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Intuitionistic Fuzzy –k- Quasi-(n,m)- Normal Operator

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Abstract:

We introduce a family of operators called the family of an IF-k-quasi-(n,m)-normal operator .Such family includws IF-n-normal and IF-(n,m)-normal operators .An operator $A \in B(H)$ is called IF-k-quasi-(n,m)-normal operator if it satisfies

$$\rho_{\mu\nu}\left(A^{*k}(A^n A^{*m} - A^{*m} A^n)A^k\right)u,t) = 0$$

where k,n,and m are natural numbers . Firstly , some basic structural properties of this family of operators are established with the help of special kind of operator matrix representation associated with such family of operators. Secondaly some properties of algebrically IF-k-quasi-(n,m)-normal operators are discussed . Thirdily , we consider the study of tensor product of IF-k-quasi-(n,m)-normal operator. Anecessary and sufficient condition for $A \otimes S$ be IF-k-quasi-(n,m)-normal operator is given when $A \neq 0$ and $S \neq 0$.

Keywords: Intuitionistic Fuzzy-n - normal , Intuitionistic Fuzzy- (n,m)- normal , Intuitionistic Fuzzy-k -quasi-(n,m)-normal)

1-Introduction

Let $B(H)$ be the algebra of all bounded linear operators defined on H, and let H be a complex Hilbert space, In 2004 [3] , Park defined intuitionistic fuzzy metric spaces, Latir in 2006 [7] , Saadati defined the intuitionistic fuzzy metric and norm .After that,[5] in 2009, Goudarzi et al. interduced intuitionistic fuzzy inner product(IFIP-space). In 2018 [1],[2] , Radharamani et al ,provied intuitionistic fuzzy adjoint and self-adjoint operators (IFA and IFSA-operators) . In 2010 Shqipe Lohaj study Quasi-normal operators [8],In 2015 Laith K.Shaakir defined quasi –normal operator of order n [4] .In 2021 Naeem Ahmad study On class of k-quasi-(n,m)- normal operators[6]. In this work There are three sections

This paper is devoted to some class of operators on the Hilbert space which is a generalization of IF-normal and IF-(n,m)-normal operators .More precisely , we intrduce a new class of operators which is called IF-k-quasi-(n,m0-normal operator .It proved in example 2.2 that there is an operator which is an IF-k-quasi-(n,m)-normal , but not IF-(n,m)-normal operator .We characterize this class of operators in terms of IF-(n,m)-normal operator on subspace $\overline{R(A^n)}$ (Lemma 2.1) and we studies some results are obtained .

2- IF-k-quasi (n, m)- normal operator

Definition 2.1

Let $(\mathcal{V},\mathcal{F}_{\mu\nu},\mathcal{T})$ be IFH space with IP:$\langle x,y\rangle = sup\{t\in R\ ,\mathcal{F}_{\mu\nu}(x,y,t)<1\}\ \forall\ x,y\in\mathcal{V}$ and if

$\rho_{\mu v}(A^{*k}(A^nA^{*m}-A^{*m}A^n)A^k)u,t)=0$. (2.1)

We state that $A\in B(K)$ an IF-k-quasi (n,m)- normal operator ,for certain positive integers k,n, and m.

Example 2.1

Let A be a matrix in $\mathcal{B}(\mathcal{V})$, such that $A=\begin{pmatrix}1&0\\0&1\end{pmatrix}$, then A an IF-k-quasi-(n,m)-normal operator .

Proof:

$A=\begin{pmatrix}1&0\\0&1\end{pmatrix}$, $A^*=\begin{pmatrix}1&0\\0&1\end{pmatrix}$

$\rho_{\mu v}(A^{*k}(A^nA^{*m}-A^{*m}A^n)A^k)u,t)$

$=\rho_{\mu v}\left(\begin{pmatrix}1&0\\0&1\end{pmatrix}^{*k}\left(\begin{pmatrix}1&0\\0&1\end{pmatrix}^{n}\begin{pmatrix}1&0\\0&1\end{pmatrix}^{*m}-\begin{pmatrix}1&0\\0&1\end{pmatrix}^{*m}\begin{pmatrix}1&0\\0&1\end{pmatrix}^{n}\right)\begin{pmatrix}1&0\\0&1\end{pmatrix}^{k}u,t\right)$

$=\rho_{\mu v}\left(\begin{pmatrix}1&0\\0&1\end{pmatrix}\left(\begin{pmatrix}1&0\\0&1\end{pmatrix}\begin{pmatrix}1&0\\0&1\end{pmatrix}-\begin{pmatrix}1&0\\0&1\end{pmatrix}\begin{pmatrix}1&0\\0&1\end{pmatrix}\right)\begin{pmatrix}1&0\\0&1\end{pmatrix}u,t\right)$

$=\rho_{\mu v}\left(\begin{pmatrix}1&0\\0&1\end{pmatrix}\left(\begin{pmatrix}1&0\\0&1\end{pmatrix}-\begin{pmatrix}1&0\\0&1\end{pmatrix}\right)\begin{pmatrix}1&0\\0&1\end{pmatrix}\right)u,t)$

$=\rho_{\mu v}\left(\begin{pmatrix}1&0\\0&1\end{pmatrix}\begin{pmatrix}0&0\\0&0\end{pmatrix}\begin{pmatrix}1&0\\0&1\end{pmatrix}\right)u,t)$

$=\rho_{\mu v}\left(\begin{pmatrix}0&0\\0&0\end{pmatrix}u,t\right)$

A is IF-k-quasi-(n, m)-normal operator

Lemma 2.1 :

Suppose that $(\mathcal{V},\mathcal{F}_{\mu\nu},\mathcal{T})$ be IFH with IP:$\langle x,y\rangle = \sup\{t\in R,\mathcal{F}_{\mu\nu}(x,y,t)<1\}\forall\ x,y\in\mathcal{V}$ and let $A\in B(K)$ is IF-k-quasi (n, m)-power normal iff it is IF-(n, m)-power normal on $\overline{R(A^k)}$

proof:

A is IF-(n, m)-power normal

$\rho_{\mu v}\Big((A^{*k}(A^nA^{*m}-A^{*m}A^n)A^k)u,t\Big)=0$

$\langle A^{*k}(A^nA^{*m}-A^{*m}A^n)A^kx,|x\rangle=0$

$\langle (A^nA^{*m} - A^{*m}A^n)A^kx, A^kx\rangle = 0$
$A^nA^{*m} - A^{*m}A^n = 0$

Proposition 2.1:
Suppose that $(\mathcal{V}, \mathcal{F}_{\mu\nu}, \mathcal{T})$ be IFH with IP:$\langle x, y\rangle = \{t \in R, \mathcal{F}_{\mu\nu}(x, y, t) < 1\}\forall\, x, y \in \mathcal{V}$ and let $A \in B(K)$ is an IF-k-quasi (n, m)-power normal .When $N(A^*) \subset N(A)$ then $A^*$is IF-k-quasi (m, n)-normal.

Proof:
As A IF-k-quasi (n, m)-normal
$\rho_{\mu\nu}\left((A^{*k}(A^nA^{*m} - A^{*m}A^n)A^k)u, t\right) = 0$
Under the assumption $N(A^{*k})I \subset N(A)$
$\rho_{\mu\nu}(A(A^nA^{*m} - A^{*m}A^n)A^k)u, t) = 0$
And hence
$\rho_{\mu\nu}\left((A^{*k}(A^mA^{*n} - A^{*n}A^m)A^*)u, t\right) = 0$
$\rho_{\mu}(A(A^mA^{*n} - A^{*n}A^m)A^*)u, t) = 0$
$\rho_{\mu\nu}\left((A^k(A^mA^{*n} - A^{*n}A^m)A^{*k})u, t\right) = 0$
Thus $A^*$is IF-k-quasi-(m, n)-power normal

Theorem2.1:
Suppose that $(\mathcal{V}, \mathcal{F}_{\mu\nu}, \mathcal{T})$ be IFH with IP:$\langle x, y\rangle = sup\{t \in R, \mathcal{F}_{\mu\nu}(x, y, t) < 1\}\ \forall\, x, y \in \mathcal{V}$, and let A be IF-k-quasi-(n, m)- power normal operators . If IF-$N(A^{*q}) = N\left(A^{*(q+1)}\right)$ for some $1 \leq q \leq k - 1$ then , A is an IF-q-quasi- (n, m)-power normal.

Proof:
Assuming that $N(A^{*q}) = N(A^{*(q+1)})$ we have $N(A^{*q}) = N(A^{*n})\forall k \in \mathbb{N}, k \geq 2$. From the hypothesis , A is IF-k-quasi (n, m)- normal operator , then
$\rho_{\mu\nu}\left((A^{*k}(A^nA^{*m} - A^{*m}A^n)A^k)u, t\right) = 0$
Since $N(A^{*q}) = N(A^{*n})$
then a simple calculation demonstrates that
$\rho_{\mu\nu}\left((A^{*q}(A^nA^{*m} - A^{*m}A^n)A^q)u, t\right) = 0$
Thus, it can be said that A is IF-q-quasi (n, m)- normal operator .

Remark 2.1:
In the example below, we demonstrate that(theorem 2.1) is false in general if $N(A^{*q}) \neq N\left(A^{*(q+1)}\right)$.

Example 2.2:
Let the operator A=$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ acting on the two dimensional Hilbert space $\mathbb{C}^2$.then a direct calculation finds that A is IF- 2-quasi- (1,1)- normal but it is not a IF-quasi (1,1)- normal .
Nevertheless $N(A^*) \neq N(A^{*q})$

Proof:

$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad , A^* = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad , A^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$A^{*2} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$\rho_{\mu v}((A^{*2}(AA^* - A^*A)A^2)u, t)$

$= \rho_{\mu v}\left(\left(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\left\{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right\}\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\right)u, t\right)$

$= \rho_{\mu v}(\left(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\left\{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right\}\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\right)u, t)$

$= \rho_{\mu v}((\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix})u, t)$

$= \rho_{\mu v}\left(\left(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\right)u, t\right)$

$= \rho_{\mu v}\left(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}u, t\right) = 0$

A is IF-2-quasi-(1,1)-power normal

$\rho_{\mu V}\left((A^*(AA^* - A^*A)A)u, t\right)$

$= \rho_{\mu v}\left(\left(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\left\{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right\}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right)u, t\right)$

$= \rho_{\mu v}\left(\left(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\left\{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right\}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right)u, t\right)$

$= \rho_{\mu v}\left(\left(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right)u, t\right)$

$= \rho_{\mu v}\left(\left(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right)u, t\right)$

$= \rho_{\mu v}\left(\left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right)u, t\right) \neq 0$

A not IF-quasi-(1,1)-power normal

$N(A^*) = \{x, y \in R / A^*(x) = 0\}$

$= \left\{x, y \in R\,, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0\right\}$

$= (x, 0) \in R^2$

$N(A^{*2}) = \{x, y \in R, A^{*2}(x) = 0\}$

$= \left\{x, y \in R, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0\right\}$

$=(0,0) \in R^2$

$\therefore N(A^*) \neq N(A^{*2})$

Proposition 2.2:

Suppose that $(\mathcal{V}, \mathcal{F}_{\mu\nu}, \mathcal{T})$ be IFH with IP:$\langle x,y\rangle = sup\{t \in R, \mathcal{F}_{\mu\nu}(x,y,t) < 1\}$ $\forall x,y \in \mathcal{V}$and Let M be a closed subspace of K that reduces A, and let A be the IF-k-quasi- (n,m)- normal operator $A\setminus M$ is then IF-k-quasi (n,m)- normal operator .

Proof:

Assuming M is a decreasing subspace of A, then

A=$\begin{pmatrix} A_1 & 0 \\ 0 & A_3 \end{pmatrix}$ $on\ K = M \oplus M^{\perp}$

As A is an IF-k-quasi (n, m)- normal, we have $\rho_{\mu\nu}\left((A^{*k}(A^nA^{*m} - A^{*m}A^n)A^k)u,t\right) = 0$

$$\Rightarrow \rho_{\mu\nu}\left(\left(\begin{pmatrix} A_1 & 0 \\ 0 & A_3 \end{pmatrix}^{*k}\left[\begin{pmatrix} A_1 & 0 \\ 0 & A_3 \end{pmatrix}^{n}\begin{pmatrix} A_1 & 0 \\ 0 & A_3 \end{pmatrix}^{*m} - \begin{pmatrix} A_1 & 0 \\ 0 & A_3 \end{pmatrix}^{*m}\begin{pmatrix} A_1 & 0 \\ 0 & A_3 \end{pmatrix}^{n}\right]\begin{pmatrix} A_1 & 0 \\ 0 & A_3 \end{pmatrix}^{k}\right)u,t\right) = 0$$

$$\begin{pmatrix} \rho_{\mu\nu}((A^{*k}(A^nA^{*m} - A^{*m}A^n)A^k)u,t) & 0 \\ 0 & V \end{pmatrix} = 0$$

For some operators V, this indicates that

$\rho_{\mu\nu}\left((A^{*k}(A^nA^{*m} - A^{*m}A^n)A^k)u,t\right) = 0$ □

Thus $A_{1=A/_M}$ *is IF-k-quasi-* (n,m)*-normal operator ..*

Corollarly 2.1:

Suppose that $(\mathcal{V}, \mathcal{F}_{\mu\nu}, \mathcal{T})$ be IFH with IP:$\langle x,y\rangle = \sup\{t \in R, \mathcal{F}_{\mu\nu}(x,y,t) < 1\}\forall\, x,y \in \mathcal{V}$ and $A \in B(K)$ is an IF-n- normal operator . The following assertions holds

1.$\overline{A^n(K)}$ reduces A

2. A is represented by the following matrix

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} on\ K = \overline{A^n(K)} \oplus N(A^{*n})$$

where$A_1 = A\setminus\overline{A^n(K)}$ is IF-n-power normal , A_2 is nilpotent.

Furthermore $\sigma(A) = \sigma(A_1) \cup \{0\}$.

Proof:

Since $\overline{R(A^n)}$ reduce A Hence A is represented by a matrix.,$A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$ $on\ K = \overline{R(A^n)} \oplus N(A^{*n})$

. P should represent the orthogonal projection of $\overline{R(A^n)}$.

Then $\begin{pmatrix} A_1 & 0 \\ 0 & 0 \end{pmatrix} = AP = PA = PAP$

$P(A^*A^n)P = \begin{pmatrix} {A_1}^n{A_1}^* & 0 \\ 0 & 0 \end{pmatrix}$.*Also* $P(A^*A^n)P$ =$\begin{pmatrix} {A_1}^*{A_1}^n & 0 \\ 0 & 0 \end{pmatrix}$.

Since A is IF-n-normal

$\rho_{\mu\nu}\left((P(A^nA^*)P - P(A^*A^n)P)u,t\right) = 0$

implying

$\rho_{\mu\nu}\left((A^nA^* - A^*A^n)u,t\right) = 0$

Hence A_1 *IF-n-power normal*

For any $z=\binom{z_1}{z_2} \in K$,
$sup\{\ t \in R, \mathcal{F}_{\mu\nu}\big((A_2{}^n z_2), z_2, \big), t < 1\big\}$
$= sup\{t \in R, \mathcal{F}_{\mu\nu}(A^n(I-P)z, (I-P)z,), t) < 1\}$
$= sup\{t \in R, \mathcal{F}_{\mu\nu}, (I-P)z, A^{*n}(I-P)z), t) < 1\}$
$= 0$
ThereforA_2^n *.Since* $R(A^n)$*reduces,*$\sigma(A) = \sigma(A_1) \cup \{0\}$

3. IF-k- quasi -(n,m)- normal operators are algebraically

Definition 3.1:

let$(\mathcal{V}, \mathcal{F}_{\mu\nu}, \mathcal{T})$ be IFH with IP: $\langle x,y\rangle = sup\{t \in R, \mathcal{F}_{\mu\nu}(x,y,t) < 1\}\forall x,y \in \mathcal{V}$ algebraically relates to an operator A$\in B(K)$ IF-k-quasi-(n,m)-normal operator if an irregular polynomial $\mathbb{Q} \in \mathbb{C}[z]$ exists such that $\mathbb{Q}(A)$ is IF-k-quasi-(n,m)- normal

Lemma 3.1:

Suppose that $(\mathcal{V}, \mathcal{F}_{\mu\nu}, \mathcal{T})$be IFH with IP:$\langle x,y\rangle = sup\{tR, \mathcal{F}_{\mu\nu}(x,y,t) < 1\}\forall x,y \in \mathcal{V}$, Given that $\alpha \in \mathbb{C}. if\ \sigma(A) = \{\alpha\}$and A$\in B(K)$ is an IF-(n,m)-power normal, there exists a positive integer j so that $A^j = \alpha^j I$

Proof:

We look at two situations:

$(\mathbf{1})\alpha = 0$ It follows that A^j is IF-normal if A is IF-(n,m)-power normal, where j is the least common multiple of n and m

A^j is hence normaloid Thus, A^j=0

$(\mathbf{2})$ $\alpha \neq 0$ In light of the fact that A is invertible and IF-(n,m)-power normal, A^{-1} is also IF-(n,m)-power normal. A^{-1} is normaloid, so that's why. In addition, $\sigma(A^{-1}) = \left\{\frac{1}{\alpha^j}\right\}$.,

therefore

$\|A^j\|\|A^{-j}\| = |\alpha^j|\left|\frac{1}{\alpha^j}\right| = 1$

We infer that A^j is convexoid, therefore$\mathbb{W}(A^j) = \{\alpha^j\}$, where $\mathbb{W}(A^j)$ is the numerical range of A^j, and $A^j = \alpha^j I$

Theorem:3.1

Suppose that $(\mathcal{V}, \mathcal{F}_{\mu\nu}, \mathcal{T})$ be IFH with IP: $\langle x,y\rangle = sup\{t \in R, \mathcal{F}_{\mu\nu}(x,y,t) < 1\}\forall x,y \in \mathcal{V}$. the algebraic IF-(n,m)-power normal operator $A \in B(K)$ be the case A is isoloid if $A-\alpha$ is an algebraically IF-(n,m)-power normal operator for $\alpha \in iso\ \sigma(A)$

Proof:

Assume that $\alpha \in iso\ \sigma(A)$ and take into account $\mathbb{P}_\alpha := \frac{1}{2i\pi}\int_{\partial\mathbb{D}(\alpha,r)^-}(\beta - A)^{-1}d\beta$ the

$Riesz\ idempotent$ of A related to α where $D(\alpha, r)^-$ is a closed disk centered at α and contains no other point of c then A can be expressed as

$A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$ $with\ \sigma(A_1) = \{\alpha\}\ and\ \sigma(A_2) = \sigma(A) - \{\alpha\}$

It follows that there exists an irregular polynomial $\mathbb{Q}$ for which $\mathbb{Q}$(A) is algebraically IF-(n, m)-power normal if A is assumed to be IF-(n, m)-power normal We derive $\sigma\big(\mathbb{Q}(A_1)\big) = \mathbb{Q}(\sigma(A_1)) = \{\mathbb{Q}(\alpha)\}$ from the equivalence $\sigma(A_1) = \{\alpha\}$

$\mathbb{Q}(A_1) - \mathbb{Q}(\alpha)$ is hence IF-quasinilpotent that there is an integer j that is positive and for which $\mathbb{Q}(A_1)^j - \mathbb{Q}(\alpha)^j = 0$ because $\mathbb{Q}(A_1)$ is IF-(n, m)-power normal

Put $\mathbb{q}(z) \coloneqq \mathbb{Q}(z)^j - \mathbb{Q}(\alpha)^j$ so that $\mathbb{q}(A_1)$=0

therefore A_1 is algebraically equal By noticing that $A_1 - \alpha$ is IF- quasinilpotent and algebraically IF-(n, m)-power normal, we can deduce that $A_1 - \alpha$ is nilpotent

Therefore, $\alpha \in (\pi_1)$ and $\alpha \in \pi(A)$

Consequently, A is an isoloid

If a nonconstant polynomial Q exists such that $Q(a)$ is, then the expression IF-k-quasi-(n, m)- normal operator .

Proposition 3.1

Suppose that $(\mathcal{V}, \mathcal{F}_{\mu\nu}, \mathcal{T})$ be IFH with IP: $\langle x, y\rangle = sup\{t \in R, \mathcal{F}_{\mu\nu}(x, y, t) < 1\} \forall x, y \in \mathcal{V}$. $A \in B(K)$ be an algebraic IF-k-quasi-(n, m)-power normal if A-α is an algebraically IF- k-quasi-(n, m)-normal operator for $\alpha \in iso\ \sigma(A)$. A is a polaroid and isoloid

Proof:

As a result that A is algebraically IF-k-quasi-(n, m)- normal operator , there exists a nonconstsnt polynomial $\mathbb{Q}$ so that $\mathbb{Q}(A)$ is IF-k-quasi-(n, m)- normal operator .

let $\alpha \in iso\ \sigma(A)$ and consider the spectral projection

$$\mathbb{P}_{\boldsymbol{\alpha}} = \frac{1}{2i\pi}\int_{\partial D_{(\alpha,r)^-}} (\beta - A)^{-1} d\beta$$

where $D(\alpha, r)^-$ is a closed disk of center m such that $D(\alpha, r)^- \cap \sigma(A) = \{\alpha\}$, we can define A as the direct sum $A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$ $with\ \sigma(A_1) = \{\beta_1\}\ and\ \sigma(A_2) = \sigma(A) - \{\alpha\}$. we see

$$\mathbb{Q}(A) = \begin{pmatrix} \mathbb{Q}(A_1) & 0 \\ 0 & \mathbb{Q}(A_2) \end{pmatrix}$$

and by the fact that $\mathbb{Q}(A)$ is an IF-k-quasi-(n, m)-power normal it follow that $\mathbb{Q}(A_1)$ is an IF-k-quasi-(n, m)- normal operator so $A_1 - \alpha$

$A_1 - \alpha$ is IF-quasinilpotent and nilpotent since $\sigma(A_1 - \alpha) = \{0\}$

This implies that $A_1 - \alpha$ has finite ascent and descent $A_2 - \alpha$ obviously has finite ascent and descent because it is invertible As a result, $A - \alpha$ has finite ascent and descent, and α is the pole of A is resolvent

Consequently, $\alpha \in iso\ \sigma(A) \Longrightarrow\ \alpha \in \pi(A)$, and $iso\ \sigma(A) \subset \sigma(A)$

So, A is polaroid

4- tenso product on IF-k-quasi-(n,m)-normal operator

Definition 2.3.1:

Let $B(K)$ be a set of on linear operator on Hilbert space K , $(B(K))^*$ be a set of dual operator on a vector space K

The map $\tau: B(K) \to (B(K))^*$

$\tau(A)(f) = f(A), A \in B(K)\, , f \in (B(K))^*$

Let $\tau_r^s \subseteq B(K)$ is the set of all maps

$\tau: B(K) \times B(K) \underbrace{\dots\dots}_{r-times} \times B(K) \times B(K)^* \times B(K)^* \underbrace{\dots\dots}_{s-times} \times B(K)^* \to \mathcal{K}$

Which is $\mathcal{K}$ linear in every argument its elements are called $r - times$ convariants and $s - times$ contravariants tensors on $B(K)$ for short called tensor of type (r, s) and denoted by τ_r^s of type (r, s)

Definition 4.1:

Define operation

$\tau_{r_1+r_2}^{s_1+s_2}(B(K)) \ \otimes: \tau_{r_1}^{s_1}(B(K)) \times \tau_{r_2}^{s_2}(B(K)) \to$

As the followings :

$A_1 \in \tau_{r_1}^{s_1}(B(K)), A_2 \in \tau_{r_2}^{s_2}(B(K))$ then

$A_1 \otimes A_2(u_1, \dots u_{r_1+r_2}, v^1, \dots, v^{s_1+s_2})$

$= A_1\big(u_1, \dots, u_{r_1}, v^1, \dots, v^{s_1}\big)$

$A_2\big(u_{r_1+1}, \dots, u_{r_1+r_2}, v^{s_1+1}, \dots, v^{s_1+s_2}\big)$

Where $u_1, \dots u_{r_1+r_2} \in B(K)$

$v^1, \dots, v^{s_1+s_2} \in (B(K))^*$

The operation $\otimes$ which has the above properties is called a tensor product

Theorem 4.1:

Suppose that $(\mathcal{V}, \mathcal{F}_{\mu\nu}, \mathcal{T})$ be IFH with IP: $\langle x, y\rangle = sup\{t \in R, \mathcal{F}_{\mu\nu}(x, y, t) < 1\} \forall x, y \in \mathcal{V}$ $A \otimes S$ is IF-k-quasi-(n, m)- normal operator iff one of the following conditions is true for $S \in B(K)$ and $A \in B(K)$, assuming that $A, S \neq 0$.

(***I***) IF-k-quasi-(n,m)- normal operators are A and S

(***II***) A constant $\mathbb{c} \in \mathbb{C}/\{0\}$ exists so that $\begin{cases} \rho_{\mu\nu}\big((A^{*k}A^nA^{*m}A^k - \mathbb{c}A^{*k+m}A^{k+m})u, t\big) = 0 \\ \rho_{\mu\nu}\big((S^{*k}S^nS^{*m}S^k - \frac{1}{\mathbb{c}}S^{*k+m}S^{k+m})u, t\big) = 0 \end{cases}$

Proof:

Direct calculation demonstrates

$\rho_{\mu\nu}(((A \otimes S)^{*k}[(A \otimes S)^n(A \otimes S)^{*m} - (A \otimes S)^{*m}(A \otimes S)^n](A \otimes S)^k$

$-(A^{*k}A^nA^{*m}A^n \otimes S^{*k}S^nS^{*m}S^k - A^{*k}A^{*m}A^nA^k \otimes S^{*k}S^{*m}S^nS^k))u, t) = 0$

$A \otimes S$ is hence definitely IF-k-quasi-(n,m)-power normal if either

(I) or (II) holds

On the other hand, suppose $A \otimes S$ is an IF-k-quasi-(n, m)-power normal operator due to the prior equality

$\rho_{\mu\nu}\big((A^{*k}A^nA^{*m}A^k \otimes S^{*k}S^nS^{*m}S^k - A^{*k}A^{*m}A^nA^k \otimes S^{*k}S^{*m}S^nS^k)u, t\big) = 0$

$\rho_{\mu v}\Big((A^{*k}A^{*m}A^{n}A^{k}\otimes S^{*k}S^{*m}S^{n}S^{k} - A^{*k+m}A^{k+m}S^{*k+m}S^{k+m})u, t\Big) = 0$

The constant $\mathbb{c} \neq 0$ exists for which

$$\begin{cases} \rho_{\mu v}\Big((A^{*k}A^{n}A^{*m}A^{k} - \mathbb{c}A^{*k+m}A^{k+m})u,t\Big)=0 \\ \rho_{\mu v}\Big(\big(S^{*k}S^{n}S^{*m}S^{k} - \frac{1}{\mathbb{c}}S^{*k+m}S^{k+m}\big)u,t\Big)=0 \end{cases}$$

A and S are IF-k-quasi-(n,m)- normal operators if $\mathbb{c} = 1$, and they satisfy the requirement(II) if $\mathbb{c} \neq 1$

Theorem 4.2:

let$(\mathcal{V}, \mathcal{F}_{\mu v}, \mathcal{T})$ be IFH with IP: $\langle x, y\rangle = sup\{t \in R, \mathcal{F}_{\mu v}(x, y, t) < 1\}\forall x, y \in \mathcal{V}$,$A\otimes S$ is IF-k-quasi-(j,p)-power normal for any $p \in N$ where $j = LCM$ if A$\in B(K)$ and $S \in B(K)$ are IF-k-quasi-(n,m)-power normal operators (n,m)

Proof:

Everyone is knows that

$A\otimes S = (A\otimes I)(I\otimes S) = (I\otimes S)(A\otimes I)$

$A\otimes I$ and $I\otimes S$ are IF-k-quasi-(n,m)-power normal applying

, since A and S are IF-k-quasi-(n,m)-power normal.

Consequently, $A\otimes S$ is an IF-k-quasi-(j,p)-power normal function

6-Conclusion

We define intuitionistic fuzzy- k-quasi-(n,m)-Normal operator on IFH-space. We also develop several theorems for the intuitionistic fuzzy-k-quasi-(n,m)-Normal operator and define it, proved an example that in general $\mathrm{N}(\mathrm{A}^{*\mathrm{q}}) \neq \mathrm{N}\big(\mathrm{A}^{*(\mathrm{q}+1)}\big)$

Definition Al-gebraically intuitionistic fuzzy- k-quasi-(n,m)-normal operator on IFH-space with some of its result , Definition tensor product on Hilbert space and proved for any two operators is intuitionistic fuzzy- k-quasi-(n,m)-normal operator.

then the tensor product for the operators is intuitionistic fuzzy- k-quasi-(n,m)-normal operator

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