THE PROBLEM OF BEST ONE SIDED APPROXIMATION IN WEIGHTED $L_{p,\beta}(X)$

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Received 22/7/2023, Accepted 16/8/2023, Published 21/9/2023

Abstract

The aim objective of this article, we introduced the problem of one-sided approximation of unbounded functions in weighted space $L_{p,\beta}(X)$ by using some linear operators in terms the average modulus of smoothness. In addition we established the relation between $K$-functional and average modulus of smoothness. Also, we show that unbounded functions in weighted space $L_{p,\beta}(X)$ to approximate by algebraic and trigonometric polynomials.

Keywords: unbounded functions, algebraic (trigonometric) polynomial, average modulus of smoothness and weighted space.

1. Introduction

Hans [1] in [1982] presented described the finite-dimensional subspaces $G$ of the space of continuous or differentiable functions which have a unique best one-sided $L_1$-approximation. Thus, Gardiner, Rogge and Armitage et al. [2]1998 studied best one-sided $L^1$ approximation by harmonic functions. Moreover, Dryanov and Petrov in 2002 studied the problem of best one-sided $L^1$-approximation by blending functions of order (2,2) [3]. So, Motornaya, Motornyi and Nitiema in 2010 found an accurate estimate of the best one-sided approximation of a step by algebraic polynomials in the space $L_1$ [4]. Thus, Al-Saidy and Husain 2011 found the Degree of best approximation of unbounded functions by Bernstein operators [5]. Also, Jorge, Jose and Reinaldo

2. Preliminaries

We shall consider unbounded functions defined on $X$, where $X = [0,1]$, or $X = [0,2\pi)$ for $2\pi$ periodic functions. We consider $R$ as a normed vector space with elements $x, y, z$, $x = (x_1, x_2, ..., x_n)$, and norm $|x| = \{|x_k|; k = 1,2, ..., n\}$.

$\alpha, \beta, \epsilon$ are multyindices. $|\alpha| = \alpha_1 + \alpha_2 + ... + \alpha_n$ is the length of $\alpha$, where $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$. $\alpha \leq \beta$ means $\alpha_k \leq \beta_k$ for every $k$ and $(\frac{\alpha}{\beta}) = \prod_{k=1}^{n} \left(\frac{\alpha_k}{\beta_k}\right)$ Where $\beta = (\beta_1, \beta_2, ..., \beta_n)$. By $D^\alpha$ denotes a differential function in $R$.

Let $H_k$ the set of all algebraic and trigonometric polynomials with degree not greater than $k$. $A_k$ denotes the set of all operators of exponential type $k$.

Let $L_{p, \beta}(X), 1 \leq p < \infty$, the space of all unbounded functions that defined on $X$, with any function in this space has the norm given by

$$\|\xi\|_{p, \beta} = \left(\int_X |\xi(x)|^p \beta(x)\right)^{\frac{1}{p}} < \infty.$$ (1)

$\tilde{E}_k(\xi, x)_{p, \beta}$ be the degree of best one-sided approximation in $L_{p, \beta}(X), 1 \leq p < \infty$, of unbounded function as $\xi$ by operators $m_k$ & $n_k$ given

$$\tilde{E}_k(\xi, x)_{p, \beta} = \inf\{\|m_k - n_k\|_{p, \beta}\}$$ (2)

such that $n_k, m_k \in H_k$, $n_k \leq \xi \leq m_k$.

Let $\Delta^k_{\delta} \xi(x)$ denote the $k^{th}$ finite difference with step $\delta$ of $\xi$ in the point $x$. We denoted by
\[ \omega_k(\xi, x, \delta) = \sup \{|\Delta^k_\delta \xi(x)| : x, x + k\delta \in X\} \]  
(3)

The local modulus of \( \xi \). The Modulus of smoothness is given by

\[ \omega_k(\xi, \delta)_{p, \beta} = \sup \|\Delta^k_\delta \xi(x)\|_{p, \beta} \text{ such that } \Delta^k_\delta(\xi, x) = \sum_{i=0}^{k} (-1)^{i+k}\left(\frac{k}{i}\right)\xi(x + i\delta). \]

And the average modulus of smoothness we denote by

\[ \tau_k(\xi, \delta)_{p, \beta} = \|\omega_k(\xi, .., \delta)\|_{p, \beta} \]  
(4)

where \( \|\xi - n_k\|_{p, \beta} \leq c\tau_k(\xi, \delta)_{p, \beta} \).

Let \( k, m \) and \( n \) are determined numbers and \( c \) is a positive constant that may depend only on \( k, m \) and \( n \). the numbers \( d = \left[\frac{-m}{n}\right]+1 \) and \( l = \max\{k, d\} \) are also determined.

We show the one-sided \( K \)-functional as quantity

\[ \tilde{K}_k(\xi, t^k)_{p, \beta} = \inf \left\{ \|m_k - n_k\|_{p, \beta} + t|\alpha| \|D^\alpha m_k\|_{p, \beta} + \|D^\alpha n_k\|_{p, \beta} \right\} \]  
(5)

Where \( n_k, m_k \in L_{p, \beta}(X) \) such that \( n_k \leq \xi \leq m_k \).

From (5) we have only the sum for \( |\alpha| = k \) when \( k > \frac{m}{n} \) and the sum for \( |\alpha| = k \) and \( |\alpha| = l \) when \( k \leq \frac{m}{n} \). The last part is of importance for the multivariate case.

3. Auxiliary Lemmas

In this part, we find an one-sided approximation of unbounded functions by trigonometric polynomials.

We consider the \( 2\pi \) periodic state, i.e. \( x = [0,2\pi) \).

Let \( F_k(u) = \frac{\sin^2 \frac{\pi}{2k} \sin \frac{2\pi u}{2}}{\sin \frac{\pi u}{2}} \in T_{k-1} \) be Fejer kernel (normalized in an appropriate way).

We shall use the following properties of \( F_k \)

\[ F_k(u) > 0, \text{ for every } x \in R, \]  
(6)

\[ F_k(u) \geq 0, \text{ for every } |u| \leq \frac{\pi}{k}, \]  
(7)

\[ \int_{-\pi}^{\pi} F_k(u) du = 2\pi \sin^2 \frac{\pi}{2k} \leq \frac{c}{k^2} \]  
(8)

\[ \sum_{i=1}^{k-1} F_k \left( u - \frac{2\pi i}{k} \right) = (k \sin^2 \frac{\pi}{2k})^2 \leq \frac{c}{k^2} \]  
(9)

For \( x = (x_1, x_2, ..., x_k) \in R \) put
\( \Psi_k(x) = \prod_{i=1}^{k} F_k(x_i) \in T_{k-1}. \) \hspace{1cm} (10)

We set

\( G(x) = \{0, 1, 2, ..., n - 1\}. \)

**Lemma 3.1:** Let \( \{\alpha_k\}_{k \in G(x)} \), be a sequence and \( 0 \leq \alpha_k < \infty. \) Then

\[ \|\alpha_k \Psi_k(x - \frac{2\pi}{k})\|_{p, \beta} \leq c \left( \frac{2\pi}{k\alpha_k^p} \right)^\frac{1}{p}. \]

**Proof:** From (9), (10) and Jensen inequality, since

\[ |\alpha_k \Psi_k(x - \frac{2\pi}{k})|^p \leq \left( k\sin\frac{\pi}{2k} \right)^{p-1} |\alpha_k|^p \Psi_k(x - \frac{2\pi}{k}), \]

which together with (8) and (10) proves the lemma.

We shall also use Jackson kernels

\[ J_{l,k}(u) = \gamma_{l,k}((\sin \frac{ku}{2\sin^{2} (\frac{\pi}{2(l+1)})}) u \in T_{(l+1)(k-1)}, \]

where \( \gamma_{l,k} \) is chosen so that \( \int_{-\pi}^{\pi} J_{l,k}(u)du = 1. \) The following property of Jackson kernels is well known (see e.g. [12, p. 193])

\[ \int_{-\pi}^{\pi} |J_{l,k}(u)|n du = ck^{-n} \text{ for } n = 0, 1, ..., 2l. \] \hspace{1cm} (11)

For \( x \in R \) put

\[ L_{l,k}(x) = J_{l,k}(x_1)J_{l,k}(x_2) ... J_{l,k}(x_n) \in T_{(l+1)(k-1)}. \]

Consider the function

\[ Q_k(\zeta, x) = \int_{X} \sum_{n=1}^{k} (-1)^{n+1} c \zeta(x + ns)L_{l,k}(s)ds. \]

\( Q_k(\zeta) \in T_{(l+1)(k-1)} \) and \( Q_k(\zeta) \) is a polynomial which understand the order of single-directional approximation of \( \zeta \)

\[ \|\zeta - Q_k(\zeta)\|_{p, \beta} = \left( \int_{X} |\zeta(x) - Q_k(\zeta)|^p \beta(x)dx \right)^\frac{1}{p} \]

\[ = \left( \int_{X} |\zeta(x) - l_k(\zeta) + l_k(\zeta) - Q_k(\zeta)|^p \beta(x)dx \right)^\frac{1}{p} \leq \left( \int_{X} |\zeta(x) - l_k(\zeta)|^p \beta(x)dx \right)^\frac{1}{p} + \left( \int_{X} |l_k(\zeta) - Q_k(\zeta)|^p \beta(x)dx \right)^\frac{1}{p} \]

and then,
\[ \| \zeta - Q_k(\zeta) \|_{p,\beta} \leq \| \zeta - l_k(\zeta) \|_{p,\beta} + \| l_k(\zeta) - Q_k(\zeta) \|_{p,\beta} \]
\[ \leq \frac{2}{(1-h)^{\beta}} \tau_k(\zeta, h)_{p,\beta} + \varphi_k \| l_k(\zeta) \|_{p,\beta} \]
\[ \leq \left( \frac{2}{(1-h)^{\beta}} + \frac{3\varphi_k}{h} \right) \tau_k(\zeta, k^{-1})_{p,\beta} \]
\[ \leq \left( \frac{2}{1-h} + \frac{3\varphi_k}{h} \right) \tau_k(\zeta, k^{-1})_{p,\beta}, \text{ where } c = \frac{2}{1-h} + \frac{3\varphi_k}{h} \]
\[ = c \omega_k(\zeta, k^{-1})_{p,\beta}. \quad (14) \]

For the validity of (14) it is enough to choose in (13) any \( l \geq k \), but for our next purposes we need \( l = \{ k, d \} \).

Using \( Q_k \) we construct our one sided operators as follows:
\[ \{ P_k(\zeta, x) = Q_k(\zeta, x) - \Psi_k(x - \frac{2\pi}{k}) \sup_{\zeta} \sup_{\zeta} \{ | \zeta(s) - Q_k(\zeta, s) | \}, \]
\[ \Psi_k(x - \frac{2\pi}{k}) \sup_{\zeta} \sup_{\zeta} \{ | \zeta(s) - Q_k(\zeta, s) | \}. \quad (15) \]

4. Main Results

In this section we show that the one-sided \( K \)-functional (5) and the average of modulus are equivalent.

**Theorem 4.1:** Let \( \zeta \in L_{p,\beta}(X), 1 \leq p < \infty, k \in N, X = [0,1] \). Then there is positive constant \( c \) such that
\[ c \tau_k(\zeta, t)_{p,\beta} \leq \tilde{R}_k(\zeta, t^k) \leq c \tau_k(\zeta, t)_{p,\beta}. \quad (16) \]

**Proof:**

We see from (5) that is enough to construct two functions \( n_k, m_k \) from \( L_{p,\beta}(X) \) satisfying the conditions:

i) \[ n_k(\zeta) \leq \zeta(x) \leq m_k(\zeta), \text{ for any } x \in X = [0,1], \]
ii) \[ \| m_k(\zeta) - n_k(\zeta) \|_{p,\beta} \leq c \tau_k(\zeta, x)_{p,\beta}, \]
iii) \[ \{ ||D^\beta m_k||_{p,\beta} \leq c t^{-|\beta|} \tau_k(\zeta, x)_{p,\beta}, \quad ||D^\beta n_k||_{p,\beta} \leq c t^{-|\beta|} \tau_k(\zeta, x)_{p,\beta}, \text{ for any } k \leq |\beta| \leq l. \]

Let \( t \) be a positive constant. Put \( \delta = \frac{t}{4} \) when \( x = R \), \( n = \left[ \frac{4}{t} \right] + 1 \), \( \delta = \frac{1}{n} \) when \( x = [0,1] \) and \( n = \left[ \frac{8\pi}{t} \right] + 1 \), \( \delta = \frac{2\pi}{n} \) when \( x = [0,2\pi] \). Denote
\[ G(x) = \{ G = \{ 0, \pm 1, \pm 2, \ldots \}, x = R, \quad \{ 0,1,2, \ldots, n \}, x = \]
\[ [0,1], \quad \{ 0,1,2, \ldots, n-1 \}, x = [0,2\pi]. \]
For every $k = (k_1, k_2, ..., k_n) \in G$ we consider the following cubs in $R$.

$$X_k = \{ x \in R : |x - k\delta| \leq \delta \},$$  \hspace{1cm} (17)

$$X_k' = \{ x \in X_k : x_i \geq \delta k_i \}.$$  

Let $U$ be a positive constant operator such that $U(x) = 0$ for $x \leq 0$, $U(x) = 1$, for $1 \leq x < \infty$, $0 < U(x) < 1$ for $0 < x < 1$. For every $X_k$ put

$$U_k(x) = \prod_{i=1}^{k} U \left( \frac{x_i}{\delta - k_{i+1}} \right) \left( 1 - U \left( \frac{x_i}{\delta - k_i} \right) \right).$$

We can find some properties of operators, which defined as:

$$0 \leq U_k(x) \leq 1 \text{ for every } x \in R, \quad U_k(k\delta) = 1,$$  \hspace{1cm} (18)

$$U_k(x) = 0, \quad x \in X_k,$$  \hspace{1cm} (19)

$$U_k(x) = 1, \quad x \in R.$$  \hspace{1cm} (20)

For $k \in G(x)$ put

$$E_k(\zeta, x)_{p,\beta} = \inf \{ ||\zeta - \eta||_{p,\beta} : \eta \in H_{k-1}, \} \quad (21)$$

where $\eta_k \in H_{k-1}$.

Remark: For the case $X = [0,1]$ and $k$ such that $k\delta$ is on the boundary of $X$, we instead $X_k$ by $X_k \cap X$ in (21).

From the Whitney’s theorem (see e.g. [6]) since

$$\tilde{E}_k(\zeta, x)_{p,\beta} \leq c \omega_k(\zeta, x)_{p,\beta}, \text{ for every } x \in X_k.$$  \hspace{1cm} (22)

Put $\Psi_k(x) = \eta_k(x) - \tilde{E}_k(\zeta, x)$ and

$$\Phi_k(x) = \eta_k(x) + \tilde{E}_k(\zeta, x). \quad \text{For every } x \in X_k = [0,1] \text{ since}$$

$$\Psi_k(x) \leq \xi(x) \leq \Phi_k(x).$$ \hspace{1cm} (23)

To finish we known

$$\{ m_k(\zeta) = U_k(x) \Psi_k(x) \in L_{p,\beta}(X), n_k(\zeta) = U_k(x) \Phi_k(x) \in L_{p,\beta}(X).$$  \hspace{1cm} (24)

Now i) follows from equations (24), (23) and (17) – (21). From (24) we obtain

$$m_k(\zeta) - n_k(\zeta) = 2 U_k(x) \tilde{E}_k(\zeta, x) \text{ and so,}$$

$$0 \leq m_k(\zeta) - n_k(\zeta) \leq c \omega_k(\zeta, x) \text{ by (22), (18), and (20).}$$
Now ii) from the above inequality. Take \( \beta, k \leq |\beta| \leq l \).

Let \( x \in X_k' \). Then from equations (19), (20) and (24) we obtain

\[
n_k(\xi) = \Phi_k(x) + U_{k+\varepsilon}(x)( \Phi_{k+\varepsilon}(x) - \Phi_k(x))
\]

and so,

\[
D^\beta n_k(x) = D^\beta \Phi_k(x) + c D^{\beta-\alpha} U_{k+\varepsilon}(x) D^\alpha( \Phi_{k+\varepsilon}(x) - \Phi_k(x)).
\]

Having in mind the definitions of \( \eta_k \) and \( U_k \), (22) and Markov's inequality we obtain

\[
\| D^\beta n_k \|_{p,\beta} \leq c \| D^{\beta-\alpha} U_{k+\varepsilon} \|_{p,\beta} \| D^\alpha( \Phi_{k+\varepsilon} - \Phi_k) \|_{p,\beta}
\]

\[
\leq c t^{-|\beta-\alpha|} x^{-|\alpha|} \| (\Phi_{k+\varepsilon} - \Phi_k) \|_{p,\beta}
\]

\[
\leq c t^{-|\beta|} \{ \| (\Phi_{k+\varepsilon} - \xi) \|_{p,\beta} + \| \Phi_k - \xi \|_{p,\beta} \}
\]

\[
\leq c t^{-|\beta|} \| \omega_k(\xi, x) \|_{p,\beta}.
\]

Summating on \( k \in G(x) \) the above inequality we obtain iii). This completes the proof of the second inequality in (16).

Let \( X = [0,1], k \in L_{p,\beta}(X), D^\alpha k \in L_{p,\beta}(X) \) (generalized derivatives) for every \( \alpha, |\alpha| = l \)
(recall \( l > \frac{m}{n} \)). The \( k \) is equivalent to \( R \in L_{p,\beta}(X) \) and

\[
\| R \|_{p,\beta} \leq c \{ \| k \|_{p,\beta} + \| D^\alpha k \|_{p,\beta} \}.
\]  

(25)

Making a linear change of the variables in (25) we obtain

for every \( x \in X \) and \( \delta \) positive constant

\[
\| R \|_{p,\beta} \leq c \| k \|_{p,\beta} + \delta^l \| D^\alpha k \|_{p,\beta}.
\]  

(26)

Let \( m_k, n_k \in L_{p,\beta}(X), m_k \leq \zeta \leq n_k \). Since

\[
\tau_k(\xi, x) \leq \tau_k(\zeta - m_k, x) + \tau_k(m_k, x).
\]  

(27)

From (26) and \( R = k = n_k - m_k \) since

\[
\omega_k(\xi - m_k, x) \leq 2k \| \xi - m_k \|_{p,\beta} \leq 2k \| n_k - m_k \|_{p,\beta}
\]

\[
\leq c t^{-\frac{m}{n}} \| n_k - m_k \|_{p,\beta}.
\]  

(28)

Noticing that
\[ t^{-\frac{m}{n}}\|R\|_{p,\beta} \leq k\frac{m}{n}\|R\|_{p,\beta} \]  

(29)

for every \( R \in L_{p,\beta}(X) \), from equation (28) we obtain

\[ \tau_k(\xi - m_k, x)_{p,\beta} \leq c \left\{ \|m_k - m_k\|_{p,\beta} + t^{|\alpha|} \left( \|D^\alpha n_k\|_{p,\beta} + \|D^\alpha m_k\|_{p,\beta} \right) \right\}. \]  

(30)

For estimating the second term in the right hand side of (27) we consider two cases.

i) \( \frac{k}{n} \geq \frac{m}{n} \), \( i.e. k \geq d \). From Theorem 1 in [5] we obtain

\[ \tau_k(m_k, x)_{p,\beta} \leq c t^\frac{m}{n} \int_0^t \omega_k(m_k, s)_{p,\beta} s^{-\frac{m}{n}} ds \]  

(31)

\[ \leq c t^\frac{m}{n} \int_0^t \|D^\alpha m_k\|_{p,\beta} s^{-\frac{m}{n}} ds \]  

\[ = c t^k \|D^\alpha m_k\|_{p,\beta}. \]

ii) \( \frac{k}{n} < \frac{m}{n} \), \( i.e. k < d \). From a generalization Whitney theorem (see [6]) there is \( \eta_k \in H_{k-1} \)

\[ \|m_k - \eta_k\|_{p,\beta} \leq c \omega_k(m_k, x)_{p,\beta}. \]

From this inequality and (26) we obtain

\[ \omega_k(m_k, x) = \omega_k(m_k - \eta_k, x) \leq 2^k \|m_k - \eta_k\|_{p,\beta} \]

\[ \leq c t^{-\frac{m}{n}} \left\{ \|m_k - \eta_k\|_{p,\beta} + t^d \|D^\alpha(m_k - \eta_k)\|_{p,\beta} \right\} \leq c t^{-\frac{m}{n}} \left\{ \omega_k(m_k, x)_{p,\beta} + t^d \|D^\alpha m_k\|_{p,\beta} \right\}. \]

Taking \( L_{p,\beta}(X) \) norm with respect to \( x \) in the above inequality and from (29) we obtain

\[ \tau_k(m_k, x)_{p,\beta} \leq c \left\{ \omega_k(m_k, x)_{p,\beta} + t^d \|D^\alpha m_k\|_{p,\beta} \right\} \]

(32)

From equations (27), (30), (31), and (32) we get

\[ \tau_k(\xi, x)_{p,\beta} \leq c \left\{ \|n_k - m_k\|_{p,\beta} + t^{|\alpha|} \left( \|D^\alpha n_k\|_{p,\beta} + \|D^\alpha m_k\|_{p,\beta} \right) \right\}. \]

Taking infimum on \( m_k, n_k \in L_{p,\beta}(X) \), \( m_k \leq \xi \leq n_k \), in the above inequality we complete the proof of (16).

**Theorem 4.2:** Let \( \xi \) an unbounded function in weighted space \( L_{p,\beta}(X) \). Then the following statements are holds

\[ P_k(\xi), q_k(\xi) \in T_{(i+1)(k-1)} \]  

(33)

\[ P_k(\xi, x) \leq q_k(\xi, x) \]  

(34)

\[ \|q_k(\xi) - P_k(\xi)\|_{p,\beta} \leq c\tau_k(\xi, x)_{p,\beta}. \]  

(35)

**Proof:** From (15), (10) and (13) we obtain (33). From (7) and (10) since

\[ \Psi_k(x) \geq 1 \] for \( |x| \leq \frac{\pi}{k} \), which together with the positivity of \( \Psi_k \) gives (34).
From (13) since
\[ \sup |\zeta(s) - Q_k(\zeta, s)| \leq \sup | \int_X \Delta_0^k \zeta(s) L_{t,k}(y) dy | \]
\[ \leq \int_X \sup |\Delta_0^k \zeta(s) L_{t,k}(y) dy | \]
\[ \leq \int_X \omega_k(\zeta, \frac{2\pi}{k}) L_{t,k}(y) dy. \]

from Lemma 3.1 and \( \alpha_k = \int_X \omega_k(\zeta, \frac{2\pi}{k}) L_{t,k}(y) dy, \)

Minkovski's inequality, (12) and (11) we obtain
\[ \|q_k(\zeta) - P_k(\zeta)\|_{p\beta} \leq 2 \|\Psi_k(\cdot, -\frac{2\pi}{k}) \sup \sup |\zeta(s) - Q_k(\zeta, s)| \|_{p\beta} \]
\[ \leq 2 \|\Psi_k(\cdot, -\frac{2\pi}{k})\|_{p\beta} \]
\[ \leq c \left( (\frac{2\pi}{k}) \alpha_k^p \right)^{ \frac{1}{p} } \]
\[ \leq c \left( \int_X ( \int_X (\omega_k(\zeta, x) L_{t,k}(y) dy) dx \right)^{ \frac{1}{p} } \]
\[ \leq c \left( \int_X ( \int_X (\omega_k(\zeta, x) dx)^{ \frac{1}{p} } L_{t,k}(y) dy \right) \]
\[ = c \int_X \tau_k(\zeta, x)_{p\beta} L_{t,k}(y) dy \]
\[ \leq c \int_X L_{t,k}(y) dy \tau_k(\zeta, \frac{1}{k})_{p\beta} \]
\[ = c \tau_k(\zeta, \frac{1}{k})_{p\beta}. \]

**Theorem 4.3:** Let \( \zeta \) be an unbounded function in weighted space \( L_{p\beta}(X), 1 \leq p < \infty \) and \( n \) natural number. Then
\[ \bar{E}_k(\Phi_n, \zeta)_{p\beta} \leq c \tau_k(\zeta, \frac{1}{n})_{p\beta}. \]

**Proof:** using Theorem 4.2 with \( k = \left[ \frac{n}{l+1} \right] + 1 \), we obtain
\[ \bar{E}_k(\Phi_n, \zeta)_{p\beta} \leq \|q_k(\zeta) - P_k(\zeta)\|_{p\beta} \leq c \tau_k(\zeta, \frac{1}{k})_{p\beta} \leq c \tau_k(\zeta, \frac{l+1}{n})_{p\beta} \]
\[ \leq c \tau_k(\zeta, \frac{1}{n})_{p\beta}. \]
\[ \tau_k(\zeta, \frac{1}{n})_{p\beta} \leq \frac{c}{n^k} \sum_{z=0}^{n} (z + 1)^{k-1} \bar{E}_k(\Phi_z, \zeta)_{p\beta}. \] (36)
5. **one-sided approximation by entire operators of exponential type.**

In this section we get for entire operators parallel to those from section 3.

Let $x \in R$, $G(x) = G$. Put

$$F_k(u) = \left(\frac{\pi}{k^2(sin(\frac{k}{u})^{\infty}+A_k}\right)$$

$$J_{l,k}(u) = \gamma_{l,k}(\left(sin(\frac{k}{2}u)^{2l+2}\right) \in A_k(l+1).$$

Now, $L_{l,k}$, $Q_k$, $P_k$ and $q_k$ are given by (22), (24), (25) and (27) respectively ($n$ replaced by $k$).

**Theorem 4.4:** Let $\zeta \in L_{p,\beta}(X)$, $\zeta$ be an unbounded and $\tau_k(\zeta, k) < \infty$. Then

$$P_k(\zeta), q_k(\zeta) \in A_k(l+1),$$

$$P_k(\zeta, x) \leq \zeta(x) \leq q_k(\zeta, x) \text{ for every } x \in R$$

$$\|q_k(\zeta) - P_k(\zeta)\|_{p,\beta} \leq c \tau_k(\zeta, k^{-1})_{p,\beta}.$$

The proof follows along the lines of the proof of Theorem 4.2.

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