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Double Weighted Exponentiated Rayleigh Distribution

(Properties and Estimation)

Intisar Elaiwi Ubaid[1], Assist. Prof. Dr.Awatif R.Mezaal[2]

[1&2]Mathematical Department, College of Education,AL-Mustanseriya University

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Abstract

This study describes the statistical characteristics of new probability distribution called Double Weighted Exponentiated Rayleigh Distribution, including probability, cumulative, reliability, hazard and reverse hazard functions, moments and the moment generating function, some important coefficients and others. The unknown scale parameter for this distribution estimated using two different styles which are the moments and the maximum likelihood estimation methods. Later, a simulation study employing the mean square errors criteria is carried out in order to determine which of these two estimators is the most accurate. The moment estimator has been determined to be the most effective for all sample sizes.

1.Introduction

Fisher (1934) [5] in his study on how determining methods could change the form of the distribution of observed data, put out the concept of weighted distributions. Rao (1965) [12] produced a common understanding of weighted distribution and defined the scenarios that it can be used to represent. When recorded observations cannot be interpreted as a random sampling from the underlying distributions, weighted distributions are typically employed to characterize data. Later, Patil and Rao (1978) [11] researched how weighted distributions can be formed through truncated distributions and damaged observations. Weighted distribution was used as a stochastic model to study predation and harvesting. Gupta and Kirmani (1990) [6] explored how the original distribution and the weighted distribution are related. Patil (2006) [9] provided the theory of weighted distribution as an all-encompassing solution to a number of problems. Using the concept of Azzalini, Gupta and Kundu (2009) [7] added a shape parameter to an exponential model, producing a new class of weighted distributions. The probability density function–of the new weighted model is quite similar to the Weibull, gamma, or generalized exponential distributions in terms of shape. Priyadarshani (2011) [10] introduced a brand-new class of weighted generalized gamma distributions and related distributions. Ye (2012) [13] introduced the second type of weighted generalized beta distribution. Hantoosh (2013) [8] used both the exponential distribution and theinverse Weibull distribution, both of which have two weight functions, to define the term "double weighted distribution". Al-Dubaicy and Nahdel (2016) [1] focused on the weighted and double weighted distributions of the generalized exponential distribution. Al-Kadim,and Fadhil, (2018) [3] proposed the double weighted Lomax distribution and established its statistical characteristics Al-Saffar and Naemah 2022) [4] established an innovative exponential double-weighted Pareto distribution and derives its statistical properties.

Suppose that x is a continuous random variable with probability density function (pdf), f(x) , and cumulative distribution function (cdf), F(x), w(x) is a non-negative weight function which takes different models, then the pdf of weighted distribution denoted by $f_w(x)$ given as :

$$f_w(x) = \frac{w(x)\cdot f(x)}{z} \quad , \text{where} \quad Z = E[w(x)] = \int_0^{\infty} w(x)\cdot f(x)dx$$

And, the double weighted distribution given by [8]:

$$f_{w_D}(x) = \frac{w_D(x)\cdot f(x)\cdot F(cx)}{Z_D} \qquad (1)$$

Where $Z_D = E[w_D(x)] = \int_0^{\infty} w_D(x)f(x)F(cx)\,dx$

and $w_D(x)$ is double weight function, c is constant .

2. Exponentiated Rayleigh Distribution

The Exponentiated Rayleigh Distribution with parameter γ and β is denoted by ERD (β,γ), with pdf and cdf define as respectively:[2]

$$f(x,\beta,\gamma) = 2\gamma\beta x e^{-\beta x^2}\left(1-e^{-\beta x^2}\right)^{\gamma-1} \qquad x,\alpha,\beta > 0 \qquad (2)$$

Where γ shape parameter and β scale parameter.

$$F(x) = \left(1-e^{-\beta x^2}\right)^{\gamma} \qquad (3)$$

3. Double Weighted Exponentiated Rayleigh Distribution

In this section, we will present the pdf and cdf for Double Weighted Exponentiated Rayleigh Distribution (DWERD).

Here, assume that the shape parameter is known as: $\gamma = 2$ and suppose the double weight function is

$$W_D(x) = x \qquad (4)$$

Then by substituting eq.'s (2), (3) and (4) in eq. (1) we get:

$$f_{W_D}(x;\beta,c) = \frac{4\beta\, x^2 e^{-\beta x^2}\left(1-e^{-\beta x^2}\right)\left(1-e^{-\beta c^2 x^2}\right)^2}{Z_D}$$

Where

$$Z_D = \int_0^{\infty} 4\beta x^2\, e^{-\beta x^2}\left(1-e^{-\beta x^2}\right)\left(1-e^{-\beta c^2 x^2}\right)^2 dx$$

$$= 4\beta \int_0^{\infty} x^2 e^{-\beta x^2}\left(1-e^{-\beta x^2}\right)\cdot\left(1-2e^{-\beta c^2 x^2}+e^{-2\beta c^2 x^2}\right) dx$$

$$= 4\beta \int_0^\infty x^2 e^{-\beta x^2} \left(1 - 2e^{-\beta c^2 x^2} + e^{-2\beta c^2 x^2} - e^{-\beta x^2} + 2e^{-\beta x^2(c^2+1)} - e^{-\beta x^2(1+2c^2)}\right) dx$$

$$= 4\beta \int_0^\infty \left(x^2 e^{-\beta x^2} - 2x^2 e^{-\beta x^2(c^2+1)} + x^2 e^{-\beta x^2(1+2c^2)} - x^2 e^{-2\beta x^2} + 2x^2 e^{-\beta x^2(c^2+2)} - x^2 e^{-2\beta x^2(c^2+1)}\right) dx$$

Now, to find the value of integration as: $\int_0^\infty x^2 e^{-\beta x^2}\, dx$

Let $y = \beta x^2 \Rightarrow x^2 = \frac{y}{\beta} \Rightarrow x = \left(\frac{y}{\beta}\right)^{\frac{1}{2}}$ $dx = \frac{1}{2(\beta y)^{\frac{1}{2}}}\, dy$

Then , $\int_0^\infty \frac{y}{\beta}\, e^{-y} \frac{1}{2(\beta y)^{\frac{1}{2}}}\, dy = \frac{1}{2\beta^{\frac{3}{2}}} \int_0^\infty y^{\frac{1}{2}}\, e^{-y}\, dy = \frac{1}{2\beta^{\frac{3}{2}}}\, \Gamma_{\frac{3}{2}} = \frac{\sqrt{\pi}}{4\beta^{\frac{3}{2}}}$

In the same technique the other integrations are:

- $2\int_0^\infty x^2 e^{-\beta x^2(c^2+1)}\, dx = \frac{\sqrt{\pi}}{2(\beta(c^2+1))^{\frac{3}{2}}}$
- $\int_0^\infty x^2 e^{-\beta x^2(2c^2+1)}\, dx = \frac{\sqrt{\pi}}{4(\beta(2c^2+1))^{\frac{3}{2}}}$
- $\int_0^\infty x^2 e^{-2\beta x^2}\, dx = \frac{\sqrt{\pi}}{8\sqrt{2}\beta^{\frac{3}{2}}}$
- $2\int_0^\infty x^2 e^{-\beta x^2(c^2+2)}\, dx = \frac{\sqrt{\pi}}{2(\beta(c^2+2))^{\frac{3}{2}}}$
- $\int_0^\infty x^2 e^{-2\beta x^2(c^2+1)}\, dx = \frac{\sqrt{\pi}}{8\sqrt{2}(\beta(c^2+1))^{\frac{3}{2}}}$

Consequently,

$$Z_D = 4\beta\left(\frac{\sqrt{\pi}}{4\beta^{\frac{3}{2}}} - \frac{\sqrt{\pi}}{2(\beta(c^2+1))^{\frac{3}{2}}} + \frac{\sqrt{\pi}}{4(\beta(2c^2+1))^{\frac{3}{2}}} - \frac{\sqrt{\pi}}{8\sqrt{2}\beta^{\frac{3}{2}}} + \frac{\sqrt{\pi}}{2(\beta(c^2+2))^{\frac{3}{2}}} - \frac{\sqrt{\pi}}{8\sqrt{2}(\beta(c^2+1))^{\frac{3}{2}}}\right)$$

$$Z_D = \frac{2\sqrt{\pi}}{\beta^{\frac{1}{2}}}\left(\frac{1}{2} - \frac{1}{(c^2+1)^{\frac{3}{2}}} + \frac{1}{2(2c^2+1)^{\frac{3}{2}}} - \frac{1}{4\sqrt{2}} + \frac{1}{(c^2+2)^{\frac{3}{2}}} - \frac{1}{4\sqrt{2}(c^2+1)^{\frac{3}{2}}}\right)$$

$$Z_D = \frac{2\sqrt{\pi}}{\beta^{\frac{1}{2}}}\left(\frac{2\sqrt{2}-1}{4\sqrt{2}} - \frac{4\sqrt{2}+1}{4\sqrt{2}(c^2+1)^{\frac{3}{2}}} + \frac{1}{2(2c^2+1)^{\frac{3}{2}}} + \frac{1}{(c^2+2)^{\frac{3}{2}}}\right)$$

Then the pdf of DWERD is given by:

$$f_{W_D}(x;\beta,c) = \frac{2\beta^{\frac{3}{2}}}{\sqrt{\pi}Q} \cdot x^2 e^{-\beta x^2}\left(1 - e^{-\beta x^2}\right) \cdot \left(1 - e^{-\beta c^2 x^2}\right)^2 \qquad (5)$$

Where x, β *and* $c > 0$, and, $Q = \frac{2\sqrt{2}-1}{4\sqrt{2}} - \frac{4\sqrt{2}+1}{4\sqrt{2}(c^2+1)^{\frac{3}{2}}} + \frac{1}{2(2c^2+1)^{\frac{3}{2}}} + \frac{1}{(c^2+2)^{\frac{3}{2}}}$

[اكتب نصاً]

The figure below shows the pdf of DWERD with fixed c=3 and different values of β.

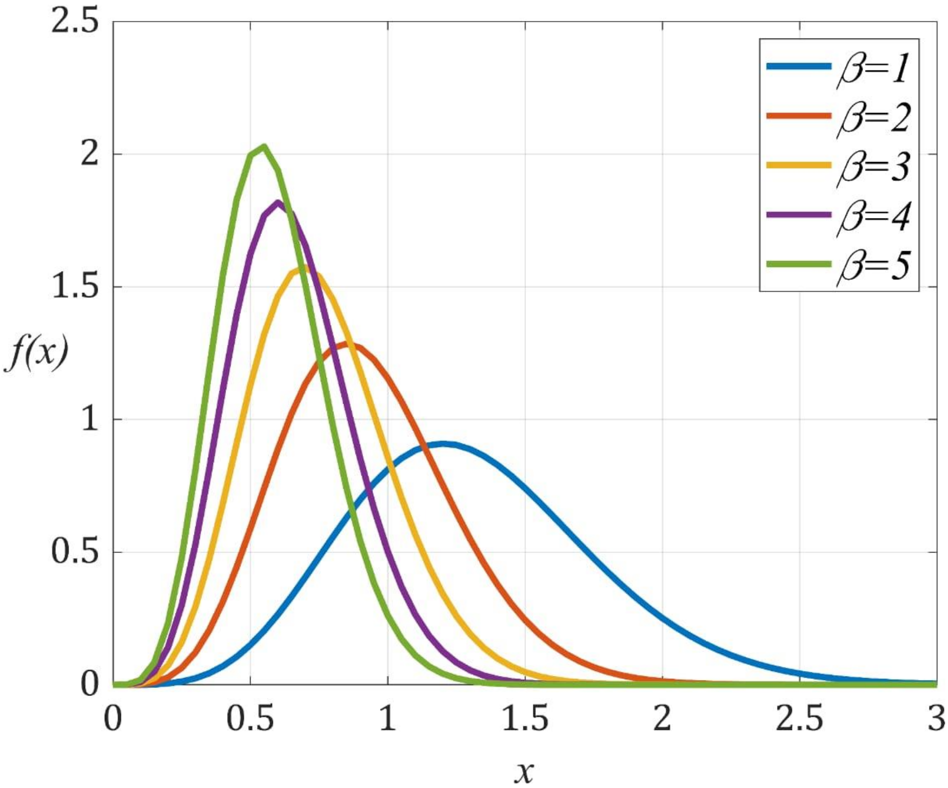

FIGURE (1): f_{w_D} of DWERD with fixed $c = 3$ and different values of β

The figure show that pdf of DWERD increases when value of the scale parameter β increases.

Consequently, the cdf of DWERD is given by:

$$F_{W_D}(x;\beta,c) = \int_0^x f_{W_D}(t;\beta,c)\ dt$$

$$= \int_0^x \frac{2}{\sqrt{\pi}Q}\beta^{\frac{3}{2}}\, t^2 e^{-\beta t^2}\left(1 - e^{-\beta t^2}\right)\left(1 - e^{-\beta c^2 t^2}\right)^2 dt$$

$$= \frac{-2\beta^{\frac{3}{2}}}{\sqrt{\pi}Q}\int_0^x t^2\left(e^{-\beta t^2} - 1\right)e^{-\beta t^2}\left(e^{-\beta c^2 t^2} - 1\right)^2 dt$$

Now solving:

$$\int_0^x t^2\left(e^{-\beta t^2} - 1\right)e^{-\beta t^2}\cdot\left(e^{-\beta c^2 t^2} - 1\right)^2 dt$$

Put terms over a common denominator:

$$= \int_0^x t^2\, e^{-2\beta t^2}\left(1 - e^{\beta t^2}\right)\left(e^{-\beta c^2 t^2} - 1\right)^2 dt$$

$$= -\int_0^x t^2\, e^{-2\beta t^2}\left(e^{\beta t^2} - 1\right)\left(e^{-\beta c^2 t^2} - 1\right)^2 dt$$

Now solving:

$$\int_0^x t^2\, e^{-2\beta t^2}\left(e^{\beta t^2}-1\right)\left(e^{-\beta c^2t^2}-1\right)^2 dt$$

$$=\int_0^x e^{-2\beta t^2}\cdot\left(t^2e^{\beta t^2}\cdot\left(e^{-\beta c^2t^2}-1\right)^2-t^2\cdot\left(e^{-\beta c^2t^2}-1\right)^2\right)dt$$

$$=\int_0^x\left(t^2e^{-\beta t^2}\cdot\left(e^{-\beta c^2t^2}-1\right)^2-t^2e^{-2\beta t^2}\cdot\left(e^{-\beta c^2t^2}-1\right)^2\right)dt$$

$$=\int_0^x\left(-2t^2e^{-\beta c^2t^2-\beta t^2}+t^2e^{-2\beta c^2t^2-\beta t^2}+t^2e^{-\beta t^2}+2t^2e^{-\beta c^2t^2-2\beta t^2}-\quad t^2e^{-2\beta c^2t^2-2\beta t^2}-t^2e^{-2\beta t^2}\right)dt$$

$$=-2\int_0^x t^2e^{-\beta c^2t^2-\beta t^2}dt+\int_0^x t^2e^{-2\beta c^2t^2-\beta t^2}+\int_0^x t^2e^{-\beta t^2}dt+\ 2\int_0^x t^2e^{-\beta c^2t^2-2\beta t^2}dt-\int_0^x t^2e^{-2\beta c^2t^2-2\beta t^2}dt-\int_0^x t^2e^{-2\beta t^2}dt$$

So, $\int_0^x t^2e^{-\beta c^2t^2-\beta t^2}dt$

Integrate by parts:

$u^*=\mathrm{t},\ \mathrm{d}v^*=\mathrm{t}\mathrm{e}^{-\beta c^2t^2-\beta t^2}$

$\mathrm{d}u^*=dt,\ v^*=-\frac{\mathrm{e}^{-\beta c^2t^2-\beta t^2}}{2\beta c^2+2\beta}$

$$=\frac{-\mathrm{t}\mathrm{e}^{-\beta c^2t^2-\beta t^2}}{2\beta c^2+2\beta}\Big|_0^x-\int_0^x-\frac{\mathrm{e}^{-\beta c^2t^2-\beta t^2}}{2\beta c^2+2\beta}\mathrm{dt}$$

Now solving: $\int_0^x-\frac{e^{-\beta c^2t^2-\beta t^2}}{2\beta c^2+2\beta}dt$

Substitute $u=\sqrt{\beta}\cdot\sqrt{c^2+1}\ \mathrm{t}\Longrightarrow du=\sqrt{\beta}\cdot\sqrt{c^2+1}\cdot dt$

$$=\int_0^x-\frac{e^{-\frac{c^2u^2}{c^2+1}-\frac{u^2}{c^2+1}}}{\sqrt{\beta}\cdot\sqrt{c^2+1}(2\beta c^2+2\beta)}du\quad\rightarrow\frac{-\sqrt{\pi}}{2\sqrt{\beta}\cdot\sqrt{c^2+1}(2\beta c^2+2\beta)}\int_0^x\frac{2e^{-u^2}}{\sqrt{\pi}}du$$

Now solving: $\int\frac{2e^{-u^2}}{\sqrt{\pi}}\mathrm{du}\ =(Gauss\ error\ function)$

Plug in solved integrals:

$$\frac{-\sqrt{\pi}}{2\sqrt{\beta}\cdot\sqrt{c^2+1}(2\beta c^2+2\beta)}\int_0^x\frac{2e^{-u^2}}{\sqrt{\pi}}\mathrm{du}\ =\frac{-\sqrt{\pi}erf(u)}{2\sqrt{\beta}\cdot\sqrt{c^2+1}(2\beta c^2+2\beta)}$$

Undo substitution $\mathrm{u}=\sqrt{\beta}\cdot\sqrt{c^2+1}\ \mathrm{t}$:

$$=\frac{-\sqrt{\pi}erf(\sqrt{\beta}\cdot\sqrt{c^2+1}\,t)}{2\sqrt{\beta}\cdot\sqrt{c^2+1}(2\beta c^2+2\beta)}\Big|_0^x$$

[اكتب نصاً]

Plug in solved integrals:

$$\frac{-te^{-\beta c^2t^2-\beta t^2}}{2\beta c^2+2\beta}\Big|_0^x-\int_0^x-\frac{e^{-\beta c^2t^2-\beta t^2}}{2\beta c^2+2\beta}dt=\frac{\sqrt{\pi}erf(\sqrt{\beta}\cdot\sqrt{c^2+1}\,t)}{2\sqrt{\beta}\cdot\sqrt{c^2+1}(2\beta c^2+2\beta)}-\frac{te^{-\beta c^2t^2-\beta t^2}}{2\beta c^2+2\beta}\Big|_0^x$$

$$=\frac{\sqrt{\pi}erf(\sqrt{\beta}\cdot\sqrt{c^2+1}\,x)}{2\sqrt{\beta}\cdot\sqrt{c^2+1}(2\beta c^2+2\beta)}-\frac{xe^{-\beta c^2x^2-\beta x^2}}{(2\beta c^2+2\beta)}$$

In the same technique the other integrations are:

- $\int_0^x t^2e^{-2\beta c^2t^2-\beta t^2}\,dt=\frac{\sqrt{\pi}erf(\sqrt{\beta}\cdot\sqrt{2c^2+1}\,x)}{2\sqrt{\beta}\cdot\sqrt{2c^2+1}(4\beta c^2+2\beta)}-\frac{xe^{-2\beta c^2x^2-\beta x^2}}{(4\beta c^2+2\beta)}$
- $\int_0^x t^2e^{-\beta t^2}dt=\frac{\sqrt{\pi}\mathrm{erf}(\sqrt{\beta}\,\mathrm{x})}{4\beta^{\frac{3}{2}}}-\frac{xe^{-\beta x^2}}{2\beta}$
- $\int_0^x t^2e^{-\beta c^2t^2-2\beta t^2}dt=\frac{\sqrt{\pi}\mathrm{erf}(\sqrt{\beta}\sqrt{c^2+2}\,\mathrm{x})}{2\sqrt{\beta}\sqrt{c^2+2}(2\beta c^2+4\beta)}-\frac{xe^{-\beta c^2x^2-2\beta x^2}}{(2\beta c^2+4\beta)}$
- $\int_0^x t^2e^{-2\beta c^2t^2-2\beta t^2}dt=\frac{\sqrt{\pi}\mathrm{erf}(\sqrt{2}\sqrt{\beta}\sqrt{c^2+1}\,\mathrm{x})}{2^{\frac{3}{2}}\sqrt{\beta}\sqrt{c^2+1}(4\beta c^2+4\beta)}-\frac{xe^{-2\beta c^2x^2-2\beta x^2}}{(4\beta c^2+4\beta)}$
- $\int_0^x t^2e^{-2\beta t^2}dt=\frac{\sqrt{\pi}\mathrm{erf}(\sqrt{2}\sqrt{\beta}x)}{2^{\frac{7}{2}}\beta^{\frac{3}{2}}}-\frac{xe^{-2\beta x^2}}{4\beta}$

So, the cdf of DWERD is given by:

$$F_{W_D}(x;\beta,c)=$$

$$\frac{\beta\mathrm{erf}(\sqrt{\beta}\cdot\sqrt{2c^2+1}\,\mathrm{x})}{\sqrt{2c^2+1}(4\beta c^2+2\beta)Q}+\frac{2\beta\mathrm{erf}(\sqrt{\beta}\sqrt{c^2+2}\,\mathrm{x})}{\sqrt{c^2+2}(2\beta c^2+4\beta)Q}-\frac{\beta\,\mathrm{erf}(\sqrt{2}\sqrt{\beta}\sqrt{c^2+1}\,\mathrm{x})}{2^{\frac{1}{2}}\sqrt{c^2+1}(4\beta c^2+4\beta)Q}-\frac{2\beta\,\mathrm{erf}(\sqrt{\beta}\cdot\sqrt{c^2+1}\,\mathrm{x})}{\sqrt{c^2+1}(2\beta c^2+2\beta)Q}-$$

$$\frac{\mathrm{erf}(\sqrt{2}\sqrt{\beta}x)}{2^{\frac{5}{2}}Q}+\frac{\mathrm{erf}(\sqrt{\beta}\,\mathrm{x})}{2Q}+\frac{4\beta^{\frac{3}{2}}xe^{-\beta c^2x^2-\beta x^2}}{\sqrt{\pi}(2\beta c^2+2\beta)Q}-\frac{4\beta^{\frac{3}{2}}xe^{-\beta c^2x^2-2\beta x^2}}{\sqrt{\pi}(2\beta c^2+4\beta)Q}-\frac{2\beta^{\frac{3}{2}}xe^{-2\beta c^2x^2-\beta x^2}}{\sqrt{\pi}(4\beta c^2+2\beta)Q}+$$

$$\frac{2\beta^{\frac{3}{2}}xe^{-2\beta c^2x^2-2\beta x^2}}{\sqrt{\pi}(4\beta c^2+4\beta)Q}-\frac{\sqrt{\beta}xe^{-\beta x^2}}{\sqrt{\pi}Q}+\frac{\sqrt{\beta}xe^{-2\beta x^2}}{2\sqrt{\pi}Q}$$

(6)

The figure below shows the cdf of DWERD with fixed c=3 and different values of β.

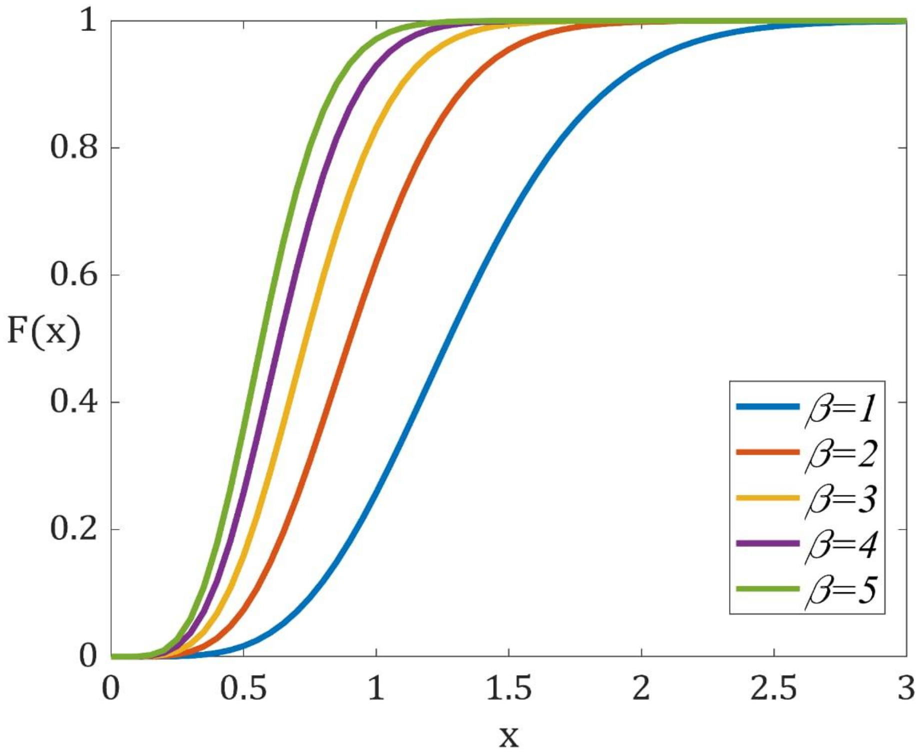

FIGURE (2): F_{W_D} of DWERD with fixed $c = 3$ and different values of β

The figure show that cdf of DWERD is non-creasing with increasing x.

4. Limits and mode of the function

The pdf in equation (5) is tested to see what happens when the variable x reaches 0 and infinite. as a result, $\lim\limits_{x\to 0}$ and $\lim\limits_{x\to\infty}$ can be obtained in the following formats:

$$\lim_{x\to 0} f_{W_D}(x;\beta,c) = \lim_{x\to 0} \frac{2\beta^{\frac{3}{2}}}{\sqrt{\pi}Q} x^2 e^{-\beta x^2}\left(1-e^{-\beta x^2}\right)\cdot\left(1-e^{-\beta c^2 x^2}\right)^2$$

$$= \frac{2\beta^{\frac{3}{2}}}{\sqrt{\pi}Q} \lim_{x\to 0} x^2 e^{-\beta x^2}\left(1-e^{-\beta x^2}\right)\cdot\left(1-e^{-\beta c^2 x^2}\right)^2 = 0$$

$$\lim_{x\to\infty} f_{W_D}(x;\beta,c) = \lim_{x\to\infty} \frac{2\beta^{\frac{3}{2}}}{\sqrt{\pi}Q} x^2 e^{-\beta x^2}\left(1-e^{-\beta x^2}\right)\cdot\left(1-e^{-\beta c^2 x^2}\right)^2$$

$$= \frac{2\beta^{\frac{3}{2}}}{\sqrt{\pi}Q} \lim_{x\to\infty} x^2 e^{-\beta x^2}\left(1-e^{-\beta x^2}\right)\cdot\left(1-e^{-\beta c^2 x^2}\right)^2$$

$$= \frac{2\beta^{\frac{3}{2}}}{\sqrt{\pi}Q} \lim_{x\to\infty} x^2 e^{-\beta x^2}\left(1-e^{-\beta x^2}\right)\left(1-2e^{-\beta c^2 x^2}+e^{-2\beta c^2 x^2}\right)$$

$$= \frac{2\beta^{\frac{3}{2}}}{\sqrt{\pi}Q} \lim_{x\to\infty} x^2 e^{-\beta x^2}\left(1 - 2e^{-\beta c^2 x^2} + e^{-2\beta c^2 x^2} - e^{-\beta x^2} + 2e^{-\beta x^2(c^2+1)} - \quad e^{-\beta x^2(1+2c^2)}\right)$$

$$= \frac{2\beta^{\frac{3}{2}}}{\sqrt{\pi}Q} \lim_{x\to\infty} \left(x^2 e^{-\beta x^2} - 2x^2 e^{-\beta x^2(c^2+1)} + x^2 e^{-\beta x^2(1+2c^2)} - x^2 e^{-2\beta x^2} + \quad 2x^2 e^{-\beta x^2(c^2+2)} - x^2 e^{-2\beta x^2(c^2+1)}\right)$$

$$= \frac{2\beta^{\frac{3}{2}}}{\sqrt{\pi}Q} \lim_{x\to\infty} \left(\frac{x^2}{e^{\beta x^2}} - \frac{2x^2}{e^{\beta x^2(c^2+1)}} + \frac{x^2}{e^{\beta x^2(1+2c^2)}} - \frac{x^2}{e^{2\beta x^2}} + \frac{2x^2}{e^{\beta x^2(c^2+2)}} - \frac{x^2}{e^{2\beta x^2(c^2+1)}}\right)$$

$$= \frac{2\beta^{\frac{3}{2}}}{\sqrt{\pi}Q}\left(\lim_{x\to\infty}\frac{x^2}{e^{\beta x^2}} - \lim_{x\to\infty}\frac{2x^2}{e^{\beta x^2(c^2+1)}} + \lim_{x\to\infty}\frac{x^2}{e^{\beta x^2(1+2c^2)}} - \lim_{x\to\infty}\frac{x^2}{e^{2\beta x^2}} + \lim_{x\to\infty}\frac{2x^2}{e^{\beta x^2(c^2+2)}} - \lim_{x\to\infty}\frac{x^2}{e^{2\beta x^2(c^2+1)}}\right)$$

$$= \frac{2\beta^{\frac{3}{2}}}{\sqrt{\pi}Q}\left(\lim_{x\to\infty}\frac{2x}{2\beta x e^{\beta x^2}} - \lim_{x\to\infty}\frac{4x}{2\beta(c^2+1)xe^{\beta x^2(c^2+1)}} + -\lim_{x\to\infty}\frac{2x}{4\beta x e^{2\beta x^2}} + \quad \lim_{x\to\infty}\frac{4x}{2\beta(c^2+2)xe^{\beta x^2(c^2+2)}} - \lim_{x\to\infty}\frac{2x}{4\beta(c^2+1)xe^{2\beta x^2(c^2+1)}}\right)$$

By Lhopital-rule, get: $\lim_{x\to\infty} f_{W_D}(x;\beta,c) = 0$

Consequently, the model most surely has a unique mode. Equation (5) provides the following results:

$$f_{W_D}(x;\beta,c) = \frac{2\beta^{\frac{3}{2}}}{\sqrt{\pi}Q} x^2 e^{-\beta x^2}\left(1 - e^{-\beta x^2}\right)\cdot\left(1 - e^{-\beta c^2 x^2}\right)^2$$

$$ln f_{W_D}(x;\beta,c) = \ln\left(\frac{2\beta^{\frac{3}{2}}}{\sqrt{\pi}Q} x^2 e^{-\beta x^2}\left(1 - e^{-\beta x^2}\right)\cdot\left(1 - e^{-\beta c^2 x^2}\right)^2\right)$$

$$= ln\left(\frac{2\beta^{\frac{3}{2}}}{\sqrt{\pi}Q}\right) + 2ln x - \beta x^2 + ln\left(1 - e^{-\beta x^2}\right) + 2ln\left(1 - e^{-\beta c^2 x^2}\right)$$

$$\frac{\partial ln f_{W_D}(x;\beta,c)}{\partial x} = \frac{2}{x} - 2\beta x + \frac{2\beta x e^{-\beta x^2}}{\left(1 - e^{-\beta x^2}\right)} + \frac{4\beta c^2 x e^{-\beta c^2 x^2}}{\left(1 - e^{-\beta c^2 x^2}\right)}$$

$$= \frac{2}{x} - 2\beta x + \frac{2\beta x e^{-\beta x^2}}{\left(1 - e^{-\beta x^2}\right)} + \frac{4\beta c^2 x e^{-\beta c^2 x^2}}{\left(1 - e^{-\beta c^2 x^2}\right)} = 0$$

The mode of DWERD is gained by solving the above nonlinear equation with respect to x.

5. The statistical properties

In this section, drive some important statistical properties of DWERD as :

5.1 The r^{th} moment

Theorem 1. The r^{th} Moment for DWERD with $w_D(x) = x$ as follows

$$E(X^r) = \frac{2}{\beta^{\frac{r}{2}}\sqrt{\pi}Q}\,\Gamma\left(\frac{r+1}{2}+1\right)\left[\frac{1}{2}-\frac{1}{(c^2+1)^{\frac{r+3}{2}}}+\frac{1}{2(1+2c^2)^{\frac{r+3}{2}}}-\frac{1}{(2)^{\frac{r+5}{2}}}+\frac{1}{(c^2+2)^{\frac{r+3}{2}}}-\frac{1}{2(2(c^2+1))^{\frac{r+3}{2}}}\right]$$

r =1, 2, …

Where $Q = \frac{2\sqrt{2}-1}{4\sqrt{2}} - \frac{4\sqrt{2}+1}{4\sqrt{2}(c^2+1)^{\frac{3}{2}}} + \frac{1}{2(2c^2+1)^{\frac{3}{2}}} + \frac{1}{(c^2+2)^{\frac{3}{2}}}$

Proof:

Using eq.(5) , The r^{th} Moment as:

$$E(X^r) = \int_0^\infty x^r f_{W_D}(x;\beta,c)dx$$

$$= \int_0^\infty x^r \frac{2\beta^{\frac{3}{2}}}{\sqrt{\pi}Q}\cdot x^2 e^{-\beta x^2}\left(1-e^{-\beta x^2}\right)\cdot\left(1-e^{-\beta c^2x^2}\right)^2 dx$$

$$= \frac{2\beta^{\frac{3}{2}}}{\sqrt{\pi}Q}\int_0^\infty x^{r+2}\, e^{-\beta x^2}\left(1-e^{-\beta x^2}\right)\cdot\left(1-e^{-\beta c^2x^2}\right)^2 dx$$

$$= \frac{2\beta^{\frac{3}{2}}}{\sqrt{\pi}Q}\int_0^\infty x^{r+2}\, e^{-\beta x^2}\left(1-e^{-\beta x^2}\right)\cdot\left(1-2e^{-\beta c^2x^2}+e^{-2\beta c^2x^2}\right) dx$$

$$= \frac{2\beta^{\frac{3}{2}}}{\sqrt{\pi}Q}\int_0^\infty x^{r+2}e^{-\beta x^2}\left(1-2e^{-\beta c^2x^2}+e^{-2\beta c^2x^2}-e^{-\beta x^2}+2e^{-\beta x^2(c^2+1)}-e^{-\beta x^2(1+2c^2)}\right)dx$$

$$= \frac{2\beta^{\frac{3}{2}}}{\sqrt{\pi}Q}\int_0^\infty\Big(x^{r+2}e^{-\beta x^2}-2x^{r+2}e^{-\beta x^2(c^2+1)}+x^{r+2}e^{-\beta x^2(1+2c^2)}-x^{r+2}e^{-2\beta x^2}+2x^{r+2}e^{-\beta x^2(c^2+2)}-x^{r+2}e^{-2\beta x^2(c^2+1)}\Big)\,dx$$

Now,

- $\int_0^\infty x^{r+2}e^{-\beta x^2}dx$

Let y= $\beta x^2 \implies x^2 = \frac{y}{\beta} \implies x = \left(\frac{y}{\beta}\right)^{\frac{1}{2}} \implies dx = \frac{1}{2(\beta y)^{\frac{1}{2}}}dy$

$$\to \int_0^\infty \left(\frac{y}{\beta}\right)^{\frac{r+2}{2}} e^{-y}\frac{1}{2(\beta y)^{\frac{1}{2}}}dy = \frac{1}{2\beta^{\frac{r+3}{2}}}\int_0^\infty y^{\frac{r+1}{2}}e^{-y}\,dy \to \frac{1}{2\beta^{\frac{r+3}{2}}}\Gamma\left(\frac{r+1}{2}+1\right)$$

In the same technique the other integrations are:

- $\int_0^\infty 2x^{r+2}e^{-\beta x^2(c^2+1)}dx = \frac{1}{(\beta(c^2+1))^{\frac{r+3}{2}}}\Gamma\left(\frac{r+1}{2}+1\right)$
- $\int_0^\infty x^{r+2}e^{-\beta x^2(1+2c^2)}dx = \frac{1}{2(\beta(1+2c^2))^{\frac{r+3}{2}}}\Gamma\left(\frac{r+1}{2}+1\right)$

[اكتب نصاً]

- $\int_0^\infty x^{r+2} e^{-2\beta x^2} dx = \frac{1}{2(2\beta)^{\frac{r+3}{2}}} \Gamma\left(\frac{r+1}{2}+1\right)$
- $\int_0^\infty 2x^{r+2} e^{-\beta x^2(c^2+2)} dx = \frac{1}{(\beta(c^2+2))^{\frac{r+3}{2}}} \Gamma\left(\frac{r+1}{2}+1\right)$
- $\int_0^\infty x^{r+2} e^{-2\beta x^2(c^2+1)} dx = \frac{1}{2(2\beta(c^2+1))^{\frac{r+3}{2}}} \Gamma\left(\frac{r+1}{2}+1\right)$

So, the r^{th}moment define as:

$$E(X^r) = \frac{2}{\beta^{\frac{r}{2}}\sqrt{\pi}Q} \Gamma\left(\frac{r+1}{2}+1\right)\left[\frac{1}{2} - \frac{1}{(c^2+1)^{\frac{r+3}{2}}} + \frac{1}{2(1+2c^2)^{\frac{r+3}{2}}} - \frac{1}{(2)^{\frac{r+5}{2}}} + \frac{1}{(c^2+2)^{\frac{r+3}{2}}} - \frac{1}{2(2(c^2+1))^{\frac{r+3}{2}}}\right]$$

Where : $Q = \frac{2\sqrt{2}-1}{4\sqrt{2}} - \frac{4\sqrt{2}+1}{4\sqrt{2}(c^2+1)^{\frac{3}{2}}} + \frac{1}{2(2c^2+1)^{\frac{3}{2}}} + \frac{1}{(c^2+2)^{\frac{3}{2}}}$

The prove is verified.

Now, when $r = 1$, we get the mean of the distribution as:

$$E(X) = \frac{2}{\beta^{\frac{1}{2}}\sqrt{\pi}Q}\left[\frac{3}{8} - \frac{1}{(c^2+1)^2} + \frac{1}{2(1+2c^2)^2} + \frac{1}{(c^2+2)^2} - \frac{1}{2(2(c^2+1))^2}\right] \quad (7)$$

when r=2

$$E(X^2) = \frac{3}{2\beta Q}\left[\frac{1}{2} - \frac{1}{(c^2+1)^{\frac{5}{2}}} + \frac{1}{2(1+2c^2)^{\frac{5}{2}}} - \frac{1}{(2)^{\frac{7}{2}}} + \frac{1}{(c^2+2)^{\frac{5}{2}}} - \frac{1}{2(2(c^2+1))^{\frac{5}{2}}}\right] \quad (8)$$

when r=3

$$E(X^3) = \frac{4}{\beta^{\frac{3}{2}}\sqrt{\pi}Q}\left[\frac{7}{16} - \frac{1}{(c^2+1)^3} + \frac{1}{2(1+2c^2)^3} + \frac{1}{(c^2+2)^3} - \frac{1}{2(2(c^2+1))^3}\right] \quad (9)$$

when r=4

$$E(X^4) = \frac{15}{4\beta^2 Q}\left[\frac{1}{2} - \frac{1}{(c^2+1)^{\frac{7}{2}}} + \frac{1}{2(1+2c^2)^{\frac{7}{2}}} - \frac{1}{(2)^{\frac{9}{2}}} + \frac{1}{(c^2+2)^{\frac{7}{2}}} - \frac{1}{2(2(c^2+1))^{\frac{7}{2}}}\right] \quad (10)$$

5.2 Moment Generating Function

Theorem 2. The moment generating function of (DWERD) with $w(x) = x$ is given by:

$$M_x(t) = \sum_{i=1}^{\infty} \frac{t^i}{i!}\left(\frac{2}{\beta^{\frac{i}{2}}\sqrt{\pi}Q} \Gamma\left(\frac{i+1}{2}+1\right)\left[\frac{1}{2} - \frac{1}{(c^2+1)^{\frac{i+3}{2}}} + \frac{1}{2(1+2c^2)^{\frac{i+3}{2}}} - \frac{1}{(2)^{\frac{i+5}{2}}} + \frac{1}{(c^2+2)^{\frac{i+3}{2}}} - \frac{1}{2(2(c^2+1))^{\frac{i+3}{2}}}\right]\right)$$

Where $Q = \frac{2\sqrt{2}-1}{4\sqrt{2}} - \frac{4\sqrt{2}+1}{4\sqrt{2}(c^2+1)^{\frac{3}{2}}} + \frac{1}{2(2c^2+1)^{\frac{3}{2}}} + \frac{1}{(c^2+2)^{\frac{3}{2}}}$

Proof.

[اكتب نصاً]

Using eq. (5):

$$M_{x(t)} = E(e^{tX}) = \int_0^\infty e^{tx}\, f_{W_D}(x;\beta,c)\, dx$$

$$\Rightarrow \int_0^\infty e^{tx} \cdot \frac{2\beta^{\frac{3}{2}}}{\sqrt{\pi}Q} \cdot x^2 e^{-\beta x^2}\left(1-e^{-\beta x^2}\right)\cdot\left(1-e^{-\beta c^2 x^2}\right)^2$$

by Taylor series

$$M_{x(t)} = \int_0^\infty \left(1+\frac{tx}{1!}+\frac{t^2x^2}{2!}+\cdots+\frac{t^nx^n}{n!}+\cdots\right) f_{W_D}(x;\beta,c)\, dx \quad = \sum_{i=0}^{\infty}\frac{t^i E(X^i)}{i!} \quad \text{then,}$$

$$\mathrm{M}_x(t) = \sum_{i=1}^{\infty}\frac{t^i}{i!}\left(\frac{2}{\beta^{\frac{i}{2}}\sqrt{\pi}Q}\,\Gamma\left(\frac{i+1}{2}+1\right)\left[\frac{1}{2}-\frac{1}{(c^2+1)^{\frac{i+3}{2}}}+\frac{1}{2(1+2c^2)^{\frac{i+3}{2}}}-\frac{1}{(2)^{\frac{i+5}{2}}}+\frac{1}{(c^2+2)^{\frac{i+3}{2}}}-\frac{1}{2(2(c^2+1))^{\frac{i+3}{2}}}\right]\right)$$

5.3 Variance v(x)

The variance for (DWERD) defines as: $v(X) = \mu_2 - (\mu_1)^2$

Then by eq.'s (7) and (8) is given by:

$$v(X) = \frac{3}{2\beta Q}\left[\frac{1}{2}-\frac{1}{(c^2+1)^{\frac{5}{2}}}+\frac{1}{2(1+2c^2)^{\frac{5}{2}}}-\frac{1}{(2)^{\frac{7}{2}}}+\frac{1}{(c^2+2)^{\frac{5}{2}}}-\frac{1}{2(2(c^2+1))^{\frac{5}{2}}}\right]-$$

$$\frac{4}{\beta\pi Q^2}\left[\frac{3}{8}-\frac{1}{(c^2+1)^2}+\frac{1}{2(1+2c^2)^2}+\frac{1}{(c^2+2)^2}-\frac{1}{2(2(c^2+1))^2}\right]^2 \qquad (11)$$

5.4 Reliability Function R(x)

The Reliability function for DWERD defines as:

$R(x) = 1 - F_{W_D}(x;\beta,c)$, then by eq. (6) we get:

$$R(x) = 1 - \left[\frac{\beta\,\mathrm{erf}(\sqrt{\beta}\cdot\sqrt{2c^2+1}\,x)}{\sqrt{2c^2+1}(4\beta c^2+2\beta)Q}+\frac{2\beta\,\mathrm{erf}(\sqrt{\beta}\sqrt{c^2+2}\,x)}{\sqrt{c^2+2}(2\beta c^2+4\beta)Q}-\frac{\beta\,\mathrm{erf}(\sqrt{2}\sqrt{\beta}\sqrt{c^2+1}\,x)}{2^{\frac{1}{2}}\sqrt{c^2+1}(4\beta c^2+4\beta)Q}-\frac{2\beta\,\mathrm{erf}(\sqrt{\beta}\cdot\sqrt{c^2+1}\,x)}{\sqrt{c^2+1}(2\beta c^2+2\beta)Q}-\right.$$

$$\frac{\mathrm{erf}(\sqrt{2}\sqrt{\beta}x)}{2^{\frac{5}{2}}Q}+\frac{\mathrm{erf}(\sqrt{\beta}\,x)}{2Q}+\frac{4\beta^{\frac{3}{2}}xe^{-\beta c^2x^2-\beta x^2}}{\sqrt{\pi}(2\beta c^2+2\beta)Q}-\frac{4\beta^{\frac{3}{2}}xe^{-\beta c^2x^2-2\beta x^2}}{\sqrt{\pi}(2\beta c^2+4\beta)Q}-\frac{2\beta^{\frac{3}{2}}xe^{-2\beta c^2x^2-\beta x^2}}{\sqrt{\pi}(4\beta c^2+2\beta)Q}+\frac{2\beta^{\frac{3}{2}}xe^{-2\beta c^2x^2-2\beta x^2}}{\sqrt{\pi}(4\beta c^2+4\beta)Q}-$$

$$\left.\frac{\sqrt{\beta}xe^{-\beta x^2}}{\sqrt{\pi}Q}+\frac{\sqrt{\beta}xe^{-2\beta x^2}}{2\sqrt{\pi}Q}\right] \qquad (12)$$

5.5 Hazard Function H(x)

Hazard function for DWERD is defines as:

$$\mathrm{H(x)} = \frac{f w_D(x;\beta,c)}{\mathrm{R(x)}}$$

Then by eq.'s (5) and (12) we get:

$$\mathrm{H(x)} = \frac{2\beta^{\frac{3}{2}}}{\sqrt{\pi}Q} \cdot x^2 e^{-\beta x^2}\left(1 - e^{-\beta x^2}\right) \cdot \left(1 - e^{-\beta c^2 x^2}\right)^2 \cdot$$

$$\left(1 - \left[\frac{\beta\,\mathrm{erf}(\sqrt{\beta}\cdot\sqrt{2c^2+1}\,\mathrm{x})}{\sqrt{2c^2+1}(4\beta c^2+2\beta)Q} + \frac{2\beta\,\mathrm{erf}(\sqrt{\beta}\sqrt{c^2+2}\,\mathrm{x})}{\sqrt{c^2+2}(2\beta c^2+4\beta)Q} - \frac{\beta\,\mathrm{erf}(\sqrt{2}\sqrt{\beta}\sqrt{c^2+1}\,\mathrm{x})}{2^{\frac{1}{2}}\sqrt{c^2+1}(4\beta c^2+4\beta)Q} - \frac{2\beta\,\mathrm{erf}(\sqrt{\beta}\cdot\sqrt{c^2+1}\,\mathrm{x})}{\sqrt{c^2+1}(2\beta c^2+2\beta)Q} - \frac{\mathrm{erf}(\sqrt{2}\sqrt{\beta}x)}{2^{\frac{5}{2}}Q} + \frac{\mathrm{erf}(\sqrt{\beta}\,\mathrm{x})}{2Q} + \frac{4\beta^{\frac{3}{2}}xe^{-\beta c^2x^2-\beta x^2}}{\sqrt{\pi}(2\beta c^2+2\beta)Q} - \frac{4\beta^{\frac{3}{2}}xe^{-\beta c^2x^2-2\beta x^2}}{\sqrt{\pi}(2\beta c^2+4\beta)Q} - \frac{2\beta^{\frac{3}{2}}xe^{-2\beta c^2x^2-\beta x^2}}{\sqrt{\pi}(4\beta c^2+2\beta)Q} + \frac{2\beta^{\frac{3}{2}}xe^{-2\beta c^2x^2-2\beta x^2}}{\sqrt{\pi}(4\beta c^2+4\beta)Q} - \frac{\sqrt{\beta}xe^{-\beta x^2}}{\sqrt{\pi}Q} + \frac{\sqrt{\beta}xe^{-2\beta x^2}}{2\sqrt{\pi}Q}\right]\right)^{-1}$$

5.6 Reverse Hazard Function $\varphi(x)$

Reverse Hazard function for DWERD is defines as:

$$\varphi(x) = \frac{f w_D(x;\beta,c)}{F w_D(x;\beta,c)}$$

Then by eq.'s (5) and (6) is given by:

$$\varphi(\mathrm{x}) = \frac{2\beta^{\frac{3}{2}}}{\sqrt{\pi}Q} \cdot x^2 e^{-\beta x^2}\left(1 - e^{-\beta x^2}\right)\left(1 - e^{-\beta c^2 x^2}\right)^2 \cdot$$

$$\left(\frac{\beta\mathrm{erf}(\sqrt{\beta}\cdot\sqrt{2c^2+1}\,\mathrm{x})}{\sqrt{2c^2+1}(4\beta c^2+2\beta)Q} + \frac{2\beta\mathrm{erf}(\sqrt{\beta}\sqrt{c^2+2}\,\mathrm{x})}{\sqrt{c^2+2}(2\beta c^2+4\beta)Q} - \frac{\beta\,\mathrm{erf}(\sqrt{2}\sqrt{\beta}\sqrt{c^2+1}\,\mathrm{x})}{2^{\frac{1}{2}}\sqrt{c^2+1}(4\beta c^2+4\beta)Q} - \frac{2\beta\,\mathrm{erf}(\sqrt{\beta}\cdot\sqrt{c^2+1}\,\mathrm{x})}{\sqrt{c^2+1}(2\beta c^2+2\beta)Q} - \frac{\mathrm{erf}(\sqrt{2}\sqrt{\beta}x)}{2^{\frac{5}{2}}Q} + \frac{\mathrm{erf}(\sqrt{\beta}\,\mathrm{x})}{2Q} + \frac{4\beta^{\frac{3}{2}}xe^{-\beta c^2x^2-\beta x^2}}{\sqrt{\pi}(2\beta c^2+2\beta)Q} - \frac{4\beta^{\frac{3}{2}}xe^{-\beta c^2x^2-2\beta x^2}}{\sqrt{\pi}(2\beta c^2+4\beta)Q} - \frac{2\beta^{\frac{3}{2}}xe^{-2\beta c^2x^2-\beta x^2}}{\sqrt{\pi}(4\beta c^2+2\beta)Q} + \frac{2\beta^{\frac{3}{2}}xe^{-2\beta c^2x^2-2\beta x^2}}{\sqrt{\pi}(4\beta c^2+4\beta)Q} - \frac{\sqrt{\beta}xe^{-\beta x^2}}{\sqrt{\pi}Q} + \frac{\sqrt{\beta}xe^{-2\beta x^2}}{2\sqrt{\pi}Q}\right)^{-1}$$

5.7 The Coefficients

- **The Coefficient of Variation**

The Coefficient of Variation for DWERD defines as: $\mathrm{CV} = \frac{\sigma}{\mu}$

Then by eq.'s (7) and (11) is given by:

$$CV = \left(\frac{3}{2\beta Q}\left[\frac{1}{2} - \frac{1}{(c^2+1)^{\frac{5}{2}}} + \frac{1}{2(1+2c^2)^{\frac{5}{2}}} - \frac{1}{(2)^{\frac{7}{2}}} + \frac{1}{(c^2+2)^{\frac{5}{2}}} - \frac{1}{2(2(c^2+1))^{\frac{5}{2}}}\right] - \frac{4}{\beta\pi Q^2}\left[\frac{3}{8} - \frac{1}{(c^2+1)^2} + \frac{1}{2(1+2c^2)^2} + \frac{1}{(c^2+2)^2} - \frac{1}{2(2(c^2+1))^2}\right]^2\right)^{\frac{1}{2}}.$$

[اكتب نصاً]

$$\left(\frac{2}{\beta^{\frac{1}{2}}\sqrt{\pi}Q}\left[\frac{3}{8}-\frac{1}{(c^2+1)^2}+\frac{1}{2(1+2c^2)^2}+\frac{1}{(c^2+2)^2}-\frac{1}{2(2(c^2+1))^2}\right]\right)^{-1}$$

- **The Coefficient of Skewness**

The Coefficient of Skewness for DWERD defines as:

$$CS = E\left(\frac{X-\mu}{\sigma}\right)^3 = \frac{E(X-\mu)^3}{\sigma^3} = \frac{\mu_3}{\sigma^3}$$

$$\mu_3 = E(X^3) - 3\mu\, E(X^2) + 3\mu^2 E(X) - \mu^3$$

Then by eq.'s (7), (8), (9), and (11) the CS is given by:

$$\text{CS} = \frac{4}{\beta^{\frac{3}{2}}\sqrt{\pi}Q}\left[\frac{7}{16}-\frac{1}{(c^2+1)^3}+\frac{1}{2(1+2c^2)^3}+\frac{1}{(c^2+2)^3}-\frac{1}{2(2(c^2+1))^3}\right]-$$

$$\frac{9}{\beta^{\frac{3}{2}}\sqrt{\pi}Q^2}\left[\frac{3}{8}-\frac{1}{(c^2+1)^2}+\frac{1}{2(1+2c^2)^2}+\frac{1}{(c^2+2)^2}-\frac{1}{2(2(c^2+1))^2}\right]\cdot$$

$$\left[\frac{1}{2}-\frac{1}{(c^2+1)^{\frac{5}{2}}}+\frac{1}{2(1+2c^2)^{\frac{5}{2}}}-\frac{1}{(2)^{\frac{7}{2}}}+\frac{1}{(c^2+2)^{\frac{5}{2}}}-\frac{1}{2(2(c^2+1))^{\frac{5}{2}}}\right]+$$

$$\frac{16}{\beta^{\frac{3}{2}}\pi^{\frac{3}{2}}Q^3}\left[\frac{3}{8}-\frac{1}{(c^2+1)^2}+\frac{1}{2(1+2c^2)^2}+\frac{1}{(c^2+2)^2}-\frac{1}{2(2(c^2+1))^2}\right]^3\cdot$$

$$\left(\frac{3}{2\beta Q}\left[\frac{1}{2}-\frac{1}{(c^2+1)^{\frac{5}{2}}}+\frac{1}{2(1+2c^2)^{\frac{5}{2}}}-\frac{1}{(2)^{\frac{7}{2}}}+\frac{1}{(c^2+2)^{\frac{5}{2}}}-\frac{1}{2(2(c^2+1))^{\frac{5}{2}}}\right]-\frac{4}{\beta\pi Q^2}\left[\frac{3}{8}-\frac{1}{(c^2+1)^2}+\frac{1}{2(1+2c^2)^2}+\frac{1}{(c^2+2)^2}-\frac{1}{2(2(c^2+1))^2}\right]^2\right)^{\frac{-3}{2}}$$

- **The Coefficient of Kurtosis**

The Coefficient of Kurtosis for DWERD is given by:

$$CK = \frac{\mu_4}{(\sigma^2)^2} - 3$$

$$\mu_4 = E(X^4) - 4\mu\, E(X^3) + 6\mu^2 E(X^2) - 4\mu^3 E(X) + \mu^4$$

Then by eq.'s (7), (8), (9), (10) and (11) the CK is given by:

$$CK = \frac{15}{4\beta^2 Q}\left[\frac{1}{2}-\frac{1}{(c^2+1)^{\frac{7}{2}}}+\frac{1}{2(1+2c^2)^{\frac{7}{2}}}-\frac{1}{(2)^{\frac{9}{2}}}+\frac{1}{(c^2+2)^{\frac{7}{2}}}-\frac{1}{2(2(c^2+1))^{\frac{7}{2}}}\right]$$

$$-\frac{32}{\beta^2\pi Q^2}\left[\frac{3}{8}-\frac{1}{(c^2+1)^2}+\frac{1}{2(1+2c^2)^2}+\frac{1}{(c^2+2)^2}-\frac{1}{2(2(c^2+1))^2}\right]\cdot$$

[اكتب نصاً]

$$\left[\frac{7}{16}-\frac{1}{(c^2+1)^3}+\frac{1}{2(1+2c^2)^3}+\frac{1}{(c^2+2)^3}-\frac{1}{2(2(c^2+1))^3}\right]+$$

$$\frac{36}{\beta^2\pi D Q^3}\left[\frac{3}{8}-\frac{1}{(c^2+1)^2}+\frac{1}{2(1+2c^2)^2}+\frac{1}{(c^2+2)^2}-\frac{1}{2(2(c^2+1))^2}\right]^2\cdot$$

$$\left[\frac{1}{2}-\frac{1}{(c^2+1)^{\frac{5}{2}}}+\frac{1}{2(1+2c^2)^{\frac{5}{2}}}-\frac{1}{(2)^{\frac{7}{2}}}+\frac{1}{(c^2+2)^{\frac{5}{2}}}-\frac{1}{2(2(c^2+1))^{\frac{5}{2}}}\right]-$$

$$\frac{48}{\beta^2\pi^2 Q^4}\left[\frac{3}{8}-\frac{1}{(c^2+1)^2}+\frac{1}{2(1+2c^2)^2}+\frac{1}{(c^2+2)^2}-\frac{1}{2(2(c^2+1))^2}\right]^4\cdot$$

$$\left(\frac{3}{2\beta Q}\left[\frac{1}{2}-\frac{1}{(c^2+1)^{\frac{5}{2}}}+\frac{1}{2(1+2c^2)^{\frac{5}{2}}}-\frac{1}{(2)^{\frac{7}{2}}}+\frac{1}{(c^2+2)^{\frac{5}{2}}}-\frac{1}{2(2(c^2+1))^{\frac{5}{2}}}\right]-\frac{4}{\beta\pi Q^2}\left[\frac{3}{8}-\frac{1}{(c^2+1)^2}+\frac{1}{2(1+2c^2)^2}+\frac{1}{(c^2+2)^2}-\frac{1}{2(2(c^2+1))^2}\right]^2\right)^{-2}-3$$

6. Estimation of the scale parameter

In this section, estimating the unknown scale parameter β of the (DWERD) by two methods as:

6.1 Method of moments (MOM)

The moments of the distribution and the moments of the samples must be equal in order to determine the moment estimates for β, by eq.(7) we get:

$$\bar{X}=\frac{2}{\beta^{\frac{1}{2}}\sqrt{\pi}Q}\left[\frac{3}{8}-\frac{1}{(c^2+1)^2}+\frac{1}{2(1+2c^2)^2}+\frac{1}{(c^2+2)^2}-\frac{1}{2(2(c^2+1))^2}\right]$$

Then by solving the equation above, we get:

$$\hat{\beta}_{MOM}=\left(2\left[\frac{3}{8}-\frac{1}{(c^2+1)^2}+\frac{1}{2(1+2c^2)^2}+\frac{1}{(c^2+2)^2}-\frac{1}{2(2(c^2+1))^2}\right]\cdot\left(\bar{X}\sqrt{\pi}Q\right)^{-1}\right)^2 \qquad (13)$$

Where c is constant.

6.2. Maximum Likelihood Method (MLE)

Take into consideration the size n random sample, x_2,..., x_n) for the DWERD with parameter β, then the likelihood function of eq. (5) given by:

$$Lf_{w_D}(x;\theta,c)=\prod_{i=1}^{n}\left(\frac{2\beta^{\frac{3}{2}}}{\sqrt{\pi}Q}\cdot x^2e^{-\beta x^2}\left(1-e^{-\beta x^2}\right)\cdot\left(1-e^{-\beta c^2x^2}\right)^2\right)$$

$$=\left(\frac{2}{\sqrt{\pi}Q}\right)^n\cdot(\beta)^{\frac{3n}{2}}\cdot\left(\prod_{i=1}^{n}x_i^2\right)\cdot e^{-\beta\sum_{i=1}^{n}x_i^2}\cdot\prod_{i=1}^{n}\left(1-e^{-\beta x^2{}_i}\right)\cdot\prod_{i=1}^{n}\left(1-\quad e^{-\beta c^2x^2{}_i}\right)^2$$

$$\ln l=n\,ln\left(\frac{2}{\sqrt{\pi}Q}\right)+\frac{3n}{2}ln\beta+\sum_{i=1}^{n}lnx_i{}^2-\beta\sum_{i=1}^{n}x_i{}^2+\sum_{i=1}^{n}ln\left(1-e^{-\beta x_i{}^2}\right)+2\sum_{i=1}^{n}ln\left(1-e^{-\beta c^2x_i{}^2}\right)$$

$$\frac{\partial \ln l}{\partial \beta}=\frac{3n}{2\beta}-\sum_{i=1}^{n} x_i{}^2+\sum_{i=1}^{n}\frac{x_i{}^2 e^{-\beta x_i{}^2}}{1-e^{-\beta x_i{}^2}}+2\sum_{i=1}^{n}\frac{c^2 x_i{}^2 e^{-\beta c^2 x_i{}^2}}{1-e^{-\beta c^2 x_i{}^2}}$$

$$\frac{3n}{2\beta}-\sum_{i=1}^{n} x_i{}^2+\sum_{i=1}^{n}\frac{x_i{}^2 e^{-\beta x_i{}^2}}{1-e^{-\beta x_i{}^2}}+2\sum_{i=1}^{n}\frac{c^2 x_i{}^2 e^{-\beta c^2 x_i{}^2}}{1-e^{-\beta c^2 x_i{}^2}}=0 \qquad (14)$$

To find $\hat{\beta}_{MLE}$ from non- linear equation (14) must be solved by one of the numerical methods.

7. Simulation

Because of the advantages that come from using simulation method to generate a particular distribution of data, in order to find the best estimate of the parameter β for this distribution, is a worthwhile approach because repeating the experiment by changing the inputs given each time provides a detailed explanation and is essential for the nature of the mathematical process used [1].

7.1. Experimentation phase of the simulation

There are three basic phases, and the following stages are used to estimate the unknown scale parameter for DWERD: [1]

First stage: Using simulation, create a certain data distribution. The outcome of numerical experiments is shown here, based on Monte Carlo in MATLAB (2018a) the performance of the two estimators with sample size (n=10, 50, 100), the parameter default values ($\beta = 2, 3, 5$), as well as(c = 2, 3) with replicated (ψ =1000) time by using equations (13) and the solving of eq. (14)

Second stage: In this stage, we generated DWERD data depend on equation (6) as following:

Assume that $U = F_{w_D}(x)$,where U is a Uniformly distributed random variable on the interval (0,1) . Since $F_{w_D}(x)$ is nonlinear function and by utilize Newton Raphson method to solve the following equation by find the root of it, then$F_{w_D}(x)$-U=0

$$\Rightarrow \Big(\frac{\beta\,\mathrm{erf}(\sqrt{\beta}\cdot\sqrt{2c^2+1}\,x)}{\sqrt{2c^2+1}(4\beta c^2+2\beta)Q}+\frac{2\beta\,\mathrm{erf}(\sqrt{\beta}\sqrt{c^2+2}\,x)}{\sqrt{c^2+2}(2\beta c^2+4\beta)Q}-\frac{\beta\,\mathrm{erf}(\sqrt{2}\sqrt{\beta}\sqrt{c^2+1}\,x)}{2^{\frac{1}{2}}\sqrt{c^2+1}(4\beta c^2+4\beta)Q}-\frac{2\beta\,\mathrm{erf}(\sqrt{\beta}\cdot\sqrt{c^2+1}\,x)}{\sqrt{c^2+1}(2\beta c^2+2\beta)Q}-\frac{\mathrm{erf}(\sqrt{2}\sqrt{\beta}x)}{2^{\frac{5}{2}}Q}+\frac{\mathrm{erf}(\sqrt{\beta}\,x)}{2Q}+\frac{4\beta^{\frac{3}{2}}xe^{-\beta c^2x^2-\beta x^2}}{\sqrt{\pi}(2\beta c^2+2\beta)Q}-\frac{4\beta^{\frac{3}{2}}xe^{-\beta c^2x^2-2\beta x^2}}{\sqrt{\pi}(2\beta c^2+4\beta)Q}-\frac{2\beta^{\frac{3}{2}}xe^{-2\beta c^2x^2-\beta x^2}}{\sqrt{\pi}(4\beta c^2+2\beta)Q}+\frac{2\beta^{\frac{3}{2}}xe^{-2\beta c^2x^2-2\beta x^2}}{\sqrt{\pi}(4\beta c^2+4\beta)Q}-\frac{\sqrt{\beta}xe^{-\beta x^2}}{\sqrt{\pi}Q}+\frac{\sqrt{\beta}xe^{-2\beta x^2}}{2\sqrt{\pi}D}\Big)-U=0$$

$$\text{Where : } Q=\frac{2\sqrt{2}-1}{4\sqrt{2}}-\frac{4\sqrt{2}+1}{4\sqrt{2}(c^2+1)^{\frac{3}{2}}}+\frac{1}{2(2c^2+1)^{\frac{3}{2}}}+\frac{1}{(c^2+2)^{\frac{3}{2}}}$$

Third stage: In this stage, two estimators for the scale parameter are compared using the Mean Square Error (MSE) criteria as follows:

$$MSE(\hat{\beta})=\frac{1}{\psi}\sum_{i=1}^{\psi}(\hat{\beta}-\beta)^2$$

7.2. Simulation Results

[اكتب نصاً]

This section shows how the simulation estimation method works in obtaining the best estimates for scale parameter disputes between the estimator's values based on comparison criteria; the MSE values are shown in tables (1) and (2).

Table (1) MSE values of scale parameter β when $c = 2$ for DWERD

$\beta = 2$			
Method	**MOM**	**MLE**	**Best**
$n = 10$	0.197713	0.203195	MOM
$n = 50$	0.033635	0.034021	MOM
$n = 100$	0.016662	0.017804	MOM
$\beta = 3$			
Method	**MOM**	**MLE**	**Best**
$n = 10$	0.434725	0.468557	MOM
$n = 50$	0.075436	0.081128	MOM
$n = 100$	0.036089	0.037314	MOM
$\beta = 5$			
Method	**MOM**	**MLE**	**Best**
$n = 10$	1.346009	1.402081	MOM
$n = 50$	0.213505	0.223571	MOM
$n = 100$	0.102643	0.106105	MOM

Table (2) MSE values of scale parameter β when $c = 3$ for DWERD

$\beta = 2$			
Method	**MOM**	**MLE**	**Best**
$n = 10$	0.162677	0.193154	MOM
$n = 50$	0.030466	0.040004	MOM
$n = 100$	0.015556	0.024821	MOM
$\beta = 3$			
Method	**MOM**	**MLE**	**Best**
$n = 10$	0.419124	0.491982	MOM
$n = 50$	0.070034	0.095835	MOM
$n = 100$	0.034874	0.051789	MOM
$\beta = 5$			

Method	MOM	MLE	Best
$n = 10$	1.127455	1.347771	MOM
$n = 50$	0.206806	0.288922	MOM
$n = 100$	0.094867	0.139222	MOM

7.3. Analysis of Results

The results of simulation in tables (1) and (2) show that:

1- When the sample size is increased, the MSE values are small and positively decrease, according to the results of using the DWERD distribution, this is consistent with the statistical theory.

2-While the results of the two methods (MOM and MLE) when compared, MOM generally produces the best results in all sample size because it has lower MSE from the MLE method.

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