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THE PROBLEM OF BEST ONE SIDED APPROXIMATION IN WEIGHTED $L_{p,\beta}(X)$

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Abstract

The aim objective of this article, we introduced the problem of one-sided approximation of unbounded functions in weighted space $L_{p,\beta}(X)$ by using some liner operators in terms the average modulus of smoothness. In addition we established the relation between K-functional and average modulus of smoothness. Also, we show that unbounded functions in weighted space $L_{p,\beta}(X)$ to approximate by algebraic and trigonometric polynomials.

Keywords: unbounded functions, algebraic (trigonometric) polynomial, average modulus of smoothness and weighted space.

1. Introduction

Hans [1] in [1982] presented described the finite-dimensional subspaces G of the space of continuous or differentiable functions which have a unique best one-sided L_1-approximation. Thus, Gardiner, Rogge and Armitage *et al.* [2]1998 studied best one-sided L^1 approximation by harmonic functions. Moreover, Dryanov and Petrov in 2002 studied the problem of best one-sided L^1-approximation by blending functions of order (2,2) [3]. So, Motornaya, Motornyi and Nitiema in 2010 found an accurate estimate of the best one-sided approximation of a step by algebraic polynomials in the space L_1 [4]. Thus, Al-Saidy and Husain 2011 found the Degree of best approximation of unbounded functions by bernstein operators [5]. Also, Jorge, Jose and Reinaldo

[6] 2012 explained the polynomial operators for one-sided approximation to functions in $W_p^r[0,1]$ by algebraic polynomials. Thus, Babenko *et al.* [7] 2013 resented the one-sided approximation in L of the characteristic function of an interval by trigonometric polynomials. Alaa and Jassim in [2014] [8] obtained the order of convergence of the weighted area by polynomial interpolation on [-π, π]. Viswanathan and Navascues [9] 2016 presented associate fractal functions in L^p-spaces and in one-sided uniform approximation. Thus, Torgashova [10] 2017 is achieved solution to the problem of one-sided approximation in L(−1,1) to the characteristic function of the interval (−√3/5, 2/5) by fifth-degree algebraic polynomials. So, Deikalova and Torgashova [11] 2020 found the problems of the best one-sided approximation (from below and from above) in the space Lυ(−1, 1) to the characteristic function of an interval (a, b), $-1 < a < b < 1$, by the set of algebraic polynomials of degree not exceeding a given number. Also, Ioannis *et al.* [2020] Hybrid Block Successive Approximation for One-Sided Non-Convex Min-Max Problems Algorithms and Applications [12] and in [2021] Al-Jawari *et al.* studied best one-sided multiplier approximation of unbounded functions by algebraic Polynomials operators in space $L_{p,\varphi n}(X)$ by terms averaged modulus[13]. Furthermore, Fedunyk and Hembars'ka [14]2022 found best orthogonal trigonometric approximations of the Nikol'skii-Besov-type classes of periodic functions of one and several variables.

2. Preliminaries

We shall consider unbounded functions defined on X, where $X = [0,1]$, or $X = [0,2\pi)$ for 2π periodic functions. We consider R as a normed vector space with elements x, y, z. $x = (x_1, x_2, \dots, x_n)$, and norm $|x| = \{|x_k|: k = 1,2,\dots,n\}$.

α, β, ϵ are multyindices. $|\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_n$ is the length of α, where $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$. $\alpha \le \beta$ means $\alpha_k \le \beta_k$ for every k and $\binom{\alpha}{\beta} = \prod_{k=1}^{n} \binom{\alpha_k}{\beta_k}$ Where $\beta = (\beta_1, \beta_2, \dots, \beta_n)$. By D^α denotes a differential function in R.

Let H_k the set of all algebraic and trigonometric polynomials with degree not greater than k. A_k denotes the set of all operators of exponential type k.

Let $L_{p,\beta}(X), 1 \le p < \infty$, the space of all unbounded functions that defined on X, with any function in this space has the norm given by

$$\|\xi\|_{p,\beta} = \left(\int_X |\xi(x)|^p \beta(x)\right)^{\frac{1}{p}} < \infty. \quad (1)$$

$\tilde{E}_k(\xi, x)_{p,\beta}$ be the degree of best one-sided approximation in $L_{p,\beta}(X), 1 \le p < \infty$, of unbounded function as ξ by operators m_k& n_k given

$$\tilde{E}_k(\xi, x)_{p,\beta} = inf\{\| m_k - n_k\|_{p,\beta}\} \quad (2)$$

such that $n_k,\ m_k \in H_k,\ n_k \le \xi \le m_k$.

Let $\Delta_\delta^k\, \xi(x)$ denote the k[th] finite difference with step k of ξ in the point x. We denoted by

$$\omega_k(\xi, x, \delta) = sup\{\left|\Delta_\delta^k\, \xi(x)\right|: x, x + k\delta \in X\} \quad (3)$$

The local modulus of ξ. The Modulus of smoothness is given by

$\omega_k(\xi, \delta)\,_{p,\beta} = sup\|\Delta_\delta^k\, \xi(x)\|_{p,\beta}$ such that

$\Delta_\delta^k(\xi, x) = \sum_{i=0}^{k} \quad (-1)^{i+k}\binom{k}{i}\xi(x + i\delta).$

And the average modulus of smoothness we denote by

$$\tau_k(\xi, \delta)\,_{p,\beta} = \|\omega_k(\xi, ., \delta)\|_{p,\beta} \quad (4)$$

where $\|\xi - n_k\|_{p,\beta} \leq c\tau_k(\xi, \delta)\,_{p,\beta}$.

Let k, m and n are determined numbers and c is a positive constant that may depend only on k, m and n. the numbers $d = [\frac{m}{n}] + 1$ and $l = max\{k, d\}$ are also determined.

We show the one-sided K-functional as quantity

$$\widetilde{K}_k(\xi, t^k)_{p,\beta} = inf\{\|\, m_k - n_k\|_{p,\beta} + t^{|\alpha|}\, \|\, D^\alpha\, m_k\|_{p,\beta} + \|\, D^\alpha n_k\|_{p,\beta}\} \quad (5)$$

Where $n_k,\, m_k \in L_{p,\beta}(X)$ such that $n_k \leq \xi \leq m_k$.

From (5) we have only the sum for $|\alpha| = k$ when $k > \frac{m}{n}$ and the sum for $|\alpha| = k$ and $|\alpha| = l$ when $k \leq \frac{m}{n}$. The last part is of importance for the multivariate case.

3. Auxiliary Lemmas

In this part, we find an one-sided approximation of unbounded functions by trigonometric polynomials.

We consider the 2π periodic state, i.e. $x = [0,2\pi)$.

Let $F_k(u) = \frac{sin^2\frac{\pi}{2k\, sin^2\frac{ku}{2}}}{sin^2\frac{u}{2}} \in T_{k-1}$ be Fejer kernel (normalized in an appropriate way).

We shall use the following properties of F_k

$$F_k(u) > 0, \quad \text{for every } x \in R, \quad (6)$$

$$F_k(u) \geq 0, \quad \text{for every } |u| \leq \frac{\pi}{k,} \quad (7)$$

$$\int_{-\pi}^{\pi} \quad F_k(u)du = 2\pi k sin^2\frac{\pi}{2k} \leq \frac{c}{k}, \quad (8)$$

$$\sum_{i=1}^{k-1} \quad F_k\left(u - \frac{2i\pi}{k}\right) = (k sin^2\frac{\pi}{2k)^2 \leq c} \quad (9)$$

For $x = (x_1, x_2, \dots, x_k) \in R$ put

$$\Psi_k(x) = \prod_{i=1}^{k} \quad F_k(xi) \in T_{k-1}. \qquad (10)$$

We set

$$G(x) = \{0,1,2, \dots, n-1\}.$$

Lemma 3.1: Let $\{\alpha_k\}_{k\in G(x)}$, be a sequence and $0 \le \alpha_k < \infty$. Then

$$\|\alpha_k \Psi_k(. - \frac{2\pi}{k)}\|_{p,\beta} \le c((\frac{2\pi}{k)\alpha_k^p)^{\frac{1}{p}}}.$$

Proof **:** From (9), (10) and Jensen inequality, since

$$|\alpha_k \Psi_k(x - \frac{2\pi}{k)|^p} \le (k sin \frac{\pi}{2k)^{2(p-1)}\alpha_k^p} \Psi_k(x - \frac{2\pi}{k)},$$

which together with (8) and (10) proves the lemma.

We shall also use Jackson kernels

$$J_{l,k}(u) = \gamma_{l,k}((sin \frac{ku}{2\frac{)}{sin}} \frac{u}{2)^{2l+2}} \in T_{(l+1)(k-1),}$$

where $\gamma_{l,k}$ is chosen so that $\int_{-\pi}^{\pi} \quad J_{l,k}(u)du = 1$. The following property of Jackson kernels is well known (see e.g. [12, p. 193])

$$\int_{-\pi}^{\pi} \quad J_{l,k}(u)|u|^n du = ck^{-n} \text{ for } n = 0,1, \dots, 2l. \qquad (11)$$

For $x \in R$ put

$$L_{l,k}(x) = J_{l,k}(x_1)J_{l,k}(x_2) \dots J_{l,k}(x_n) \in T_{(l+1)(k-1).} \qquad (12)$$

Consider the function

$$Q_k(\zeta, x) = \int_X \quad \sum_{n=1}^{k} \quad (-1)^{n+1}\, c\, \zeta(x + ns)L_{l,k}(s)ds. \qquad (13)$$

$Q_k(\zeta) \in T_{(l+1)(k-1)}$ and $Q_k(\zeta)$ is a polynomial which understand the order of single-directional approximation of ζ

$$\|\zeta - Q_k(\zeta)\|_{p,\beta} = (\int_X \quad |\zeta(x) - Q_k(\zeta)|^p \beta(x)dx)^p)^{\frac{1}{p}}$$

$$= (\int_X \quad |\zeta(x) - l_k(\zeta) + l_k(\zeta) - Q_k(\zeta)|^p\, \beta(x)dx)^{\frac{1}{p}}$$

$$\le (\int_X \quad |\zeta(x) - l_k(\zeta)|^p \beta(x)dx)|^{\frac{1}{p}} + (\int_X \quad |l_k(\zeta) - Q_k(\zeta)|^p\, \beta(x)dx)^{\frac{1}{p}}$$

and then,

$$\|\zeta - Q_k(\zeta)\|_{p,\beta} \le \|\zeta - l_k(\zeta)\|_{p,\beta} + \|l_k(\zeta) - Q_k(\zeta)\|_{p,\beta}$$

$$\le \frac{2}{(1-h)^{\frac{1}{p}}}\tau_k(\zeta,h)_{p,\beta} + \varphi_k\|l_k(\zeta)\|_{p,\beta}$$

$$\le \left(\frac{2}{(1-h)^{\frac{1}{p}}} + \frac{3\varphi_k}{h}\right)\tau_k(\zeta,k^{-1})_{p,\beta}$$

$$\le \left(\frac{2}{1-h} + \frac{3\varphi_k}{h}\right)\tau_k(\zeta,k^{-1})_{p,\beta}, \text{ where } c = \frac{2}{1-h} + \frac{3\varphi_k}{h}$$

$$= c\,\omega_k(\zeta,k^{-1})_{p,\beta}. \qquad (14)$$

For the validity of (14) it is enough to choose in (13) any $l \ge k$, but for our next purposes we need $l = \{k,d\}$.

Using Q_k we construct our one sided operators as follows:

$$\{P_k(\zeta,x) = Q_k(\zeta,x) - \Psi_k(x - \frac{2\pi}{k)}\, sup\, sup\,\{|\zeta(s) - Q_k(\zeta,s)|\}\,, q_k(\zeta,x) = Q_k(\zeta,x) + \Psi_k(x - \frac{2\pi}{k)}\, sup\, sup\,\{|\zeta(s) - Q_k(\zeta,s)|\}. \qquad (15)$$

4. Main Results

In this section we show that the one-sided K-functional (5) and the average of modulus are equivalent.

Theorem 4.1: Let $\zeta \in L_{p,\beta}(X), 1 \le p < \infty,\ k \in N,\ X = [0,1]$. Then there is positive constant c such that

$$c\tau_k(\zeta,t)_{p,\beta} \le \widetilde{K}_k(\zeta,t^k) \le c\tau_k(\zeta,t)_{p,\beta.} \qquad (16)$$

Proof:

We see from (5) that is enough to construct two functions n_k, m_k from $L_{p,\beta}(X)$ satisfying the conditions :

i) $n_k(\zeta) \le \zeta(x) \le m_k(\zeta)$, for any $x \in X = [0,1]$,
ii) $\|m_k(\zeta) - n_k(\zeta)\|_{p,\beta} \le c\tau_k(\zeta,x)_{p,\beta}$,
iii) $\{\|D^\beta m_k\|_{p,\beta} \le c\,t^{-|\beta|}\,\tau_k(\zeta,x)_{p,\beta}, \qquad \|D^\beta n_k\|_{p,\beta} \le c\,t^{-|\beta|}\,\tau_k(\zeta,x)_{p,\beta},\ for\ any\ k \le |\beta| \le l.$

Let t be a positive constant. Put $\delta = \frac{t}{4}$ when $x = R$, $n = \left[\frac{4}{t}\right] + 1$, $\delta = \frac{1}{n}$ when $x = [0,1]$ and $n = \left[\frac{8\pi}{t}\right] + 1$, $\delta = \frac{2\pi}{n}$ when $x = [0,2\pi]$. Denote

$$G(x) = \{G = \{0, \pm 1, \pm 2, \dots\},\ x = R, \qquad \{0,1,2,\dots,n\}, \qquad x = [0,1], \qquad \{0,1,2,\dots,n-1\}, \qquad x = [0,2\pi].$$

For every $k = (k_1, k_2, \dots, k_n) \in G$ we consider the following cubs in R.

$$X_k = \{ x \in R : |x - k\delta| \leq \delta\}, \quad (17)$$

$$X'_k = \{ x \in X_k : x_i \geq \delta k_i\}.$$

Let U be a positive constant operator such that $U(x) = 0$ for $x \leq 0$, $U(x) = 1$, $for\ 1 \leq x < \infty$, $0 < U(x) < 1$ for $0 < x < 1$. For every X_k put

$$U_k(x) = \prod_{i=1}^{k} \quad U\left(\frac{xi}{\delta - ki + 1}\right)(1 - U\left(\frac{xi}{\delta - ki}\right)).$$

We can find some properties of operators, which defined as:

$$0 \leq U_k(x) \leq 1 \text{ for every } x \in R,\ U_k(k\delta) = 1, \quad (18)$$

$$U_k(x) = 0,\ x \in X_k, \quad (19)$$

$$U_k(x) = 1,\ x \in R. \quad (20)$$

For $k \in G(x)$ put

$$\tilde{E}_k(\zeta, x)_{p,\beta} = inf\{\|\zeta - \eta\|_{p,\beta} : \eta \in H_{k-1}, \quad (21)$$

where $\eta_k \in H_{k-1}$.

Remark: For the case $X = [0,1]$ and k such that $k\delta$ is on the boundary of X, we instead X_k by $X_k \cap X$ in (21).

From the Whitney's theorem (see e.g. [6]) since

$$\tilde{E}_k(\zeta, x)_{p,\beta} \leq c\ \omega_k(\zeta, x)_{p,\beta},\ \text{ for every } x \in X_k. \quad (22)$$

Put $\Psi_k(x) = \eta_k(x) - \tilde{E}_k(\zeta, x)$ and

$\Phi_k(x) = \eta_k(x) + \tilde{E}_k(\zeta, x)$. For every $x \in X_k = [0,1]$ since

$$\Psi_k(x) \leq \zeta(x) \leq \Phi_k(x). \quad (23)$$

To finish we known

$$\{ m_k(\zeta) = U_k(x)\, \Psi_k(x) \in L_{p,\beta}(X), n_k(\zeta) = U_k(x)\, \Phi_k(x) \in L_{p,\beta}(X).$$
(24)

Now i) follows from equations (24), (23) and (17) – (21). From (24) we obtain

$m_k(\zeta) - n_k(\zeta) = 2\, U_k(x)\, \tilde{E}_k(\zeta, x)$ and so,

$0 \leq m_k(\zeta) - n_k(\zeta) \leq c\ \omega_k(\zeta, x)$ by (22), (18), and (20).

Now ii) from the above inequality. Take $\beta, k \le |\beta| \le l$.

Let $x \in X'_k$. Then from equations (19), (20) and (24) we obtain

$$n_k(\zeta) = \Phi_k(x) + U_{k+\epsilon}(x)(\Phi_{k+\epsilon}(x) - \Phi_k(x))$$

and so,

$$D^\beta n_k(x) = D^\beta \Phi_k(x) + c\, D^{\beta-\alpha} U_{k+\epsilon}(x)\, D^\alpha\big(\Phi_{k+\epsilon}(x) - \Phi_k(x)\big).$$

Having in mind the definitions of η_k and U_k, (22) and Markov's inequality

we obtain

$$\|D^\beta n_k\|_{p,\beta} \le c\, \|D^{\beta-\alpha} U_{k+\epsilon}\|_{p,\beta}\ \|D^\alpha(\Phi_{k+\epsilon} - \Phi_k)\|_{p,\beta}$$

$$\le c\, t^{-|\beta-\alpha|}\ x^{-|\alpha|}\ \|(\Phi_{k+\epsilon} - \Phi_k)\|_{p,\beta}$$

$$\le c\, t^{-|\beta|}\, \{\|(\Phi_{k+\epsilon} - \zeta)\|_{p,\beta} + \|\Phi_k - \zeta\|_{p,\beta}\}$$

$$\le c\, t^{-|\beta|}\, \|\omega_k(\zeta, x)\|_{p,\beta}.$$

Summating on $k \in G(x)$ the above inequality we obtain iii). This completes the proof of the second inequality in (16).

Let $X = [0,1]$, $k \in L_{p,\beta}(X)$, $D^\alpha k \in L_{p,\beta}(X)$ (generalized derivatives) for every α, $|\alpha| = l$ (recall $l > \frac{m}{n}$). The k is equivalent to $R \in L_{p,\beta}(X)$ and

$$\|R\|_{p,\beta} \le c\,\{\|k\|_{p,\beta} + \|D^\alpha k\|_{p,\beta}\}. \tag{25}$$

Making a liner change of the variables in (25) we obtain

for every $x \in X$ and δ positive constant

$$\|R\|_{p,\beta} \le c\, \delta^{\frac{-m}{n}}\{\|k\|_{p,\beta} + \delta^l\, \|D^\alpha k\|_{p,\beta}\}. \tag{26}$$

Let $m_k, n_k \in L_{p,\beta}(X)$, $m_k \le \zeta \le n_k$. Since

$$\tau_k(\zeta, x)_{p,\beta} \le \tau_k(\zeta - m_k, x)_{p,\beta} + \tau_k(m_k, x)_{p,\beta}. \tag{27}$$

From (26) and $R = k = n_k - m_k$ since

$$\omega_k(\zeta - m_k, x) \le 2^k\, \|\zeta - m_k\|_{p,\beta} \le 2^k\, \|n_k - m_k\|_{p,\beta} \tag{28}$$

$$\le c\, t^{-\frac{m}{n}}\, \|n_k - m_k\|_{p,\beta} + t^l\, \|D^\alpha(n_k - m_k)\|_{p,\beta}.$$

Noticing that

$$t^{-\frac{m}{n}}\|R\|_{p,\beta} \leq k^{\frac{m}{n}}\|R\|_{p,\beta} \tag{29}$$

for every $R \in L_{p,\beta}(X)$, from equation (28) we obtain

$$\tau_k(\zeta - m_k, x)_{p,\beta} \leq c\, \{\|n_k - m_k\|_{p,\beta} + t^{|\alpha|}\left(\| D^\alpha n_k\|_{p,\beta} + \|D^\alpha m_k\|_{p,\beta}\right)\}. \tag{30}$$

For estimating the second term in the right hand side of (27) we consider two cases.

$i)$ $\frac{k>m}{n}$, $i.e.\ k \geq d$. From Theorem 1 in [5] we obtain

$$\begin{aligned}\tau_k(m_k, x)_{p,\beta} &\leq c\, t^{\frac{m}{n}} \int_0^t \omega_k(m_k, s)_{p,\beta}\, s^{\frac{-m}{n-1}} ds \\ &\leq c\, t^{\frac{m}{n}} \int_0^t \|D^\alpha m_k\|_{p,\beta} s^{k\frac{-m}{n-1}} ds \\ &= c\, t^k \|D^\alpha m_k\|_{p,\beta}.\end{aligned} \tag{31}$$

$ii)$ $\frac{k\leq m}{n,\ \ i.e.\ k<d=l.}$ From a generalization Whitney theorem (see [6]) there is $\eta_k \in H_{k-1}$

$$\|m_k - \eta_k\|_{p,\beta} \leq c\, \omega_k(m_k, x)_{p,\beta}.$$

From this inequality and (26) we obtain

$$\omega_k(m_k, x) = \omega_k(m_k - \eta_k, x) \leq 2^k \|m_k - \eta_k\|_{p,\beta}$$

$$\leq c\, t^{\frac{-m}{n}} \{\|m_k - \eta_k\|_{p,\beta} + t^d \| D^\alpha(m_k - \eta_k)\|_{p,\beta}\} \leq c\, t^{\frac{-m}{n}} \left\{\omega_k(m_k, x)_{p,\beta} + t^d \| D^\alpha m_k\|_{p,\beta}\right\}.$$

Taking $L_{p,\beta}(X)$ norm with respect to *x* in the above inequality and from (29) we obtain

$$\begin{aligned}\tau_k(m_k, x)_{p,\beta} &\leq c\left\{\omega_k(m_k, x)_{p,\beta} + t^d \| D^\alpha m_k\|_{p,\beta}\right\} \\ &\leq c\left\{t^k \| D^\beta m_k\|_{p,\beta} + t^d \| D^\alpha m_k\|_{p,\beta}\right\}.\end{aligned} \tag{32}$$

From equations (27), (30), (31), and (32) we get

$$\tau_k(\zeta, x)_{p,\beta} \leq c\, \{\|n_k - m_k\|_{p,\beta} + t^{|\alpha|}(\| D^\alpha n_k\|_{p,\beta} + \| D^\alpha m_k\|_{p,\beta}\}.$$

Taking infimum on $m_k, n_k \in L_{p,\beta}(X)$, $m_k \leq \zeta \leq n_k$, in the above inequality we compete the proof of (16).

<u>Theorem 4.2</u>: Let ζ an unbounded function in weighted space $L_{p,\beta}(X)$. Then the following statements are holds

$$P_k(\zeta),\ q_k(\zeta) \in T_{(l+1)(k-1)}, \tag{33}$$

$$P_k(\zeta, x) \leq \zeta(x) \leq q_k(\zeta, x) \text{ for every } x \in X, \tag{34}$$

$$\|q_k(\zeta) - P_k(\zeta)\|_{p,\beta} \leq c\tau_k(\zeta, x)_{p,\beta}. \tag{35}$$

<u>Proof</u> : From (15), (10) and (13) we obtain (33). From (7) and (10) since

$\Psi_k(x) \geq 1$ for $|x| \leq \frac{\pi}{k),}$ which together with the positivity of Ψ_k gives (34).

From (13) since

$$sup|\zeta(s) - Q_k(\zeta,s)| \leq sup\,|\int_X \Delta_\delta^k \zeta(s) L_{l,k}(y)dy|$$

$$\leq \int_X sup\,|\Delta_\delta^k\,\zeta(s)|L_{l,k}(y)dy$$

$$\leq \int_X \omega_k(\zeta,\frac{2\pi}{k)} L_{l,k}(y)dy.$$

from Lemma 3.1 and $\alpha_k = \int_X \omega_k(\zeta,\frac{2\pi}{k)} L_{l,k}(y)dy,$

Minkovski's inequality, (12) and (11) we obtain

$$\|q_k(\zeta) - P_k(\zeta)\|_{p,\beta} \leq 2\,\|\Psi_k(.-\frac{2\pi}{k)}\,sup\,sup|\zeta(s) - Q_k(\zeta,s)|\,\|_{p,\beta}$$

$$\leq 2\,\|\Psi_k(.-\frac{2\pi}{k)}\|_{p,\beta}$$

$$\leq c\,((\frac{2\pi}{k)}\alpha_k^p)^{\frac{1}{p}}$$

$$\leq c\,(\int_X (\int_X (\omega_k(\zeta,x)L_{l,k}(y)dy)^p dx)^{\frac{1}{p}}$$

$$\leq c\,(\int_X (\int_X \omega_k(\zeta,x)^p dx)^{\frac{1}{p}} L_{l,k}(y)dy$$

$$= c\,\int_X \tau_k(\zeta,x)_{p,\beta} L_{l,k}(y)dy$$

$$\leq c\,\int_X L_{l,k}(y)dy\,\tau_k(\zeta,\frac{1}{k})_{p,\beta}$$

$$= c\,\tau_k(\zeta,\frac{1}{k})_{p,\beta}.$$

Theorem 4.3: Let ζ be an unbounded function in weighted space $L_{p,\beta}(X)$,$1 \leq p < \infty$ and n natural number. Then

$$\tilde{E}_k(\Phi_n,\zeta)_{p,\beta} \leq c\,\tau_k(\zeta,\frac{1}{n})_{p,\beta}.$$

Proof : using Theorem 4.2 with $k = \left[\frac{n}{l+1}\right] + 1$, we obtain

$$\tilde{E}_k(\Phi_n,\zeta)_{p,\beta} \leq \|q_k(\zeta) - P_k(\zeta)\|_{p,\beta} \leq c\,\tau_k(\zeta,\frac{1}{k})_{p,\beta} \leq c\,\tau_k(\zeta,\frac{l+1}{n})_{p,\beta}$$

$$\leq c\,\tau_k(\zeta,\frac{1}{n})_{p,\beta}.$$

$$\tau_k(\zeta,\frac{1}{n})_{p,\beta} \leq \frac{c}{n^k}\sum_{z=0}^{n} (z+1)^{k-1}\tilde{E}_k(\Phi_z,\zeta)_{p,\beta}. \quad (36)$$

5. one-sided approximation by entire operators of exponential type.

In this section we get for entire operators parallel to those from section 3.

Let $x \in R,\ G(x) = G$. Put

$F_k(u) = (\frac{\pi}{k)^2(sin(\frac{ku}{u)^2} \in A_k}$ and

$J_{l,k}(u) = \gamma_{l,k}((sin\frac{(ku}{2}\frac{)}{u)^{2l+2}} \in A_{k(l+1)}$.

Now, $L_{l,k},\ Q_k,\ P_k$ and q_k are given by (22), (24), (25) and (27) respectively (n replaced by k).

<u>Theorem 4.4</u>: Let $\zeta \in L_{p,\beta}(X),\ \zeta$ be an unbounded and $\tau_k(\zeta, k) < \infty$. Then

$P_k(\zeta),\ q_k(\zeta) \in A_{k(l+1)}$,

$P_k(\zeta, x) \leq \zeta(x) \leq q_k(\zeta, x)$ for every $x \in R$ and

$\|q_k(\zeta) - P_k(\zeta)\|_{p,\beta} \leq c\ \tau_k(\zeta, k^{-1})_{p,\beta}$.

The proof follows along the lines of the proof of Theorem 4.2.

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