$$\leq \psi(\varkappa_n,S\varkappa_n) - \psi(r,Sr) + \psi(r,Sr) - \psi(t,t)$$

$\leq \psi(\varkappa_n,S\varkappa_n) - \psi(t,St)$.

That means for each $n \in \mathbb{N}$, $t \in D(\varkappa_n)$ and thus $\alpha(\varkappa_n) < \psi(t,St)$. One can easily obtain $L \leq \psi(t,St)$ by taking the limit when n goes to infinity, which is inconsistent with $\psi(t,St) < \psi(r,Sr) = L$. Thus, for each $\varkappa \in \mathcal{M}$ with $\varkappa \neq r$ implies $\varkappa \notin D(r)$ that is, $\varkappa \neq r$ implies,

$G_p(r,\varkappa,S\varkappa) > \psi(r,Sr) - \psi(\varkappa,S\varkappa)$. □

The next corollary is a direct collection of Theorem 1.3.1. and Corollary 1.3.2.

Corollary 1.3.3. Assume that $(\mathcal{M},G_p)$, (Y,G_p) be two complete G_p- metric spaces, $S:\mathcal{M} \to \mathcal{M}$ be a self-mapping, and that S is a Caristi map on $(\mathcal{M},G_p)$. Let $K:\mathcal{M} \to Y$ be a closed function, $\psi:\mathcal{M}^2 \to \mathbb{R}^+$is S -l.s.c., and $i > 0$ is a constant, given that:

$$Max\left\{G_p(\varkappa,r,Sr), iG_p(K\varkappa,Kr,KSr)\right\} \leq \psi(K\varkappa,KS\varkappa) - \psi(Kr,KSr),$$

for every $\varkappa \in \mathcal{M}$. (3.13)

Therefore, S has a fixed point.

Proof. Let $D(\varkappa)$ and $\alpha(\varkappa)$ symbolize the following two sets for $\varkappa \in \mathcal{M}$:

$$D(\varkappa) = \{\partial \in \mathcal{M}: max\{G_p(\varkappa,\partial,S\partial), iG_p(K\varkappa,K\partial,KS\partial) \leq \psi(K\varkappa,KS\varkappa) -$$

$\psi(K\partial,KS\partial)\}$ (3.14)

And,

$\alpha(\varkappa) = inf\ \{\psi(K\partial,KS\partial): y \in D(\varkappa)\}$ (3.15)

Since $\varkappa \in D(\varkappa)$, then $D(\varkappa) \neq \emptyset$. $\partial \in D(\varkappa)$ implies that

$$0 \leq \alpha(\varkappa) \leq \psi(K\partial,KS\partial) \leq \psi(K\varkappa,KS\varkappa).$$

The sequence $\{\varkappa_n\}_{n\geq 1}$can be founded in the following method:

$$\varkappa_1 := \varkappa,$$

For any $n \in \mathbb{N}$, $\varkappa_{n+1} \in D(\varkappa_n)$ such that $\psi(K\varkappa_{n+1}, KS\varkappa_{n+1}) \leq \alpha(\varkappa_n) + \frac{1}{n}$.

As per in Theorem 1.3.1. It is simple to understand that $\{\varkappa_n\}_{n\geq1}$ is G_p-convergent sequence to $r_0 \in \mathcal{M}$.

Similarly, $\{K\varkappa_n\}_{n\geq1}$ is Cauchy sequence in Y and converges to some z. $\{Kr_0, KSr_0\} = \{z, Sz\}$ due to K is closed mapping. Consequently, just like in the proof of Theorem 3.3.1, we get:

$$\psi(z, Sz) = \psi\{Kr_0, KSr_0\} = L = \lim_{n\to\infty} \alpha(\varkappa_n).$$

Via Corollary 1.3.2, we obtain that $\varkappa \neq r_0$ that led to $\varkappa \notin D(r_0)$.

From (3.13), $Sr_0 \in D(r_0)$, we have

$Sr_0 = r_0$. □

Example 1.3.4. In the case where $X = \mathbb{R}^+$ and $G_p(\varkappa, \partial, z) = max\{\varkappa, \partial, z\}$, then (X, G_p) is a G_p- metric space [22]. Supposing $\mathcal{H}$ is a family of mapping $h: \mathbb{R}^+ \to \mathbb{R}^+$ such that $h(\varkappa) = \frac{\varkappa}{5}$, and let $S: \mathbb{R}^+ \to \mathbb{R}^+$ such that $S\varkappa = \frac{\varkappa}{3}$ for every $\varkappa \in X$, and $\psi(a, b): \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ such that $\psi(a, b) = 3(2a + b)$. Then,

$G_p(\varkappa, h(\varkappa), Sh(\varkappa)) = max\left\{\varkappa, \frac{\varkappa}{5}, \frac{\varkappa}{5}\right\} = \varkappa$ and $\psi(\varkappa, S\varkappa) - \psi(h(\varkappa), Sh(\varkappa)) = \frac{26\varkappa}{5}$.

If $h(\varkappa) = r$, where r is a common fixed point of $\mathcal{H}$, then

$$G_p(\varkappa, r, Sr) = \max\{\varkappa, \frac{\varkappa}{5}, \frac{\varkappa}{5}\}$$

$= \varkappa$.

And

$$\psi(\varkappa, S\varkappa) - u = \psi(\varkappa, S\varkappa) - inf\ \{\psi(\varkappa, S\varkappa),\ \varkappa \in X\}$$

$$= \psi\left(\varkappa, \frac{\varkappa}{3}\right) - inf\ \psi\left(\varkappa, \frac{\varkappa}{3}\right)$$

$$= 7\varkappa - \frac{\varkappa}{3}$$

$= \frac{20x}{3}$.

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