Journal of Education for Pure Science- University of Thi-Qar
Vol.13, No.3 (Sept.2023)
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DOI: http://doi.org/10.32792/utq.jceps.10.01.01

Fixed Point Theorems in Fuzzy Soft Rectangular b- Metric Space

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Received 4/6/2023, Accepted 12/6/2023, Published 21/9/2023

Abstract.
In this study, we construct the fuzzy soft rectangular b-metric space first, and then define the fuzzy soft convergence sequence, also known as the fuzzy soft Cauchy sequence, in this space. In addition, we defined fuzzy soft contraction mapping and proved its fixed point in fuzzy soft rectangular b-metric space.

Keywords: fuzzy soft sets, fuzzy soft rectangular b- metric space, fuzzy soft Hausdorff *b- metric space*, fuzzy soft limit point, fuzzy soft Cauchy sequence.

1- Introduction

Fuzzy set theory, developed by Zadeh [1]as well as the theory of soft sets, developed by Molodtsov [2], are categories of mathematical instruments that may be used to deal with uncertainties and aid with difficulties in a variety of fields. One of the most important tools is fixed point theory in many scientific fields, including the advancement of nonlinear analysis, engineering, computer science, and economics. B-metric space was first presented by Back htin[3]in 1989. Czer wik [4]expanded the b-metric spaces results in 1993. Branciari [5]first proposed the idea of a rectangle metric space in 2000, replacing the triangle inequality of a metric space with a different inequality known as the "rectangular inequality." The idea of rectangular b-metric space, which generalizes the concepts of metric space, rectangle metric space, and b-metric space, was introduced in [6]by George et al.

Maji et al. [7], [8] presented a number of soft set operations, Sonam, et al. in [9] presented fixed point in Soft Rectangular B-metric Space, while Biswas and Roy [8] came up with the term fuzzy soft sets As an extension of soft metric space, Bea ula et al. [10] recently developed the idea of fuzzy soft metric space.

In this article, we introduce the definition of fuzzy soft rectangular b- metric space and defined some concepts in this space After that, we established fuzzy,soft contractive mappings on fuzzy soft rectangular b- metric.spaces and proved some fuzzy-soft contractive mapping fixed.point theorems.

2.Basic definitions

Let us begin with some basic definitions. U refers to an initial universe throughout this work, and I^U be the family of all fuzzy sets over U .

Definition (2.1),[1]:

The fuzzy set $\tilde{X}$ under a universal set U is a set describe by function of membership $\mu_{\tilde{X}}: U \rightarrow I$, $where\ I = [0,1]$ and $\tilde{X}$ an ordered pair collection defined by $\tilde{X} = \{ (u, \mu_{\tilde{X}}(u)) : u \in U, \mu_{\tilde{X}}(u) \in I \}$, where $\mu_{\tilde{X}}(u)$ is namely degree membership of u in $\tilde{X}$.

Definition (2.2),[2]:

Assume that U is an universal set, E is a set of parameters, and $A \subseteq E$. So the pair $(\mathcal{G}, A)$ is soft set under U and defined as a set $\mathcal{G}_A = \{ (e, \mathcal{G}_A(e)): e \in E, \mathcal{G}_A(e) \in P(U)\}$, such that $\mathcal{G}$ a mapping provided as $\mathcal{G}: A \rightarrow P(U)$ and $P(U)$ is a power set of U.

Definition (2.3),[8]:

A set $(\mathcal{G}, A)$ is refer to be fuzzy soft set over U , whenever $\mathcal{G}$ is mapping $\mathcal{G}: A \rightarrow I^U$ and $\{ \mathcal{G}(e) \in I^U : e \in A\}$ The collection of all fuzzy soft set , is symbolized by $\mathcal{F}_{ss}(U)$ and fuzzy soft in short denoted by $\mathcal{F}_{ss}$.

Definition (2.4),[11]:

A $\mathcal{F}_{ss} -$ set $(\mathcal{G}, A)$ over a universal set U *is* namely .

1) A set of absolute $\mathcal{F}_{ss} -$ set, represented by C_A , if $\mu\, \mathcal{G}(e) = 1$ for each $e \in A$.

2) A null $\mathcal{F}_{ss} -$ set, symbolized by $\tilde{\Phi}$, if for all $e \in A$, we have $\mu\, \mathcal{G}(e) = 0$.

Definition (2.5), [9]:

Let $(\mathcal{G}_1, \mathrm{A})$ and $(\mathcal{G}_2, \mathrm{B})$ be two $\mathcal{F}_{\mathrm{ss}} -$ sets over a common universal set U , Then

(1) $(\mathcal{G}_1, \mathrm{A})$ is said to be a $\mathcal{F}_{\mathrm{ss}} -$ subset of $(\mathcal{G}_2, \mathrm{B})$ if $\mathrm{A} \subseteq \mathrm{B}$, and $\mathcal{G}_1(\mathrm{e}) \subseteq \mathcal{G}_2(\mathrm{e})$ that is $\mu\mathcal{G}_1(\mathrm{e}) \leq \mu\mathcal{G}_2(\mathrm{e})$ for all $\mathrm{e} \in \mathrm{A}$. We write $(\mathcal{G}_1, \mathrm{A}) \tilde{\subseteq} (\mathcal{G}_2, \mathrm{B})$.

(2) The two $\mathcal{F}_{\mathrm{ss}} -$ sets $(\mathcal{G}_1, \mathrm{A})$ and $(\mathcal{G}_2, \mathrm{B})$ are said to be equal $\mathcal{F}_{\mathrm{ss}} -$ set , and denoted by $(\mathcal{G}_1, \mathrm{A}) \cong (\mathcal{G}_2, \mathrm{B})$, if $(\mathcal{G}_1, \mathrm{A}) \tilde{\subseteq} (\mathcal{G}_2, \mathrm{B})$ and $(\mathcal{G}_2, \mathrm{B}) \tilde{\subseteq} (\mathcal{G}_1, \mathrm{A})$.

Definition (2.6),[12]:

Consider the $\mathcal{F}_{ss} -$ sets $(\mathcal{G}_2, \mathrm{B})$ and $(\mathcal{G}_1, \mathrm{A})$ over the same universal set U, then

1) $(\mathcal{G}_1, A)\ \tilde{\cup}\ (\mathcal{G}_2, B) \cong (\mathcal{G}_3, C)$, where $B \cup A = C$ and for all $e \in C, u \in U$

$$\mu\mathcal{G}_{3\,(\mathrm{e})}(\mathrm{u}) = \begin{cases} \mu\ \mathcal{G}_1(\mathrm{e})(u), if e \in A - B, u \in \mathrm{U} \\ \mu\ \mathcal{G}_2(\mathrm{e})(u), if e \in B - A, u \in \mathrm{U} \\ max[\mu\ \mathcal{G}_1(\mathrm{e})(u), \mu\ \mathcal{G}_2(\mathrm{e})(u)], if e \in A \cap B, u \in \mathrm{U} \end{cases}$$

2) $(\mathcal{G}_1, A)\ \tilde{\cap}\ (\mathcal{G}_2, B) \cong (\mathcal{G}_3, C)$, where $C = A \cup B$ and for all $\in C$,

$$\mu\mathcal{G}_{3\,(\mathrm{e})}(\mathrm{u}) = \begin{cases} \mu\ \mathcal{G}_1(\mathrm{e})(u), if\ e \in A - B, u \in \mathrm{U} \\ \mu\ \mathcal{G}_2(\mathrm{e})(u), if\ e \in B - A, u \in \mathrm{U} \\ min[\mu\ \mathcal{G}_1(\mathrm{e})(u), \mu\ \mathcal{G}_2(\mathrm{e})(u)], if\ e \in A \cap B, u \in \mathrm{U}. \end{cases}$$

Definition(2.7), [8]:
The complement of a $\mathcal{F}_{ss}-$ set $(\mathcal{G},A)$ symbolized by $(\mathcal{G},A)^C = (\mathcal{G}^C,A)$, where $\mathcal{G}^C : E \to I^U$ is a map defined by $\mu\,\mathcal{G}^C(e) = 1 - \mu\,\mathcal{G}(e)$ $for\ all\ e \in A \subseteq E$.

Definition(2.8),[12]:
The $\mathcal{F}_{ss}-$ set $(\mathcal{G},A)$ over U is called $\mathcal{F}_{ss}-$ point and symbolized by $u_{\mu_{\mathcal{G}(e)}}$, if $e \in A$ and $u \in U$,

$$\mu_{\mathcal{G}(e)} = \begin{cases} \sigma \text{ if } u = u_0 \in U \text{ and } e = e_0 \in A, \\ 0 \text{ if } u \in U - \{u_0\} \text{ or } e \in A - \{e_0\}, \text{where } \sigma \in (0,1] \end{cases}$$

Definition (2.9), ([7],[13], [12]):
Let $u^1{}_{\mu_{\mathcal{G}(e_1)}}, u_{\mu_{\mathcal{G}(e)}}$ *and* $u^2{}_{\mu_{\mathcal{G}(e_2)}}$ are three $\mathcal{F}_{ss}-$ points over common universal set U and $(\mathcal{G}_1,A)$ $\mathcal{F}_{ss}-$ set then:

1. The $\mathcal{F}_{ss}-$ point $u_{\mu_{\mathcal{G}(e)}}$ is said to belongs to $(\mathcal{G}_1,A)$ if for the element $e \in A$, we have $\mathcal{G}(e) \subseteq \mathcal{G}_1(e)$. And denoted by $u_{\mu_{\mathcal{G}(e)}} \tilde{\tilde{\in}} (\mathcal{G}_1,A)$.
2. $u^1{}_{\mu_{\mathcal{G}(e_1)}}$ and $u^2{}_{\mu_{\mathcal{G}(e_2)}}$ over a common universal set U are considered equal if $u^1 = u^2$, $e_1 = e_2$ and $\mu_{\mathcal{G}(e_1)} = \mu_{\mathcal{G}(e_2)}$.
3. The $\mathcal{F}_{ss}-$ point $u_{\mu_{\mathcal{G}^c(e)}}$ is called complement $\mathcal{F}_{ss}-$ point of $\mathcal{F}_{ss}-$ point $u_{\mu_{\mathcal{G}(e)}}$ if for $e \in A$ and $u \in U$

$$\mu_{\mathcal{G}^c(e)} = \begin{cases} 1 - \mu_{\mathcal{G}(e)}, if\ u = u_0 \in U\ and\ e = e_0 \in A \\ 0, if\ u \in U - \{u_0\}\ or\ e \in A - \{e_0\}. \end{cases}$$

Definition (2.10), [14]:
Consider the set of all real integers $\mathbb{R}$, where E is a parameter set, $A \subseteq E$ and $\mathcal{F}^{\mathbb{B}(\mathbb{R})}$ be the set of all non-empty bounded Fuzzy subsets of $\mathbb{R}$, then (R,A) namely $\mathcal{F}_{ss}-$ real set over $\mathbb{R}$ and is defined as a set of $R_A = \{(e, R_A(e)): e \in A, R_A(e) \in \mathcal{F}^{\mathbb{B}(\mathbb{R})}\}$, where R is a mapping provide as $R: A \to \mathcal{F}^{\mathbb{B}(\mathbb{R})}$. A is referred to as the support of R_A.

Definition (2.11), [14]:
Let(R,A) is namely a $\mathcal{F}_{ss}-$ real number in $\mathbb{R}$, with describe as (r,A) (shortly $\bar{r}$), whenever is a singleton $\mathcal{F}_{ss}-$ real set, such as $\mathbb{R}(A)$ represent the set of each $\mathcal{F}_{ss}-$ real values and $\mathbb{R}^+(A)$ represents the collection of all $\mathcal{F}_{ss}-$ real values that are not negative.

Definition (2.12), [15]:
Let $\mathcal{F}_{ss}(U)$ be the family of all a nonempty $\mathcal{F}_{ss}-$ set over U, a $\mathcal{F}_{ss}-$ metric space is a pair $(\mathcal{F}_{ss}(U), \widetilde{\widetilde{d}})$, where $\widetilde{\widetilde{d}} : \mathcal{F}_{ss}(U) \times \mathcal{F}_{ss}(U) \to \mathbb{R}^+(A)$ be a function satisfy the following statements for all

$u^1{}_{\mu_{\mathcal{G}}(e_1)}, u^2{}_{\mu_{\mathcal{G}}(e_2)}, u^3{}_{\mu_{\mathcal{G}}(e_3)} \tilde{\tilde{\in}} \mathcal{F}_{ss}(U)$

(1) $\tilde{\tilde{d}}\left(u^1{}_{\mu_{\mathcal{G}}(e_1)}, u^2{}_{\mu_{\mathcal{G}}(e_2)}\right) \tilde{\geqq} \tilde{0}$,

(2) $\tilde{\tilde{d}}\left(u^1{}_{\mu_{\mathcal{G}}(e_1)}, u^2{}_{\mu_{\mathcal{G}}(e_2)}\right) \cong \tilde{0}$ if and only if $u^1{}_{\mu_{\mathcal{G}}(e_1)} \cong u^2{}_{\mu_{\mathcal{G}}(e_2)}$,

(3) $\tilde{\tilde{d}}\left(u^1{}_{\mu_{\mathcal{G}}(e_1)}, u^2{}_{\mu_{\mathcal{G}}(e_2)}\right) \cong \tilde{\tilde{d}}\left(u^2{}_{\mu_{\mathcal{G}}(e_2)}, u^1{}_{\mu_{\mathcal{G}}(e_1)}\right)$ *and*

(4) $\tilde{\tilde{d}}\left(u^1{}_{\mu_{\mathcal{G}}(e_1)}, u^3{}_{\mu_{\mathcal{G}}(e_3)}\right) \tilde{\leqq}$
$\tilde{\tilde{d}}\,(u^1{}_{\mu_{\mathcal{G}}(e_1)}, u^2{}_{\mu_{\mathcal{G}}(e_2)}) + \tilde{\tilde{d}}\,(u^2{}_{\mu_{\mathcal{G}}(e_2)}, u^3{}_{\mu_{\mathcal{G}}(e_3)})$.

Definition (2.13),[16]:
Let $(\mathcal{F}_{ss}(U_1), E_1)$ and $(\mathcal{F}_{ss}(U_2), E_2)$ be two $\mathcal{F}_{ss}-sets$, respectively. If $T: U_1 \to U_2$ *and* $\psi: E_1 \to E_2$ are both mappings, where E_1 and E_2 are the parameter sets for the over a universal sets U_1 and U_2, respectively. then the mapping
$T\psi = (T,\psi): (\mathcal{F}_{ss}(U_1), E_1) \to (\mathcal{F}_{ss}(U_2), E_2)$ is Known as $\mathcal{F}_{ss}-$ mapping.

Definition (2.14), [16]:
Let the two $\mathcal{F}_{ss}-$ set $(\mathcal{G}_1, A_1) \tilde{\subseteq} \mathcal{F}_{ss}(U_1)$ and $(\mathcal{G}_2, A_2) \tilde{\subseteq} \mathcal{F}_{ss}(U_2)$ when $A_1 \subseteq E_1, A_2 \subseteq E_2$ and let
$T\psi = (T,\psi): (\mathcal{F}_{ss}(U_1), E_1) \to (\mathcal{F}_{ss}(U_2), E_2)$ be $\mathcal{F}_{ss}-$ mapping.
(1) The image of $(\mathcal{G}_1, A_1)$ under a $\mathcal{F}_{ss}-$ mapping $T\psi$ denoted by $T\psi((\mathcal{G}_1, A_1))$, is the $\mathcal{F}_{ss}-$ set on U_2 for all $y \in U_2$, $\beta \in \psi(E_1) \subseteq E_2$ *and* $x \in U_1, e \in E_1$ defined by

$$T\psi\big((\mathcal{G}_1, A_1)\big)(\beta)(y) = \begin{cases} \tilde{\cup}_{x \in T^{-1}(y)}\left(\tilde{\cup}_{e \in \psi^{-1}(\beta)\cap A_1} \mathcal{G}_1(e)\right)(x) \\ \quad if\ T^{-1}(y) \neq \emptyset, \psi^{-1}(\beta) \neq \emptyset, \\ 0, otherwise. \end{cases}$$

(2) $$T\psi^{-1}\big((\mathcal{G}_2, A_2)\big)(e)(x)\begin{cases} \mathcal{G}_2(\psi(e))T(x)), for\ \psi(e) \in A_2 \subseteq E_2 \\ 0, otherwise. \end{cases}.$$

3.1 $\mathcal{F}_{ss}-$ Rectangle b-Metric Space.

In this section, we will present some definitions and lemmas in $\mathcal{F}_{ss}-$ rectangle b-metric space, such as convergent sequences, Cauchy sequence and $\mathcal{F}_{ss}-$ open ball.

Take a non-null set U and non-null collection of parameters E. Consider C_A as an absolute $\mathcal{F}_{ss}-$ set and a group of a $\mathcal{F}_{ss}-$points of C_A be signified as $\mathcal{F}_{ss}(C_A)$. Also, the collection of non-negative $\mathcal{F}_{ss}-$real-numbers is signified as $\mathbb{R}^+(E)$ and $[0,\infty)E$ indicates all $\mathcal{F}_{ss}-$ real numbers in the interval $[0,\infty)$. Then the $\mathcal{F}_{ss}-$Rectangular b-Metric space using the $\mathcal{F}_{ss}-$points is defined as below :
Any map $D_{fs\,rb}: \mathcal{F}_{ss}(C_A) \times \mathcal{F}_{ss}(C_A) \to \mathbb{R}^+(E)$ to be claimed as a $\mathcal{F}_{ss}-$Rectangular b-Metric over $\mathcal{F}_{ss}-$ set C_A with coefficient $\tilde{S} \geq 1$ condition to the following are fulfilled:
(1) $D_{fs\,rb}\left(u^1{}_{\mu_{\mathcal{G}}(e_1)}, u^2{}_{\mu_{\mathcal{G}}(e_2)}\right) \tilde{\geqq} \bar{0}$ for all
$u^1{}_{\mu_{\mathcal{G}}(e_1)}, u^2{}_{\mu_{\mathcal{G}}(e_2)} \tilde{\tilde{\in}} \mathcal{F}_{ss}(C_A)$.

(2) $D_{fs\,rb}\left(u^1{}_{\mu_{\mathcal{G}}(e_1)}, u^2{}_{\mu_{\mathcal{G}}(e_2)}\right) = \bar{0}$ if and only if $u^1{}_{\mu_{\mathcal{G}}(e_1)} = u^2{}_{\mu_{\mathcal{G}}(e_2)}$ for all $u^1{}_{\mu_{\mathcal{G}}(e_1)}, u^2{}_{\mu_{\mathcal{G}}(e_2)} \tilde{\tilde{\in}} \mathcal{F}_{ss}(C_A)$

(3) $D_{fs\,rb}\left(u^1{}_{\mu_{\mathcal{G}}(e_1)}, u^2{}_{\mu_{\mathcal{G}}(e_2)}\right) = D_{fs\,rb}\left(u^2{}_{\mu_{\mathcal{G}}(e_2)}, u^1{}_{\mu_{\mathcal{G}}(e_1)}\right)$ for all $u^1{}_{\mu_{\mathcal{G}}(e_1)}, u^2{}_{\mu_{\mathcal{G}}(e_2)} \tilde{\tilde{\in}} \mathcal{F}_{ss}(C_A)$

(4) $D_{fs\,rb}\left(u^1{}_{\mu_{\mathcal{G}}(e_1)}, u^2{}_{\mu_{\mathcal{G}}(e_2)}\right) \tilde{\leq} \tilde{s}\left[D_{fs\,rb}\left(u^1{}_{\mu_{\mathcal{G}}(e_1)}, u^3{}_{\mu_{\mathcal{G}}(e_3)}\right) + D_{fs\,rb}\left(u^3{}_{\mu_{\mathcal{G}}(e_3)}, u^4{}_{\mu_{\mathcal{G}}(e_4)}\right) + D_{fs\,rb}\left(u^4{}_{\mu_{\mathcal{G}}(e_4)}, u^2{}_{\mu_{\mathcal{G}}(e_2)}\right)\right]$ for all $u^1{}_{\mu_{\mathcal{G}}(e_1)}, u^2{}_{\mu_{\mathcal{G}}(e_2)} \tilde{\tilde{\in}} \mathcal{F}_{ss}(C_A)$ and each different $\mathcal{F}_{ss}$ −points $u^3{}_{\mu_{\mathcal{G}}(e_3)}, u^4{}_{\mu_{\mathcal{G}}(e_4)} \tilde{\tilde{\in}} \mathcal{F}_{ss}(C_A) \setminus \{u^1{}_{\mu_{\mathcal{G}}(e_1)}, u^2{}_{\mu_{\mathcal{G}}(e_2)}\}$.

The $\mathcal{F}_{ss}$ −Rectangular b-Metric $(\mathcal{F}_{ss}(C_A), D_{fs\,rb})$ sometimes called $\mathcal{F}_{ss}$ −b-generalized metric space with the $\tilde{s} \geq 1$ (in short, $\mathcal{F}_{ss}$ − Rbms).

To illustrate this definition , one can see the following example .

Example(3.1.1) :

Let $X = \{x, y, z\}$ *a finite set and* $E = \{e_1, e_2\}$, *a set of parameters. Then,* $\mathcal{F}_{ss}(C_A) = \{x_{\mu_{\mathcal{G}}(e_1)}, y_{\mu_{\mathcal{G}}(e_1)}, z_{\mu_{\mathcal{G}}(e_1)}, x_{\mu_{\mathcal{G}}(e_2)}, y_{\mu_{\mathcal{G}}(e_2)}, z_{\mu_{\mathcal{G}}(e_2)}\}$. *Consider a mapping* $D_{fs\,rb} : \mathcal{F}_{ss}(C_A). \times \mathcal{F}_{ss}(C_A) \to \mathbb{R}^+(E)$ *by*

$D_{fs\,rb}(u_{\mu_{\mathcal{G}}(e_i)}, v_{\mu_{\mathcal{G}}(e_j)}) = D_{fs\,rb}(v_{\mu_{\mathcal{G}}(e_j)}, u_{\mu_{\mathcal{G}}(e_i)})$ *for all distinct* $u_{\mu_{\mathcal{G}}(e_i)}, v_{\mu_{\mathcal{G}}(e_j)} \tilde{\tilde{\in}} \mathcal{F}_{ss}(C_A)$ *for* $i, j \in \{1, 2\}$.

$D_{fs\,rb}(w_{\mu_{\mathcal{G}}(e_i)}, w_{\mu_{\mathcal{G}}(e_i)}) = \bar{0}$ *for all* $w_{\mu_{\mathcal{G}}(e_i)} \tilde{\tilde{\in}} \mathcal{F}_{ss}(C_A)$.

$D_{fs\,rb}(x_{\mu_{\mathcal{G}}(e_1)}, x_{\mu_{\mathcal{G}}(e_2)}) = D_{fs\,rb}(x_{\mu_{\mathcal{G}}(e_2)}, y_{\mu_{\mathcal{G}}(e_1)}) = D_{fs\,rb}(y_{\mu_{\mathcal{G}}(e_2)}, z_{\mu_{\mathcal{G}}(e_2)}) = \overline{40}$,

$D_{fs\,rb}\left(x_{\mu_{\mathcal{G}}(e_1)}, y_{\mu_{\mathcal{G}}(e_1)}\right) = D_{fs\,rb}\left(y_{\mu_{\mathcal{G}}(e_2)}, z_{\mu_{\mathcal{G}}(e_1)}\right) = D_{fs\,rb}(y_{\mu_{\mathcal{G}}(e_2)}, y_{\mu_{\mathcal{G}}(e_1)}) = \overline{43}$,

$D_{fs\,rb}(x_{\mu_{\mathcal{G}}(e_2)}, z_{\mu_{\mathcal{G}}(e_1)}) = D_{fs\,rb}(z_{\mu_{\mathcal{G}}(e_2)}, z_{\mu_{\mathcal{G}}(e_1)}) = D_{fs\,rb}(y_{\mu_{\mathcal{G}}(e_1)}, z_{\mu_{\mathcal{G}}(e_1)}) = \overline{63}$,

$D_{fs\,rb}(y_{\mu_{\mathcal{G}}(e_2)}, x_{\mu_{\mathcal{G}}(e_1)}) = D_{fs\,rb}(x_{\mu_{\mathcal{G}}(e_2)}, y_{\mu_{\mathcal{G}}(e_2)}) = \overline{109}$,

$D_{fs\,rb}(x_{\mu_{\mathcal{G}}(e_1)}, z_{\mu_{\mathcal{G}}(e_2)}) = \overline{1543}$,

$D_{fs\,rb}(y_{\mu_{\mathcal{G}}(e_1)}, z_{\mu_{\mathcal{G}}(e_2)}) = \overline{120}$, $D_{fs\,rb}(x_{\mu_{\mathcal{G}}(e_2)}, z_{\mu_{\mathcal{G}}(e_2)}) = \overline{196}$, $D_{fs\,rb}(z_{\mu_{\mathcal{G}}(e_1)}, x_{\mu_{\mathcal{G}}(e_1)}) = \overline{259}$

Here, $D_{fs\,rb}(z_{\mu_{\mathcal{G}}(e_2)}, x_{\mu_{\mathcal{G}}(e_1)}) = \overline{1543}$ *and*

$D_{fs\,rb}(z_{\mu_{\mathcal{G}(e_2)}}, x_{\mu_{\mathcal{G}(e_2)}}) + D_{fs\,rb}(x_{\mu_{\mathcal{G}(e_2)}}, y_{\mu_{\mathcal{G}(e_1)}}) + D_{fs\,rb}(y_{\mu_{\mathcal{G}(e_1)}}, x_{\mu_{\mathcal{G}(e_1)}}) = \overline{196} + \overline{40} + \overline{43} = \overline{279}$

Then, $(\mathcal{F}_{ss}(C_A), D_{fs\,rb})$ *is a* $\mathcal{F}_{ss} -$ *Rbms with coefficient* $\tilde{s} = 5.54$.

Definition (3.1.2):

Let $(\mathcal{F}_{ss}(C_A), D_{fs\,rb})$ *be* $\mathcal{F}_{ss}$ *− Rbms , then for all* $\mathcal{F}_{ss}$ *−point* $u_{\mu_{\mathcal{G}(e)}} \tilde{\tilde{\in}} C_A$, $\bar{r} \in \mathbb{R}^{+}(A)$ *the* $\mathcal{F}_{ss}$ *− open ball define by*

$\widetilde{\widetilde{B}}_{\bar{r}}\left(u_{\mu_{\mathcal{G}(e_0)}}\right) = \{u_{\mu_{\mathcal{G}(e)}} \tilde{\tilde{\in}} C_A : D_{fs\,rb}\left(u_{\mu_{\mathcal{G}(e)}}, u_{\mu_{\mathcal{G}(e_0)}}\right) < \bar{r}\}$ *and the* $\mathcal{F}_{ss}$ *− Closed ball in* $\mathcal{F}_{ss}$ *− Rbms defined as*

$\widetilde{\widetilde{B}}_{\bar{r}}[u_{\mu_{\mathcal{G}(e_0)}}] = \{u_{\mu_{\mathcal{G}(e)}} \tilde{\tilde{\in}} C_A : D_{fs\,rb}\left(u_{\mu_{\mathcal{G}(e)}}, u_{\mu_{\mathcal{G}(e_0)}}\right) \leq \bar{r}\}$.

Definition (3.1.3):

Let $\langle u^n{}_{\mu_{\mathcal{G}(e_n)}}\rangle$ *be a* $\mathcal{F}_{ss}$ *− sequence in a* $\mathcal{F}_{ss}$ *− Rbms* $(\mathcal{F}_{ss}(C_A), D_{fsr\,b})$ *, we say*

(1) $\langle u^n{}_{\mu_{\mathcal{G}(e_n)}}\rangle$ *converges to* $u_{\mu_{\mathcal{G}(e_0)}} \tilde{\tilde{\in}} \mathcal{F}_{ss}(C_A)$ *if there exists* $\bar{0} < \bar{\delta} \in \mathbb{R}^{+}(A)$ *and a positive integer* $N = N(\tilde{\epsilon})$ *such that* $D_{fsr\,b}\left(u^n{}_{\mu_{\mathcal{G}(e_n)}}, u_{\mu_{\mathcal{G}(e_0)}}\right) \leq \bar{\delta}$ *whenever* $n \geq N$*. Equivalently,*
$lim_{n\to\infty} D_{fsr\,b}\left(u^n{}_{\mu_{\mathcal{G}(e_n)}}, u_{\mu_{\mathcal{G}(e_0)}}\right) = 0$.

(2) $\langle u^n{}_{\mu_{\mathcal{G}(e_n)}}\rangle$ *Cauchy sequence if there exists* $\bar{0} < \bar{\delta} \in \mathbb{R}^{+}(A)$ *and a positive integer* $N = N(\tilde{\epsilon})$ *such that* $D_{fs\,rb}\left(u^n{}_{\mu_{\mathcal{G}(e_n)}}, u^m{}_{\mu_{\mathcal{G}(e_m)}}\right) \leq \bar{\delta}$ *whenever* $n, m \geq N$ *equivalently,*
$lim_{n\to\infty} D_{fsr\,b}\left(u^n{}_{\mu_{\mathcal{G}(e_n)}}, u^m{}_{\mu_{\mathcal{G}(e_m)}}\right) = \bar{0}$ *for* $m > n$.

Definition (3.1.4):

A $\mathcal{F}_{ss}$ *− Rbms* $(\mathcal{F}_{ss}(C_A), D_{fsr\,b})$ *is called* $\mathcal{F}_{ss}$ *− complete if each* $\mathcal{F}_{ss}$ *− Cauchy sequence* $\langle u^n{}_{\mu_{\mathcal{G}(e_n)}}\rangle$ *in* $\mathcal{F}_{ss}(C_A)$ *converges in* $\mathcal{F}_{ss}(C_A)$.

Remarks(3.1.5)

(1) the sequence limit in $\mathcal{F}_{ss}$ *− Rbms It doesn't have to be unique,*

(2) nor does every convergent sequence in $\mathcal{F}_{ss}$ *− Rbms have to be Cauchy sequence.*

(3) An open ball is not an open set in $\mathcal{F}_{ss}$ *− Rbms and* $\mathcal{F}_{ss}$ *− Rbms is not Hausdorff . To illustrate this Remarks , one can see the following example.*

Example(3.1.6) :

Let $D = \{1/z, z \in N\}\setminus\{1\}$ *and* $\mathcal{F} = \{1,2,3\}$*. Let* $U = D \cup \mathcal{F}$ *and* $E \subseteq R$*. define the a map* $D_{fs\,rb} : \mathcal{F}_{ss}(C_A). \times \mathcal{F}_{ss}(C_A) \to [0, \infty)E$ *by*

$D_{fs\,rb}\left(u^1{}_{\mu_{\mathcal{G}(e_1)}}, u^2{}_{\mu_{\mathcal{G}(e_2)}}\right) = D_{fs\,rb}\left(u^2{}_{\mu_{\mathcal{G}(e_2)}}, u^1{}_{\mu_{\mathcal{G}(e_1)}}\right)$

for evere $u^1{}_{\mu_{\mathcal{G}(e_1)}}, u^2{}_{\mu_{\mathcal{G}(e_2)}} \tilde{\tilde{\in}} \mathcal{F}_{ss}(C_A)$.

$$D_{fs\,rb}\left(u^1{}_{\mu_{\mathcal{G}(e_1)}} , u^2{}_{\mu_{\mathcal{G}(e_2)}}\right) = \begin{cases} \bar{0} & if\ u^1{}_{\mu_{\mathcal{G}(e_1)}} = u^2{}_{\mu_{\mathcal{G}(e_2)}} \\ 6\bar{t}\ \ if\ u^1{}_{\mu_{\mathcal{G}(e_1)}} , u^2{}_{\mu_{\mathcal{G}(e_2)}} \widetilde{\widetilde{\in}} \mathcal{F}_{ss}(D_E),\ u^1{}_{\mu_{\mathcal{G}(e_1)}} \neq u^2{}_{\mu_{\mathcal{G}(e_2)}} & \\ \frac{\bar{t}}{3z} & if\ u^1{}_{\mu_{\mathcal{G}(e_1)}} \widetilde{\widetilde{\in}} \mathcal{F}_{ss}(D_E),\ u^2{}_{\mu_{\mathcal{G}(e_2)}} \notin \mathcal{F}_{ss}(D_E) \\ \frac{\bar{t}}{2} & if\ otherwise \end{cases}$$

when $\bar{t} > \bar{0}$ *denotes a constant* $\mathcal{F}_{ss} -$ *real number*

The sequence $\left\{\left(\frac{\tilde{1}}{z}\right)^{z}_{\lambda z}\right\} z \in N$ *converges to each of the* $\mathcal{F}_{ss} -$ *point* $\mathcal{F}_{ss}$ $(\mathcal{F}_E)$ *. Therefore , limit is not unique.*

Because $D_{fs\,rb}(\frac{1}{z}, \frac{1}{p}) = 6\bar{t} \neq \bar{0}$ *where* $z, p \in N$ *and* $p > z$ *, then* $\left\{\left(\frac{\tilde{1}}{z}\right)^{z}_{\lambda z}\right\} z \in N$ *is not* $\mathcal{F}_{ss} -$ *Cauchy sequence in* $(\mathcal{F}_{ss} - Rbms)$

$\widetilde{\widetilde{B}}_{\frac{\bar{t}}{3}}\left(\left(\frac{1}{2}\right)_6\right) = \{\left(\frac{1}{2}\right)_s : s \in E \cup \mathcal{F}_{ss}(\mathcal{F}_E)$ *There isn't an open ball* $\widetilde{\widetilde{B}}_{\bar{r}}((2)_5)$ *the around* $\mathcal{F}_{ss} -$ *point* $(2)_5$ *contained in* $B_{\frac{\bar{t}}{3}}\left(\left(\frac{1}{2}\right)_6\right)$ *for just a fuzzy soft point* $(2)_5$ *in* $\mathcal{F}_{ss}(C_A)$*, . As a result,* $\widetilde{\widetilde{B}}_{\frac{\bar{t}}{3}}\left(\left(\frac{1}{2}\right)_6\right)$ *is not open in*

$(\mathcal{F}_{ss}(C_A), D_{fs\,rb})$ *. Because there are no* $\bar{r}_1, \bar{r}_2$ *values greater than zero such that* $\widetilde{\widetilde{B}}_{\bar{r}_1}(2)\ \widetilde{\cap}\ \widetilde{\widetilde{B}}_{\bar{r}_2}(3) = \widetilde{\Phi}$

, $(\mathcal{F}_{ss}(C_A), D_{fs\,rb})$ *is not Hausdorff.*

In the above example, it was shown that the point of convergence of a sequence cannot be unique in general. The following Lemma shows the condition to get the point of convergence is unique with the same space if the sequence is Cauchy.

Lemma (3.1.7):

Let $(\mathcal{F}_{ss}(C_A), D_{fs\,rb})$ *be an* $(\mathcal{F}_{ss} - Rbms)$ *and* $\langle u^n{}_{\mu_{\mathcal{G}(e_n)}} \rangle$ *represent a* $\mathcal{F}_{ss} -$ *Cauchy sequence in* $\mathcal{F}_{ss}(C_A)$ *such that* $u^n{}_{\mu_{\mathcal{G}(e_n)}} \neq u^m{}_{\mu_{\mathcal{G}(e_m)}}$ *whenever* $m \neq n$*. then* $u^n{}_{\mu_{\mathcal{G}(e_n)}}$ *can converge to only one point.*

<u>Proof.</u> *Assume* $lim_{n\to\infty}\ u^n{}_{\mu_{\mathcal{G}(e_n)}} = u_{\mu_{\mathcal{G}(e_1)}}$, $lim_{n\to\infty}\ u^n{}_{\mu_{\mathcal{G}(e_n)}} = u_{\mu_{\mathcal{G}(e_2)}}$ *and* $u_{\mu_{\mathcal{G}(e_1)}} \neq u_{\mu_{\mathcal{G}(e_2)}}$ *Since* $u^n{}_{\mu_{\mathcal{G}(e_n)}}$ *and* $u^m{}_{\mu_{\mathcal{G}(e_m)}}$*, in addition to* $u_{\mu_{\mathcal{G}(e_1)}}$ *and* $u_{\mu_{\mathcal{G}(e_2)}}$*, are distinct elements, It is obvious that* k *exists such that* $u_{\mu_{\mathcal{G}(e_1)}}$ *and* $u_{\mu_{\mathcal{G}(e_2)}}$ *different from* $u^n{}_{\mu_{\mathcal{G}(e_n)}}$ *for any* $n > k$*. The rectangular inequality indicates that for* $m, n > k$,

$D_{fs\,rb}\left(u_{\mu_{\mathcal{G}(e_1)}} , u_{\mu_{\mathcal{G}(e_2)}} \right) \widetilde{\leq}$

$\tilde{s}\left[D_{fs\,rb}\left(u_{\mu_{\mathcal{G}(e_1)}} , u^n{}_{\mu_{\mathcal{G}(e_n)}}\right) + D_{fs\,rb}\left(u^n{}_{\mu_{\mathcal{G}(e_n)}} , u^m{}_{\mu_{\mathcal{G}(e_m)}}\right) + D_{fs\,rb}\left(u^m{}_{\mu_{\mathcal{G}(e_m)}} , u_{\mu_{\mathcal{G}(e_2)}}\right)\right].$

Now

$\lim_{n,m\to\infty} \left(D_{fs\,rb}\left(u_{\mu_{\mathcal{G}(e_1)}} , u_{\mu_{\mathcal{G}(e_2)}}\right) \widetilde{\leq}\right.$

$\tilde{s} \lim_{n,m\to\infty} \Big[D_{fs\,rb}\big(u_{\mu_{\mathcal{G}}(e_1)}, u^n{}_{\mu_{\mathcal{G}}(e_n)} \big) + D_{fs\,rb}\big(u^n{}_{\mu_{\mathcal{G}}(e_n)}, u^m{}_{\mu_{\mathcal{G}}(e_m)} \big)$

$+ D_{fs\,rb}\big(u^m{}_{\mu_{\mathcal{G}}(e_m)}, u_{\mu_{\mathcal{G}}(e_2)} \big)\Big]$

$= D_{fs\,rb}\big(u_{\mu_{\mathcal{G}}(e_1)}, u_{\mu_{\mathcal{G}}(e_2)}\big) \cong \tilde{s}[0+0+0] = 0$ *so* $u_{\mu_{\mathcal{G}}(e_1)} = u_{\mu_{\mathcal{G}}(e_2)}$. *this a contradiction* ■.

4.Fixed Point in $\mathcal{F}_{ss}$ − Rbms.

In this part, we will discuss the fixed point theorem for some types of mappings in $\mathcal{F}_{ss}$ − Rbms.

Definition (4.1.1): *Let* $(\mathcal{F}_{ss}(C_A), D_{fs\,rb})$ *be a* $\mathcal{F}_{ss}$ *−RbMS .Then the* $\mathcal{F}_{ss}$ *−mapping* $T: (\mathcal{F}_{ss}(C_A), D_{fs\,rb}) \to (\mathcal{F}_{ss}(C_A), D_{fs\,rb})$ *is called a* $\mathcal{F}_{ss}$ *−contraction mapping if there exist,* $\bar{\lambda} \in \mathbb{R}^+(A)$, $\bar{0} \le \bar{\lambda} < \bar{1}$ *such that for each*
$u^1{}_{\mu_{\mathcal{G}}(e_1)}, u^2{}_{\mu_{\mathcal{G}}(e_2)} \in \mathcal{F}_{ss}(C_A)$ *we have*

$$D_{fs\,rb}\Big(T\big(u^1{}_{\mu_{\mathcal{G}}(e_1)}\big), T\big(u^2{}_{\mu_{\mathcal{G}}(e_2)}\big)\Big) \le \bar{\lambda}\, D_{fs\,rb}\big(u^1{}_{\mu_{\mathcal{G}}(e_1)}, u^2{}_{\mu_{\mathcal{G}}(e_2)}\big). \qquad (4.1)$$

The fixed point in a contraction mapping in a $\mathcal{F}_{ss}$ *−RbMS is discussed in the following theorem .*

Theorem(4.1.2): *Let* $(\mathcal{F}_{ss}(C_A), D_{fs\,rb})$ *be a complete* $\mathcal{F}_{ss}$ *−RbMS with coefficients* $S > 1$ *and* $T: \mathcal{F}_{ss}(C_A) \to \mathcal{F}_{ss}(C_A)$ *be a* $\mathcal{F}_{ss}$ *− mapping satisfying*

$$D_{fs\,rb}\Big(T\big(u^1{}_{\mu_{\mathcal{G}}(e_1)}\big), T\big(u^2{}_{\mu_{\mathcal{G}}(e_2)}\big)\Big) \le \bar{\lambda}\, D_{fs\,rb}\big(u^1{}_{\mu_{\mathcal{G}}(e_1)}, u^2{}_{\mu_{\mathcal{G}}(e_2)}\big)$$

for each $u^1{}_{\mu_{\mathcal{G}}(e_1)}, u^2{}_{\mu_{\mathcal{G}}(e_2)} \,\tilde{\tilde{\in}}\, \mathcal{F}_{ss}(C_A)$ *where* $\bar{\lambda} \in [\bar{0}, \frac{\bar{1}}{s}]$. *Then T has a unique* $\mathcal{F}_{ss}$ *− fixed pint .*

proof: Let $u_{\mu_{\mathcal{G}}(e_0)} \,\tilde{\tilde{\in}}\, \mathcal{F}_{ss}(C_A)$ *be arbitrary. We define a* $\mathcal{F}_{ss}$ *− sequence* $\langle u^n{}_{\mu_{\mathcal{G}}(e_n)}\rangle$ *by* $T\big(u^n{}_{\mu_{\mathcal{G}}(e_n)}\big) = u^{n+1}{}_{\mu_{\mathcal{G}}(e_{n+1})}$ *. We will prove that* $\langle u^n{}_{\mu_{\mathcal{G}}(e_n)}\rangle$ *is a Cauchy sequence. If* $u^n{}_{\mu_{\mathcal{G}}(e_n)} = u^{n+1}{}_{\mu_{\mathcal{G}}(e_{n+1})}$ *, then* $u^n{}_{\mu_{\mathcal{G}}(e_n)}$ *is a* $\mathcal{F}_{ss}$ *− fixed point of the* $\mathcal{F}_{ss}$ *− mapping T. Now let's assume that for any* $n \ge 0$, $u^n{}_{\mu_{\mathcal{G}}(e_n)} \ne u^{n+1}{}_{\mu_{\mathcal{G}}(e_{n+1})}$ *setting* $D_{fs\,rb}\big(u^n{}_{\mu_{\mathcal{G}}(e_n)}, u^{n+1}{}_{\mu_{\mathcal{G}}(e_{n+1})}\big) = D_n$ *it follows from equation (4.1) that*

$D_{fs\,rb}\big(u^n{}_{\mu_{\mathcal{G}}(e_n)}, u^{n+1}{}_{\mu_{\mathcal{G}}(e_{n+1})}\big) =$

$$D_{fs\,rb}\Big(T\big(u^{n-1}{}_{\mu_{\mathcal{G}}(e_{n-1})}\big), T\big(u^n{}_{\mu_{\mathcal{G}}(e_n)}\big)\Big) \le \bar{\lambda} D_{fs\,rb}\big(u^{n-1}{}_{\mu_{\mathcal{G}}(e_{n-1})}, u^n{}_{\mu_{\mathcal{G}}(e_n)}\big)$$

$D_n \le \bar{\lambda} D_{n-1}$. *We obtain by repeating this process* $D_n \le (\bar{\lambda})^n D_0$ *(4.2)*

We can also assume that $u_{\mu_{\mathcal{G}}(e_0)}$ *is not a* $\mathcal{F}_{ss}$ *− periodic point of T. Indeed, if* $u_{\mu_{\mathcal{G}}(e_0)} = u^n{}_{\mu_{\mathcal{G}}(e_n)}$ *, then using (4.2), we have for any* $n \ge 2$,

$$D_{fs\,rb}\Big(u_{\mu_{\mathcal{G}}(e_0)}, T\big(u_{\mu_{\mathcal{G}}(e_0)}\big)\Big) = D_{fs\,rb}\Big(u^n{}_{\mu_{\mathcal{G}}(e_n)}, T\big(u^n{}_{\mu_{\mathcal{G}}(e_n)}\big)\Big)$$

$D_{fs\,rb}\left(u_{\mu_{\mathcal{G}(e_0)}}, u_{\mu_{\mathcal{G}(e_1)}}\right) = D_{fs\,rb}\left(u^n{}_{\mu_{\mathcal{G}(e_n)}} , u^{n+1}{}_{\mu_{\mathcal{G}(e_{n+1})}}\right)$

$D_0 = D_n \le (\bar{\lambda})^n D_0$. *a contradiction. As a result, we must have* $D_0 = 0$, *implying that* $u_{\mu_{\mathcal{G}(e_0)}} = u_{\mu_{\mathcal{G}(e_1)}}$ *and thus* $u_{\mu_{\mathcal{G}(e_0)}}$ *is a* $\mathcal{F}_{ss} -$ *fixed point of* T.

Assume that for all distinct $n, m \in \mathbb{N}$, $u^n{}_{\mu_{\mathcal{G}(e_n)}} \ne u^m{}_{\mu_{\mathcal{G}(e_m)}}$. *Again, consider* $D_{fs\,rb}\left(u^n{}_{\mu_{\mathcal{G}(e_n)}} , u^{n+2}{}_{\mu_{\mathcal{G}(e_{n+2})}}\right) = D^*{}_n$ *and use equation (4.1) for any* $n \in \mathbb{N}$, *we obtain*

$$D_{fs\,rb}\left(u^n{}_{\mu_{\mathcal{G}(e_n)}} , u^{n+2}{}_{\mu_{\mathcal{G}(e_{n+2})}}\right) = D_{fs\,rb}\left(T\left(u^{n-1}{}_{\mu_{\mathcal{G}(e_{n-1})}}\right) ,T\left(u^{n+1}{}_{\mu_{\mathcal{G}(e_{n+1})}}\right)\right)$$
$$\le \bar{\lambda} D_{fs\,rb}\left(u^{n-1}{}_{\mu_{\mathcal{G}(e_{n-1})}} , u^{n+1}{}_{\mu_{\mathcal{G}(e_{n+1})}}\right)$$

$D^*{}_n \le \bar{\lambda} D^*{}_{n-1}$. *We obtain by repeating this process*

$$D_{fs\,rb}\left(u^n{}_{\mu_{\mathcal{G}(e_n)}} , u^{n+2}{}_{\mu_{\mathcal{G}(e_{n+2})}}\right) \le (\bar{\lambda})^n D^*{}_0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4.3)$$

In the following Two cases, we consider $D_{fs\,rb}\left(u^n{}_{\mu_{\mathcal{G}(e_n)}} , u^{n+p}{}_{\mu_{\mathcal{G}(e_{n+p})}}\right)$ *for the sequence* $u^n{}_{\mu_{\mathcal{G}(e_n)}}$.

(i) If p *is an odd number, such that.* $p = 2m + 1$, *then for some* $m \in N$, *by use equation (4.2) we have*

$$D_{fs\,rb}\left(u^n{}_{\mu_{\mathcal{G}(e_n)}} , u^{n+2m+1}{}_{\mu_{\mathcal{G}(e_{n+2m+1})}}\right)$$
$$\le S\left[D_{fs\,rb}\left(u^n{}_{\mu_{\mathcal{G}(e_n)}} , u^{n+1}{}_{\mu_{\mathcal{G}(e_{n+1})}}\right) + D_{fs\,rb}\left(u^{n+1}{}_{\mu_{\mathcal{G}(e_{n+1})}} , u^{n+2}{}_{\mu_{\mathcal{G}(e_{n+2})}}\right) + D_{fs\,rb}\left(u^{n+2}{}_{\mu_{\mathcal{G}(e_{n+2})}} , u^{n+2m+1}{}_{\mu_{\mathcal{G}(e_{n+2m+1})}}\right)\right]$$
$$\le S\left[D_n + D_{n+1}\right] + S^2\left[D_{fs\,rb}\left(u^{n+2}{}_{\mu_{\mathcal{G}(e_{n+2})}} , u^{n+3}{}_{\mu_{\mathcal{G}(e_{n+3})}}\right) + D_{fs\,rb}\left(u^{n+3}{}_{\mu_{\mathcal{G}(e_{n+3})}} , u^{n+4}{}_{\mu_{\mathcal{G}(e_{n+4})}}\right) + D_{fs\,rb}\left(u^{n+4}{}_{\mu_{\mathcal{G}(e_{n+4})}} , u^{n+2m+1}{}_{\mu_{\mathcal{G}(e_{n+2m+1})}}\right)\right]$$
$$= S\left[D_n + D_{n+1}\right] + S^2\left[D_{n+2} + D_{n+3} + D_{fs\,rb}\left(u^{n+4}{}_{\mu_{\mathcal{G}(e_{n+4})}} , u^{n+2m+1}{}_{\mu_{\mathcal{G}(e_{n+2m+1})}}\right)\right]$$

⋮

⋮

$$\le S\left[D_n + D_{n+1}\right] + S^2\left[D_{n+2} + D_{n+3}\right] + S^3\left[D_{n+4} + D_{n+5}\right] + \cdots + S^m D_{n+2m}$$
$$\le S\left[(\bar{\lambda})^n D_0 + (\bar{\lambda})^{n+1} D_0\right] + S^2\left[(\bar{\lambda})^{n+2} D_0 + (\bar{\lambda})^{n+3} D_0\right] + S^3\left[(\bar{\lambda})^{n+4} D_0 + (\bar{\lambda})^{n+5} D_0\right] + \cdots + S^m (\bar{\lambda})^{n+2m} D_0$$
$$\le S.(\bar{\lambda})^n\left[\bar{1} + S.(\bar{\lambda})^2 + S^2.(\bar{\lambda})^4 + \cdots..\right] D_0 + S.(\bar{\lambda})^{n+1}\left[\bar{1} + S.(\bar{\lambda})^2 + S^2.(\bar{\lambda})^4 + \cdots..\right] D_0$$
$$= \frac{\bar{1}+\bar{\lambda}}{\bar{1}-S.(\bar{\lambda})^2} S.(\bar{\lambda})^n D_0 \quad (as\ S.(\bar{\lambda})^2 < \bar{1}.$$

Therefore $D_{fs\,rb}\left(u^n{}_{\mu_{\mathcal{G}(e_n)}} , u^{n+2m+1}{}_{\mu_{\mathcal{G}(e_{n+2m+1})}}\right) \le \frac{\bar{1}+\bar{\lambda}}{\bar{1}-S.(\bar{\lambda})^2} S.(\bar{\lambda})^n D_0$ ………….(4.4)

(ii) If p *is even ,such that* $p = 2m$, *we can use (4.2) and (4.3) to get.*

$$D_{fs\,rb}\left(u^n{}_{\mu_{\mathcal{G}(e_n)}} , u^{n+p}{}_{\mu_{\mathcal{G}(e_{n+p})}}\right) = D_{fs\,rb}\left(u^n{}_{\mu_{\mathcal{G}(e_n)}} , u^{n+2m}{}_{\mu_{\mathcal{G}(e_{n+2m})}}\right)$$

$$\leq S\left[D_{fs\,rb}\left(u^{n}{}_{\mu_{\mathcal{G}}(e_n)}\,,\,u^{n+1}{}_{\mu_{\mathcal{G}}(e_{n+1})}\right)+D_{fs\,rb}\left(u^{n+1}{}_{\mu_{\mathcal{G}}(e_{n+1})}\,,\,u^{n+2}{}_{\mu_{\mathcal{G}}(e_{n+2})}\right)\right.$$
$$\left.+D_{fs\,rb}\left(u^{n+2}{}_{\mu_{\mathcal{G}}(e_{n+2})}\,,\,u^{n+2m}{}_{\mu_{\mathcal{G}}(e_{n+2m})}\right)\right]$$
$$\leq S\left[D_n+D_{n+1}\right]$$
$$+S^2\left[D_{fs\,rb}\left(u^{n+2}{}_{\mu_{\mathcal{G}}(e_{n+2})}\,,\,u^{n+3}{}_{\mu_{\mathcal{G}}(e_{n+3})}\right)+D_{fs\,rb}\left(u^{n+3}{}_{\mu_{\mathcal{G}}(e_{n+3})}\,,\,u^{n+4}{}_{\mu_{\mathcal{G}}(e_{n+4})}\right)\right.$$
$$\left.+D_{fs\,rb}\left(u^{n+4}{}_{\mu_{\mathcal{G}}(e_{n+4})}\,,\,u^{n+2m}{}_{\mu_{\mathcal{G}}(e_{n+2m})}\right)\right]$$
$$=S\left[D_n+D_{n+1}\right]+S^2\left[D_{n+2}+D_{n+3}+D_{fs\,rb}\left(u^{n+4}{}_{\mu_{\mathcal{G}}(e_{n+4})}\,,\,u^{n+2m}{}_{\mu_{\mathcal{G}}(e_{n+2m})}\right)\right]$$
$$\vdots$$
$$\vdots$$
$$\leq S\left[D_n+D_{n+1}\right]+S^2\left[D_{n+2}+D_{n+3}\right]+S^3\left[D_{n+4}+D_{n+5}\right]+\cdots+S^{m-1}\left[D_{n+2m-4}+D_{n+2m-3}\right]$$
$$+S^{m-1}\left[D_{fs\,rb}\left(u^{n+2m-2}{}_{\mu_{\mathcal{G}}(e_{n+2m-2})}\,,\,u^{n+2m}{}_{\mu_{\mathcal{G}}(e_{n+2m})}\right)\right]$$
$$\leq S\left[\left(\bar{\lambda}\right)^{n}D_0+\left(\bar{\lambda}\right)^{n+1}D_0\right]+S^2\left[\left(\bar{\lambda}\right)^{n+2}D_0+\left(\bar{\lambda}\right)^{n+3}D_0\right]+S^3\left[\left(\bar{\lambda}\right)^{n+4}D_0+\left(\bar{\lambda}\right)^{n+5}D_0\right]+\cdots$$
$$+S^{m-1}\left[\left(\bar{\lambda}\right)^{n+2m-4}D_0+\left(\bar{\lambda}\right)^{n+2m-3}D_0\right]+S^{m-1}.\left(\bar{\lambda}\right)^{n+2m-2}D_0$$
$$\leq S.\left(\bar{\lambda}\right)^{n}\left[\bar{1}+S.\left(\bar{\lambda}\right)^{2}+S^2.\left(\bar{\lambda}\right)^{4}+\cdots..\right]D_0+S.\left(\bar{\lambda}\right)^{n+1}\left[\bar{1}+S.\left(\bar{\lambda}\right)^{2}+S^2.\left(\bar{\lambda}\right)^{4}+\cdots..\right]D_0$$
$$+S^{m-1}.\left(\bar{\lambda}\right)^{n+2m-2}D_0$$

$$=\frac{\bar{1}+\bar{\lambda}}{\bar{1}-S.\left(\bar{\lambda}\right)^{2}}\,S.\left(\bar{\lambda}\right)^{n}D_0+S^{m-1}.\left(\bar{\lambda}\right)^{n+2m-2}D_0\quad\left(\,as\ S.\left(\bar{\lambda}\right)^{2}<\bar{1}\right)$$
$$<\frac{\bar{1}+\bar{\lambda}}{\bar{1}-S.\left(\bar{\lambda}\right)^{2}}\,S.\left(\bar{\lambda}\right)^{n}D_0+S^{2m}.\left(\bar{\lambda}\right)^{n+2m-2}D_0\quad\left(as\ S>\bar{1}\right)$$
$$<\frac{\bar{1}+\bar{\lambda}}{\bar{1}-S.\left(\bar{\lambda}\right)^{2}}\,S.\left(\bar{\lambda}\right)^{n}D_0+\left(\bar{\lambda}\right)^{n-2}D_0\qquad\left(\,as\ S^{2m}.\left(\bar{\lambda}\right)^{2m}<\bar{1}\ \right)$$

So we have

$$D_{fs\,rb}\left(u^{n}{}_{\mu_{\mathcal{G}}(e_n)}\,,\,u^{n+2m}{}_{\mu_{\mathcal{G}}(e_{n+2m})}\right)<\frac{\bar{1}+\bar{\lambda}}{\bar{1}-S.\left(\bar{\lambda}\right)^{2}}\,S.\left(\bar{\lambda}\right)^{n}D_0+\left(\bar{\lambda}\right)^{n-2}D_0\quad\ldots\ldots\ldots(4.5)$$

Since $\bar{\lambda}\in[0,\frac{\bar{1}}{s})$ *from (4.4) and (4.5) we have*

$$lim_{n\to\infty}\ D_{fs\,rb}\left(u^{n}{}_{\mu_{\mathcal{G}}(e_n)}\,,\,u^{n+p}{}_{\mu_{\mathcal{G}}(e_{n+p})}\right)=\bar{0}\ \ for\ all\ p\geq 1\qquad\ldots\ldots\ldots(4.6)$$

So, the $\mathcal{F}_{ss}-$ *sequence* $\langle\, u^{n}{}_{\mu_{\mathcal{G}}(e_n)}\rangle$ *in* $\left(\mathcal{F}_{ss}(C_A),D_{fs\,rb}\right)$ *is a* $\mathcal{F}_{ss}-$ *Cauchy sequence. By the fact that* $\left(\mathcal{F}_{ss}(C_A),D_{fs\,rb}\right)$ *is complete, there is* $u_{\mu_{\mathcal{G}}(a)}\ \tilde{\tilde{\in}}\ \mathcal{F}_{ss}(C_A)$ *such that*

$$lim_{n\to\infty}\ u^{n}{}_{\mu_{\mathcal{G}}(e_n)}\cong u_{\mu_{\mathcal{G}}(a)}\qquad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4.7)$$

We will prove that $u_{\mu_{\mathcal{G}}(a)}$ *is a* $\mathcal{F}_{ss}-$*fixed point of* T. *Again, for any* $n\in N$, *we have.*

$$D_{fs\,rb}\left(u_{\mu_{\mathcal{G}}(a)}\,,T\left(u_{\mu_{\mathcal{G}}(a)}\right)\right)$$
$$\leq S\left[D_{fs\,rb}\left(u_{\mu_{\mathcal{G}}(a)}\,,\,u^{n}{}_{\mu_{\mathcal{G}}(e_n)}\right)+D_{fs\,rb}\left(u^{n}{}_{\mu_{\mathcal{G}}(e_n)}\,,\,u^{n+1}{}_{\mu_{\mathcal{G}}(e_{n+1})}\right)\right.$$
$$\left.+D_{fs\,rb}\left(u^{n+1}{}_{\mu_{\mathcal{G}}(e_{n+1})}\,,T\left(u_{\mu_{\mathcal{G}}(a)}\right)\right)\right]$$

$$= S\left[D_{fs\,rb}\left(u_{\mu_{\mathcal{G}}(a)} , u^n{}_{\mu_{\mathcal{G}}(e_n)}\right) + D_n + D_{fs\,rb}\left(T\left(u^n{}_{\mu_{\mathcal{G}}(e_n)}\right) ,T\left(u_{\mu_{\mathcal{G}}(a)}\right)\right)\right]$$
$$\le S\left[D_{fs\,rb}\left(u_{\mu_{\mathcal{G}}(a)} , u^n{}_{\mu_{\mathcal{G}}(e_n)}\right) + D_n + \bar{\lambda} D_{fs\,rb}\left(u^n{}_{\mu_{\mathcal{G}}(e_n)} , u_{\mu_{\mathcal{G}}(a)}\right)\right]$$

By using (4.6)and (4.7)we have $D_{fs\,rb}\left(u_{\mu_{\mathcal{G}}(a)} ,T\left(u_{\mu_{\mathcal{G}}(a)}\right)\right) = \bar{0}$

i.e., $u_{\mu_{\mathcal{G}}(a)} = T\left(u_{\mu_{\mathcal{G}}(a)}\right)$. *As a result,* $u_{\mu_{\mathcal{G}}(a)}$ *is a fixed point of* T.
Now, we will prove $u_{\mu_{\mathcal{G}}(a)}$ *is unique* $\mathcal{F}_{ss}-$*fixed point of* T.
Let $u_{\mu_{\mathcal{G}}(b)}$ *be another* $\mathcal{F}_{ss}-$*fixed point in* T. *Then, from (4.1), it follows that*

$$D_{fs\,rb}\left(u_{\mu_{\mathcal{G}}(a)} , u_{\mu_{\mathcal{G}}(b)}\right) = D_{fs\,rb}\left(T\left(u_{\mu_{\mathcal{G}}(a)}\right) ,T\left(u_{\mu_{\mathcal{G}}(b)}\right)\right) \le \bar{\lambda}\, D_{fs\,rb}\left(u_{\mu_{\mathcal{G}}(a)} , u_{\mu_{\mathcal{G}}(b)}\right)$$

$< D_{fs\,rb}\left(u_{\mu_{\mathcal{G}}(a)} , u_{\mu_{\mathcal{G}}(b)}\right)$ *(as* $\bar{\lambda} < \bar{1}$ *) which is not possible. Therefore*
We must have $D_{fs\,rb}\left(u_{\mu_{\mathcal{G}}(a)} , u_{\mu_{\mathcal{G}}(b)}\right) = 0$, *implying that* $u_{\mu_{\mathcal{G}}(a)} = u_{\mu_{\mathcal{G}}(b)}$. *As a result,* $\mathcal{F}_{ss}-$ *fixed point is unique.* ■

Theorem(4.1.3): *Let* $\left(\mathcal{F}_{ss}(C_A), D_{fs\,rb}\right)$ *be a complete* $\mathcal{F}_{ss}-$*RbMS with coefficients* $S > 1$ *and* $T\colon \mathcal{F}_{ss}(C_A) \to \mathcal{F}_{ss}(C_A)$ *be a* $\mathcal{F}_{ss}-$ *mapping satisfying*

$$D_{fs\,rb}\left(T\left(u^1{}_{\mu_{\mathcal{G}}(e_1)}\right) ,T\left(u^2{}_{\mu_{\mathcal{G}}(e_2)}\right)\right) \le \bar{\lambda}\left[D_{fs\,rb}\left(u^1{}_{\mu_{\mathcal{G}}(e_1)} ,T\left(u^1{}_{\mu_{\mathcal{G}}(e_1)}\right)\right) + D_{fs\,rb}\left(u^2{}_{\mu_{\mathcal{G}}(e_2)} ,T\left(u^2{}_{\mu_{\mathcal{G}}(e_2)}\right)\right)\right] ..(4.8)$$

for each $u^1{}_{\mu_{\mathcal{G}}(e_1)} , u^2{}_{\mu_{\mathcal{G}}(e_2)} \,\widetilde{\widetilde{\in}}\, \mathcal{F}_{ss}(C_A)$ *where* $\bar{\lambda} \in [\bar{0}, \frac{\bar{1}}{S+1})$. *Then* T *has a unique* $\mathcal{F}_{ss}-fixed\ pint$.

proof *Let* $u_{\mu_{\mathcal{G}}(e_0)} \,\widetilde{\widetilde{\in}}\, \mathcal{F}_{ss}(C_A)$ *be arbitrary. We define a* $\mathcal{F}_{ss}-$ *sequence* $\langle u^n{}_{\mu_{\mathcal{G}}(e_n)}\rangle$ *by* $T\left(u^n{}_{\mu_{\mathcal{G}}(e_n)}\right) = u^{n+1}{}_{\mu_{\mathcal{G}}(e_{n+1})}$. *We will prove that* $\langle u^n{}_{\mu_{\mathcal{G}}(e_n)}\rangle$ *is a Cauchy sequence. If* $u^n{}_{\mu_{\mathcal{G}}(e_n)} = u^{n+1}{}_{\mu_{\mathcal{G}}(e_{n+1})}$ *, then* $u^n{}_{\mu_{\mathcal{G}}(e_n)}$ *is a* $\mathcal{F}_{ss}-$*fixed point of the* $\mathcal{F}_{ss}-$ *mapping* T. *Now let's assume that for any* $n \ge 0$, $u^n{}_{\mu_{\mathcal{G}}(e_n)} \ne u^{n+1}{}_{\mu_{\mathcal{G}}(e_{n+1})}$ *setting* $D_{fs\,rb}\left(u^n{}_{\mu_{\mathcal{G}}(e_n)} , u^{n+1}{}_{\mu_{\mathcal{G}}(e_{n+1})}\right) = D_n$ *So from (4.8) we have*

$$D_{fs\,rb}\left(u^n{}_{\mu_{\mathcal{G}}(e_n)} , u^{n+1}{}_{\mu_{\mathcal{G}}(e_{n+1})}\right) = D_{fs\,rb}\left(T\left(u^{n-1}{}_{\mu_{\mathcal{G}}(e_{n-1})}\right) ,T\left(u^n{}_{\mu_{\mathcal{G}}(e_n)}\right)\right)$$
$$\le \bar{\lambda}\left[D_{fs\,rb}\left(u^{n-1}{}_{\mu_{\mathcal{G}}(e_{n-1})} ,T\left(u^{n-1}{}_{\mu_{\mathcal{G}}(e_{n-1})}\right)\right) + D_{fs\,rb}\left(u^n{}_{\mu_{\mathcal{G}}(e_n)} ,T\left(u^n{}_{\mu_{\mathcal{G}}(e_n)}\right)\right)\right]$$
$$= \bar{\lambda}\left[D_{fs\,rb}\left(u^{n-1}{}_{\mu_{\mathcal{G}}(e_{n-1})} , u^n{}_{\mu_{\mathcal{G}}(e_n)}\right) + D_{fs\,rb}\left(u^n{}_{\mu_{\mathcal{G}}(e_n)} , u^{n+1}{}_{\mu_{\mathcal{G}}(e_{n+1})}\right)\right] = \bar{\lambda}[D_{n-1} + D_n]\ \ SO$$

$D_n \le \bar{\lambda}[D_{n-1} + D_n]$ *We obtain* $D_n \le \frac{\bar{\lambda}}{1-\bar{\lambda}} D_{n-1} = \bar{\beta} D_{n-1}$ *where* $\bar{\beta} = \frac{\bar{\lambda}}{1-\bar{\lambda}} < \frac{\bar{1}}{s}$ *(as* $\bar{\lambda} < \frac{\bar{1}}{S+1}$*). We obtain by repeating this process* $D_n \le \left(\bar{\beta}\right)^n D_0$ *……(4.9)*
We can also assume that $u_{\mu_{\mathcal{G}}(e_0)}$ *is not a* $\mathcal{F}_{ss}-$ *periodic point of* T. *Indeed, if* $u_{\mu_{\mathcal{G}}(e_0)} = u^n{}_{\mu_{\mathcal{G}}(e_n)}$, *then using (4.9), we have for any* $n \ge 2$,

$D_{fs\,rb}\left(u_{\mu_{\mathcal{G}(e_0)}}, T\left(u_{\mu_{\mathcal{G}(e_0)}}\right)\right) = D_{fs\,rb}\left(u^n{}_{\mu_{\mathcal{G}(e_n)}}, T\left(u^n{}_{\mu_{\mathcal{G}(e_n)}}\right)\right)$

$D_{fs\,rb}\left(u_{\mu_{\mathcal{G}(e_0)}}, u_{\mu_{\mathcal{G}(e_1)}}\right) = D_{fs\,rb}\left(u^n{}_{\mu_{\mathcal{G}(e_n)}}, u^{n+1}{}_{\mu_{\mathcal{G}(e_{n+1})}}\right)$

$D_0 = D_n \le (\bar{\beta})^n D_0$. *a contradiction. As a result, we must have* $D_0 = 0$, *implying that* $u_{\mu_{\mathcal{G}(e_0)}} = u_{\mu_{\mathcal{G}(e_1)}}$ *and thus* $u_{\mu_{\mathcal{G}(e_0)}}$ *is a* $\mathcal{F}_{ss}-$ *fixed point of* T.

As a result, we suppose that , $u^n{}_{\mu_{\mathcal{G}(e_n)}} \neq u^m{}_{\mu_{\mathcal{G}(e_m)}}$ *for all distinct* $n, m \in N$. *Using* (4.8) *and* (4.9) *again for each* $n \in N$, *we get*

$D_{fs\,rb}\left(u^n{}_{\mu_{\mathcal{G}(e_n)}}, u^{n+2}{}_{\mu_{\mathcal{G}(e_{n+2})}}\right) = D_{fs\,rb}\left(T\left(u^{n-1}{}_{\mu_{\mathcal{G}(e_{n-1})}}\right), T\left(u^{n+1}{}_{\mu_{\mathcal{G}(e_{n+1})}}\right)\right) \le$
$\bar{\lambda}\left[D_{fs\,rb}\left(u^{n-1}{}_{\mu_{\mathcal{G}(e_{n-1})}}, T\left(u^{n-1}{}_{\mu_{\mathcal{G}(e_{n-1})}}\right)\right) + D_{fs\,rb}\left(u^{n+1}{}_{\mu_{\mathcal{G}(e_{n+1})}}, T\left(\left(u^{n+1}{}_{\mu_{\mathcal{G}(e_{n+1})}}\right)\right)\right)\right] =$
$\bar{\lambda}\left[D_{fs\,rb}\left(u^{n-1}{}_{\mu_{\mathcal{G}(e_{n-1})}}, u^n{}_{\mu_{\mathcal{G}(e_n)}}\right) + D_{fs\,rb}\left(u^{n+1}{}_{\mu_{\mathcal{G}(e_{n+1})}}, u^{n+2}{}_{\mu_{\mathcal{G}(e_{n+2})}}\right)\right] = \bar{\lambda}\,[D_{n-1}+D_{n+1}] \le$
$\bar{\lambda}\left[(\bar{\beta})^{n-1} D_0 + (\bar{\beta})^{n+1} D_0\right] = \bar{\lambda}\,(\bar{\beta})^{n-1}[1 + (\bar{\beta})^2]\,D_0$ *therefore* $D_{fs\,rb}\left(u^n{}_{\mu_{\mathcal{G}(e_n)}}, u^{n+2}{}_{\mu_{\mathcal{G}(e_{n+2})}}\right) \le \bar{\gamma}\,(\bar{\beta})^{n-1} D_0$ …………(4.10)

where $\bar{\gamma} = \bar{\lambda}\,[\,1 + (\bar{\beta})^2]$. *We consider* $D_{fs\,rb}\left(u^n{}_{\mu_{\mathcal{G}(e_n)}}, u^{n+p}{}_{\mu_{\mathcal{G}(e_{n+p})}}\right)$ *in two cases for the* $\mathcal{F}_{ss}-$ *sequence* $u^n{}_{\mu_{\mathcal{G}(e_n)}}$ *then using (4.9) we get*

$D_{fs\,rb}\left(u^n{}_{\mu_{\mathcal{G}(e_n)}}, u^{n+2m+1}{}_{\mu_{\mathcal{G}(e_{n+2m+1})}}\right)$
$\le S\left[D_{fs\,rb}\left(u^n{}_{\mu_{\mathcal{G}(e_n)}}, u^{n+1}{}_{\mu_{\mathcal{G}(e_{n+1})}}\right) + D_{fs\,rb}\left(u^{n+1}{}_{\mu_{\mathcal{G}(e_{n+1})}}, u^{n+2}{}_{\mu_{\mathcal{G}(e_{n+2})}}\right) + D_{fs\,rb}\left(u^{n+2}{}_{\mu_{\mathcal{G}(e_{n+2})}}, u^{n+2m+1}{}_{\mu_{\mathcal{G}(e_{n+2m+1})}}\right)\right]$
$\le S\,[D_n + D_{n+1}] + S^2\left[D_{fs\,rb}\left(u^{n+2}{}_{\mu_{\mathcal{G}(e_{n+2})}}, u^{n+3}{}_{\mu_{\mathcal{G}(e_{n+3})}}\right) + D_{fs\,rb}\left(u^{n+3}{}_{\mu_{\mathcal{G}(e_{n+3})}}, u^{n+4}{}_{\mu_{\mathcal{G}(e_{n+4})}}\right) + D_{fs\,rb}\left(u^{n+4}{}_{\mu_{\mathcal{G}(e_{n+4})}}, u^{n+2m+1}{}_{\mu_{\mathcal{G}(e_{n+2m+1})}}\right)\right]$
$\vdots$
$\le S\,[D_n + D_{n+1}] + S^2\,[D_{n+2} + D_{n+3}] + S^3[D_{n+4} + D_{n+5}] + \cdots + S^m D_{n+2m}$
$\le S\left[(\bar{\beta})^n D_0 + (\bar{\beta})^{n+1} D_0\right] + S^2\left[(\bar{\beta})^{n+2} D_0 + (\bar{\beta})^{n+3} D_0\right] + S^3\left[(\bar{\beta})^{n+4} D_0 + (\bar{\beta})^{n+5} D_0\right] + \cdots + S^m(\bar{\beta})^{n+2m} D_0$
$\le S.(\bar{\beta})^n\left[\bar{1} + S.(\bar{\beta})^2 + S^2.(\bar{\beta})^4 + \cdots..\right] D_0 + S.(\bar{\beta})^{n+1}\left[\bar{1} + S.(\bar{\beta})^2 + S^2.(\bar{\beta})^4 + \cdots..\right] D_0$
$= \frac{\bar{1}+\bar{\beta}}{\bar{1}-S.(\bar{\beta})^2}\, S.(\bar{\beta})^n D_0 \quad (as\ S.(\bar{\beta})^2 < \bar{1})$

Therefore $D_{fs\,rb}\left(u^n{}_{\mu_{\mathcal{G}(e_n)}}, u^{n+2m+1}{}_{\mu_{\mathcal{G}(e_{n+2m+1})}}\right) \le \frac{\bar{1}+\bar{\beta}}{\bar{1}-S.(\bar{\beta})^2}\, S.(\bar{\beta})^n D_0$ …….(4.11)

The second case If p *is even i.e* $p = 2m$, *we can use (4.9) and (4.10) to get.*

$D_{fs\,rb}\left(u^n{}_{\mu_{\mathcal{G}(e_n)}}, u^{n+p}{}_{\mu_{\mathcal{G}(e_{n+p})}}\right) = D_{fs\,rb}\left(u^n{}_{\mu_{\mathcal{G}(e_n)}}, u^{n+2m}{}_{\mu_{\mathcal{G}(e_{n+2m})}}\right)$

$$\leq S\Big[D_{fs\,rb}\Big(\ u^{n}{}_{\mu_{\mathcal{G}}(e_n)}\ ,\ u^{n+1}{}_{\mu_{\mathcal{G}}(e_{n+1})}\Big)+D_{fs\,rb}\Big(\ u^{n+1}{}_{\mu_{\mathcal{G}}(e_{n+1})}\ ,\ u^{n+2}{}_{\mu_{\mathcal{G}}(e_{n+2})}\Big)$$
$$+D_{fs\,rb}\Big(\ u^{n+2}{}_{\mu_{\mathcal{G}}(e_{n+2})}\ ,\ u^{n+2m}{}_{\mu_{\mathcal{G}}(e_{n+2m})}\Big)\Big]$$
$$\leq S\,[D_n+D_{n+1}]$$
$$+S^2\Big[D_{fs\,rb}\Big(\ u^{n+2}{}_{\mu_{\mathcal{G}}(e_{n+2})}\ ,\ u^{n+3}{}_{\mu_{\mathcal{G}}(e_{n+3})}\Big)+D_{fs\,rb}\Big(\ u^{n+3}{}_{\mu_{\mathcal{G}}(e_{n+3})}\ ,\ u^{n+4}{}_{\mu_{\mathcal{G}}(e_{n+4})}\Big)$$
$$+D_{fs\,rb}\Big(\ u^{n+4}{}_{\mu_{\mathcal{G}}(e_{n+4})}\ ,\ u^{n+2m}{}_{\mu_{\mathcal{G}}(e_{n+2m})}\Big)\Big]$$
$$=S\,[D_n+D_{n+1}]+S^2\Big[D_{n+2}+D_{n+3}+D_{fs\,rb}\Big(\ u^{n+4}{}_{\mu_{\mathcal{G}}(e_{n+4})}\ ,\ u^{n+2m}{}_{\mu_{\mathcal{G}}(e_{n+2m})}\Big)\Big]$$
$$\vdots$$
$$\vdots$$
$$\leq S\,[D_n+D_{n+1}]+S^2\,[D_{n+2}+D_{n+3}]+S^3[D_{n+4}+D_{n+5}]+\cdots+S^{m-1}[D_{n+2m-4}+D_{n+2m-3}]$$
$$+S^{m-1}[D_{fs\,rb}\Big(\ u^{n+2m-2}{}_{\mu_{\mathcal{G}}(e_{n+2m-2})}\ ,\ u^{n+2m}{}_{\mu_{\mathcal{G}}(e_{n+2m})}\Big)]$$
$$\leq S\,\Big[(\bar\beta\,)^{n}\,D_0+(\bar\beta\,)^{n+1}\,D_0\Big]+S^2\,\Big[(\bar\beta\,)^{n+2}\,D_0+(\bar\beta\,)^{n+3}\,D_0\Big]+S^3\,\Big[(\bar\beta\,)^{n+4}\,D_0+(\bar\beta\,)^{n+5}\,D_0\Big]+\cdots$$
$$+S^{m-1}[(\bar\beta\,)^{n+2m-4}\,D_0+(\bar\beta\,)^{n+2m-3}\,D_0]+S^{m-1}.(\bar\beta\,)^{n+2m-2}\,D_0$$
$$\leq S.(\bar\beta\,)^{n}\Big[\bar1+S.(\bar\beta\,)^{2}+S^2.(\bar\beta\,)^{4}+\cdots..\Big]\,D_0+S.(\bar\beta\,)^{n+1}\Big[\bar1+S.(\bar\beta\,)^{2}+S^2.(\bar\beta\,)^{4}+\cdots..\Big]\,D_0$$
$$+S^{m-1}.(\bar\beta\,)^{n+2m-2}\,D_0$$

$$=\frac{\bar1+\bar\beta}{\bar1-S.(\bar\beta)^2}\,S.(\bar\beta)^{n}\,D_0+S^{m-1}.(\bar\beta)^{n+2m-2}\,D_0\qquad(\ as\ S.(\bar\beta)^2<\bar1)$$
$$<\frac{\bar1+\bar\beta}{\bar1-S.(\bar\beta)^2}\,S.(\bar\beta)^{n}\,D_0+S^{2m}.(\bar\beta)^{n+2m-2}\,D_0\quad(as\ \ S\ >\bar1)$$
$$<\frac{\bar1+\bar\beta}{\bar1-S.(\bar\beta)^2}\,S.(\bar\beta)^{n}\,D_0+(\bar\beta)^{n-2}\,D_0\qquad(\ as\ S^{2m}.(\bar\beta)^{2m}<\bar1\ )$$

So we have

$$D_{fs\,rb}\Big(\ u^{n}{}_{\mu_{\mathcal{G}}(e_n)}\ ,\ u^{n+2m}{}_{\mu_{\mathcal{G}}(e_{n+2m})}\Big)<\frac{\bar1+\bar\lambda}{\bar1-S.(\bar\lambda)^2}\,S.(\bar\beta)^{n}\,D_0+(\bar\beta)^{n-2}\,D_0\quad\ldots\ldots\ldots(4.12)$$

Since $\bar\beta\in[0,\frac{\bar1}{s})$ *from (4.11) and (4.12) we have*

$$lim_{n\to\infty}\ D_{fs\,rb}\Big(\ u^{n}{}_{\mu_{\mathcal{G}}(e_n)}\ ,\ u^{n+p}{}_{\mu_{\mathcal{G}}(e_{n+p})}\Big)=\bar0\ \ for\ all\ p\geq1\qquad\ldots\ldots\ldots(4.13)$$

So, the $\mathcal{F}_{ss}-$ *sequence* $\langle\ u^{n}{}_{\mu_{\mathcal{G}}(e_n)}\rangle$ *in* $(\mathcal{F}_{ss}(C_A),D_{fs\,rb})$ *is a* $\mathcal{F}_{ss}-$ *Cauchy sequence. By the fact that* $(\mathcal{F}_{ss}(C_A),D_{fs\,rb})$ *is complete, there is* $u_{\mu_{\mathcal{G}}(a)}\ \tilde{\tilde{\in}}\ \mathcal{F}_{ss}(C_A)$ *such that*

$$lim_{n\to\infty}\ \ u^{n}{}_{\mu_{\mathcal{G}}(e_n)}\cong\ u_{\mu_{\mathcal{G}}(a)}\qquad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4.14)$$

We will prove that $u_{\mu_{\mathcal{G}}(a)}$ *is a* $\mathcal{F}_{ss}-$*fixed point of* T. *Again, for any* $n\in N$, *we have.*

$$D_{fs\,rb}\Big(\ u_{\mu_{\mathcal{G}}(a)}\ ,T\ \Big(\ u_{\mu_{\mathcal{G}}(a)}\Big)\Big)$$
$$\leq S\Big[D_{fs\,rb}\Big(\ u_{\mu_{\mathcal{G}}(a)}\ ,\ u^{n}{}_{\mu_{\mathcal{G}}(e_n)}\Big)+D_{fs\,rb}\Big(\ u^{n}{}_{\mu_{\mathcal{G}}(e_n)}\ ,\ u^{n+1}{}_{\mu_{\mathcal{G}}(e_{n+1})}\Big)$$
$$+D_{fs\,rb}\Big(\ u^{n+1}{}_{\mu_{\mathcal{G}}(e_{n+1})}\ ,T\ \Big(\ u_{\mu_{\mathcal{G}}(a)}\Big)\Big)\Big]$$

$$= S\left[D_{fs\,rb}\left(u_{\mu_{\mathcal{G}(a)}}, u^n{}_{\mu_{\mathcal{G}(e_n)}}\right) + D_n + D_{fs\,rb}\left(T\left(u^n{}_{\mu_{\mathcal{G}(e_n)}}\right), T\left(u_{\mu_{\mathcal{G}(a)}}\right)\right)\right]$$

$$\leq S\Big[D_{fs\,rb}\left(u_{\mu_{\mathcal{G}(a)}}, u^n{}_{\mu_{\mathcal{G}(e_n)}}\right) + D_n + \bar{\lambda}\{D_{fs\,rb}\left(u^n{}_{\mu_{\mathcal{G}(e_n)}}, T\left(u^n{}_{\mu_{\mathcal{G}(e_n)}}\right)\right)$$

$$+ D_{fs\,rb}\left(u_{\mu_{\mathcal{G}(a)}}, T\left(u_{\mu_{\mathcal{G}(a)}}\right)\right)\}\Big]$$

$$(1 - S\bar{\lambda}).D_{fs\,rb}\left(u_{\mu_{\mathcal{G}(a)}}, T\left(u_{\mu_{\mathcal{G}(a)}}\right)\right)$$

$$\leq S\left[D_{fs\,rb}\left(u_{\mu_{\mathcal{G}(a)}}, u^n{}_{\mu_{\mathcal{G}(e_n)}}\right) + (\bar{\beta})^n D_0 + \bar{\lambda} D_{fs\,rb}\left(u^n{}_{\mu_{\mathcal{G}(e_n)}}, T\left(u^n{}_{\mu_{\mathcal{G}(e_n)}}\right)\right)\right]$$

By using (4.13) and (4.14) and the fact $\bar{\lambda} < \frac{\bar{1}}{S+\bar{1}}$ *we have*

$D_{fs\,rb}\left(u_{\mu_{\mathcal{G}(a)}}, T\left(u_{\mu_{\mathcal{G}(a)}}\right)\right) = \bar{0}$ *so* $u_{\mu_{\mathcal{G}(a)}} = T\left(u_{\mu_{\mathcal{G}(a)}}\right)$.

Then $u_{\mu_{\mathcal{G}(a)}}$ *is* $\mathcal{F}_{ss}-$*fixed point of* T

Now, we will prove $u_{\mu_{\mathcal{G}(a)}}$ *is unique* $\mathcal{F}_{ss}-$*fixed point of* T

Let $u_{\mu_{\mathcal{G}(b)}}$ *be another* $\mathcal{F}_{ss}-$*fixed point in* T. *Then, from* (4.8), *it follows that*

$$D_{fs\,rb}\left(u_{\mu_{\mathcal{G}(a)}}, u_{\mu_{\mathcal{G}(b)}}\right) = D_{fs\,rb}\left(T\left(u_{\mu_{\mathcal{G}(a)}}\right), T\left(u_{\mu_{\mathcal{G}(b)}}\right)\right)$$

$$\leq \bar{\lambda}\left[D_{fs\,rb}\left(u_{\mu_{\mathcal{G}(a)}}, T\left(u_{\mu_{\mathcal{G}(a)}}\right)\right) + D_{fs\,rb}\left(u_{\mu_{\mathcal{G}(b)}}, T\left(u_{\mu_{\mathcal{G}(b)}}\right)\right)\right]$$

$$= \bar{\lambda}\left[D_{fs\,rb}\left(u_{\mu_{\mathcal{G}(a)}}, u_{\mu_{\mathcal{G}(a)}}\right) + D_{fs\,rb}\left(u_{\mu_{\mathcal{G}(b)}}, u_{\mu_{\mathcal{G}(b)}}\right)\right] = \bar{0}$$

Therefore $D_{fs\,rb}\left(u_{\mu_{\mathcal{G}(a)}}, u_{\mu_{\mathcal{G}(b)}}\right) = 0$, *implying that* $u_{\mu_{\mathcal{G}(a)}} = u_{\mu_{\mathcal{G}(b)}}$. *As a result,* $\mathcal{F}_{ss}-$ *fixed point is unique.* ■

Theorem(4.1.4): *Let* $(\mathcal{F}_{ss}(C_A), D_{fs\,rb})$ *be a complete* $\mathcal{F}_{ss}-$*RbMS with coefficients* $S > 1$ *and* $T: \mathcal{F}_{ss}(C_A) \to \mathcal{F}_{ss}(C_A)$ *be a* $\mathcal{F}_{ss}-$ *mapping satisfying*

$$(4.15) \qquad D_{fs\,rb}\left(T\left(u^1{}_{\mu_{\mathcal{G}(e_1)}}\right), T\left(u^2{}_{\mu_{\mathcal{G}(e_2)}}\right)\right) \leq$$

$$\bar{\lambda}.\max\left[\begin{array}{c} D_{fs\,rb}\left(u^1{}_{\mu_{\mathcal{G}(e_1)}}, u^2{}_{\mu_{\mathcal{G}(e_2)}}\right), D_{fs\,rb}\left(u^1{}_{\mu_{\mathcal{G}(e_1)}}, T\left(u^1{}_{\mu_{\mathcal{G}(e_1)}}\right)\right), \\ D_{fs\,rb}\left(u^2{}_{\mu_{\mathcal{G}(e_2)}}, T\left(u^2{}_{\mu_{\mathcal{G}(e_2)}}\right)\right), \frac{D_{fs\,rb}\left(u^1{}_{\mu_{\mathcal{G}(e_1)}}, T\left(u^1{}_{\mu_{\mathcal{G}(e_1)}}\right)\right) + D_{fs\,rb}\left(u^2{}_{\mu_{\mathcal{G}(e_2)}}, T\left(u^2{}_{\mu_{\mathcal{G}(e_2)}}\right)\right)}{2} \end{array}\right] \quad \text{for some}$$

$\bar{\lambda} \in (\bar{0}, \bar{1})$ *every two different* $\mathcal{F}_{ss}-$*points* $u^1{}_{\mu_{\mathcal{G}(e_1)}}$ *and* $u^2{}_{\mu_{\mathcal{G}(e_2)}}$ *in* $\mathcal{F}_{ss}(C_A)$. *Hence,* T *has an indistinctive fixed point.*

Proof *Let* $u_{\mu_{\mathcal{G}(e_0)}} \,\tilde{\tilde{\in}}\, \mathcal{F}_{ss}(C_A)$ *be arbitrary. We define a* $\mathcal{F}_{ss}-$ *sequence* $\langle u^n{}_{\mu_{\mathcal{G}(e_n)}} \rangle$ *by* $T\left(u^n{}_{\mu_{\mathcal{G}(e_n)}}\right) = u^{n+1}{}_{\mu_{\mathcal{G}(e_{n+1})}}$. *If an integer* $n > 0$ *such that* $u^{n+1}{}_{\mu_{\mathcal{G}(e_{n+1})}} = u^n{}_{\mu_{\mathcal{G}(e_n)}}$, *then the element* $u^n{}_{\mu_{\mathcal{G}(e_n)}}$ *is clearly a* $\mathcal{F}_{ss}-$ *fixed point of a* $\mathcal{F}_{ss}-$ *mapping* T. *Now let's assume that,* $u^n{}_{\mu_{\mathcal{G}(e_n)}} \neq u^{n+1}{}_{\mu_{\mathcal{G}(e_{n+1})}}$ *for any* $n \geq 0$ *from* (4,15) *we have*

$$D_{fs\,rb}\left(u^{n}{}_{\mu_{\mathcal{G}}(e_n)} ,\ u^{n+1}{}_{\mu_{\mathcal{G}}(e_{n+1})}\right) =$$
$$D_{fs\,rb}\left(T\left(u^{n-1}{}_{\mu_{\mathcal{G}}(e_{n-1})}\right) ,T\left(u^{n}{}_{\mu_{\mathcal{G}}(e_n)}\right)\right)$$
$$\leq \bar{\lambda}\max\left[\begin{array}{c} D_{fs\,rb}\left(u^{n-1}{}_{\mu_{\mathcal{G}}(e_{n-1})},\ u^{n}{}_{\mu_{\mathcal{G}}(e_n)}\right), D_{fs\,rb}\left(u^{n-1}{}_{\mu_{\mathcal{G}}(e_{n-1})} ,T\left(u^{n-1}{}_{\mu_{\mathcal{G}}(e_{n-1})}\right)\right) \\ ,D_{fs\,rb}\left(u^{n}{}_{\mu_{\mathcal{G}}(e_n)} ,T\left(u^{n}{}_{\mu_{\mathcal{G}}(e_n)}\right)\right), \\ \dfrac{D_{fs\,rb}\left(u^{n-1}{}_{\mu_{\mathcal{G}}(e_{n-1})} ,T\left(u^{n-1}{}_{\mu_{\mathcal{G}}(e_{n-1})}\right)\right) + D_{fs\,rb}\left(u^{n}{}_{\mu_{\mathcal{G}}(e_n)} ,T\left(u^{n}{}_{\mu_{\mathcal{G}}(e_n)}\right)\right)}{2} \end{array}\right]$$

$$= \bar{\lambda}\max\left[\begin{array}{c} D_{fs\,rb}\left(u^{n-1}{}_{\mu_{\mathcal{G}}(e_{n-1})},\ u^{n}{}_{\mu_{\mathcal{G}}(e_n)}\right), D_{fs\,rb}\left(u^{n-1}{}_{\mu_{\mathcal{G}}(e_{n-1})} ,T\left(u^{n-1}{}_{\mu_{\mathcal{G}}(e_{n-1})}\right)\right) \\ ,D_{fs\,rb}\left(u^{n}{}_{\mu_{\mathcal{G}}(e_n)} ,T\left(u^{n}{}_{\mu_{\mathcal{G}}(e_n)}\right)\right) \end{array}\right]$$
$$= \bar{\lambda}\max\left[\begin{array}{c} D_{fs\,rb}\left(u^{n-1}{}_{\mu_{\mathcal{G}}(e_{n-1})},\ u^{n}{}_{\mu_{\mathcal{G}}(e_n)}\right), D_{fs\,rb}\left(u^{n-1}{}_{\mu_{\mathcal{G}}(e_{n-1})} ,\ u^{n}{}_{\mu_{\mathcal{G}}(e_n)}\right) \\ ,D_{fs\,rb}\left(u^{n}{}_{\mu_{\mathcal{G}}(e_n)} ,\ u^{n+1}{}_{\mu_{\mathcal{G}}(e_{n+1})}\right) \end{array}\right]$$

If $D_{fs\,rb}\left(u^{n-1}{}_{\mu_{\mathcal{G}}(e_{n-1})} ,\ u^{n}{}_{\mu_{\mathcal{G}}(e_n)}\right) \leq D_{fs\,rb}\left(u^{n}{}_{\mu_{\mathcal{G}}(e_n)} ,\ u^{n+1}{}_{\mu_{\mathcal{G}}(e_{n+1})}\right)$ *we have*

$$D_{fs\,rb}\left(u^{n}{}_{\mu_{\mathcal{G}}(e_n)} ,\ u^{n+1}{}_{\mu_{\mathcal{G}}(e_{n+1})}\right) \leq \bar{\lambda} D_{fs\,rb}\left(u^{n}{}_{\mu_{\mathcal{G}}(e_n)} ,\ u^{n+1}{}_{\mu_{\mathcal{G}}(e_{n+1})}\right)$$

it is impossible. Thus,

$$D_{fs\,rb}\left(u^{n}{}_{\mu_{\mathcal{G}}(e_n)} ,\ u^{n+1}{}_{\mu_{\mathcal{G}}(e_{n+1})}\right) \leq D_{fs\,rb}\left(u^{n-1}{}_{\mu_{\mathcal{G}}(e_{n-1})} ,\ u^{n}{}_{\mu_{\mathcal{G}}(e_n)}\right).$$

hence $\langle D_{fs\,rb}\left(u^{n}{}_{\mu_{\mathcal{G}}(e_n)} ,\ u^{n+1}{}_{\mu_{\mathcal{G}}(e_{n+1})}\right)\rangle$ *is a decreasing sequence of converging to* $\bar{l} \in \mathbb{R}^{+}(A)$, *say* , *that is* $\lim_{n\to\infty} D_{fs\,rb}\left(u^{n}{}_{\mu_{\mathcal{G}}(e_n)} ,\ u^{n+1}{}_{\mu_{\mathcal{G}}(e_{n+1})}\right) = \bar{l}$ *and*

$$\lim_{n\to\infty} D_{fs\,rb}\left(u^{n}{}_{\mu_{\mathcal{G}}(e_n)} ,\ u^{n+1}{}_{\mu_{\mathcal{G}}(e_{n+1})}\right) = \lim_{n\to\infty} D_{fs\,rb}\left(T\left(u^{n-1}{}_{\mu_{\mathcal{G}}(e_{n-1})}\right) ,T\left(u^{n}{}_{\mu_{\mathcal{G}}(e_n)}\right)\right)$$
$$\leq \bar{\lambda}\lim_{n\to\infty}\max\left[\begin{array}{c} D_{fs\,rb}\left(u^{n-1}{}_{\mu_{\mathcal{G}}(e_{n-1})},\ u^{n}{}_{\mu_{\mathcal{G}}(e_n)}\right), D_{fs\,rb}\left(u^{n-1}{}_{\mu_{\mathcal{G}}(e_{n-1})} ,T\left(u^{n-1}{}_{\mu_{\mathcal{G}}(e_{n-1})}\right)\right) \\ ,D_{fs\,rb}\left(u^{n}{}_{\mu_{\mathcal{G}}(e_n)} ,T\left(u^{n}{}_{\mu_{\mathcal{G}}(e_n)}\right)\right), \\ \dfrac{D_{fs\,rb}\left(u^{n-1}{}_{\mu_{\mathcal{G}}(e_{n-1})} ,T\left(u^{n-1}{}_{\mu_{\mathcal{G}}(e_{n-1})}\right)\right) + D_{fs\,rb}\left(u^{n}{}_{\mu_{\mathcal{G}}(e_n)} ,T\left(u^{n}{}_{\mu_{\mathcal{G}}(e_n)}\right)\right)}{2} \end{array}\right]$$

$$= \bar{\lambda}\lim_{n\to\infty}\max\left[\begin{array}{c} D_{fs\,rb}\left(u^{n-1}{}_{\mu_{\mathcal{G}}(e_{n-1})},\ u^{n}{}_{\mu_{\mathcal{G}}(e_n)}\right), D_{fs\,rb}\left(u^{n-1}{}_{\mu_{\mathcal{G}}(e_{n-1})} ,T\left(u^{n-1}{}_{\mu_{\mathcal{G}}(e_{n-1})}\right)\right) \\ ,D_{fs\,rb}\left(u^{n}{}_{\mu_{\mathcal{G}}(e_n)} ,T\left(u^{n}{}_{\mu_{\mathcal{G}}(e_n)}\right)\right) \end{array}\right]$$
$$= \bar{\lambda}\lim_{n\to\infty}\max\left[\begin{array}{c} D_{fs\,rb}\left(u^{n-1}{}_{\mu_{\mathcal{G}}(e_{n-1})},\ u^{n}{}_{\mu_{\mathcal{G}}(e_n)}\right), D_{fs\,rb}\left(u^{n-1}{}_{\mu_{\mathcal{G}}(e_{n-1})} ,\ u^{n}{}_{\mu_{\mathcal{G}}(e_n)}\right) \\ ,D_{fs\,rb}\left(u^{n}{}_{\mu_{\mathcal{G}}(e_n)} ,\ u^{n+1}{}_{\mu_{\mathcal{G}}(e_{n+1})}\right) \end{array}\right]$$

$= \bar{\lambda}\, lim_{n\to\infty}\left[D_{fs\,rb}\left(u^{n-1}{}_{\mu_{\mathcal{G}}(e_{n-1})} , u^{n}{}_{\mu_{\mathcal{G}}(e_{n})}\right)\right] = \bar{\lambda}.\bar{l}$ *so* $\bar{l} = \bar{0}$ ……..(4.16)

To show the Cauchyness of the $\mathcal{F}_{ss}-$ *sequence, assume* P *is an odd number, i.e.* $P = 2m + 1$, *and apply the decreasing property to some* m *in* N.

$$D_{fs\,rb}\left(u^{n}{}_{\mu_{\mathcal{G}}(e_{n})} , u^{n+P}{}_{\mu_{\mathcal{G}}(e_{n+P})}\right) = D_{fs\,rb}\left(u^{n}{}_{\mu_{\mathcal{G}}(e_{n})} , u^{n+2m+1}{}_{\mu_{\mathcal{G}}(e_{n+2m+1})}\right)$$
$$\le S\left[D_{fs\,rb}\left(u^{n}{}_{\mu_{\mathcal{G}}(e_{n})} , u^{n+1}{}_{\mu_{\mathcal{G}}(e_{n+1})}\right) + D_{fs\,rb}\left(u^{n+1}{}_{\mu_{\mathcal{G}}(e_{n+1})} , u^{n+2}{}_{\mu_{\mathcal{G}}(e_{n+2})}\right) + D_{fs\,rb}\left(u^{n+2}{}_{\mu_{\mathcal{G}}(e_{n+2})} , u^{n+2m+1}{}_{\mu_{\mathcal{G}}(e_{n+2m+1})}\right)\right]$$
$$\le 2.S\, D_{fs\,rb}\left(u^{n}{}_{\mu_{\mathcal{G}}(e_{n})} , u^{n+1}{}_{\mu_{\mathcal{G}}(e_{n+1})}\right) + S^2\left[D_{fs\,rb}\left(u^{n+2}{}_{\mu_{\mathcal{G}}(e_{n+2})} , u^{n+3}{}_{\mu_{\mathcal{G}}(e_{n+3})}\right) + D_{fs\,rb}\left(u^{n+3}{}_{\mu_{\mathcal{G}}(e_{n+3})} , u^{n+4}{}_{\mu_{\mathcal{G}}(e_{n+4})}\right) + D_{fs\,rb}\left(u^{n+4}{}_{\mu_{\mathcal{G}}(e_{n+4})} , u^{n+2m+1}{}_{\mu_{\mathcal{G}}(e_{n+2m+1})}\right)\right]$$
$$\le 2.S\, D_{fs\,rb}\left(u^{n}{}_{\mu_{\mathcal{G}}(e_{n})} , u^{n+1}{}_{\mu_{\mathcal{G}}(e_{n+1})}\right) + 2.S^2 D_{fs\,rb}\left(u^{n+2}{}_{\mu_{\mathcal{G}}(e_{n+2})} , u^{n+3}{}_{\mu_{\mathcal{G}}(e_{n+3})}\right) + \cdots + 2.S^m D_{fs\,rb}\left(u^{n+2m}{}_{\mu_{\mathcal{G}}(e_{n+4})} , u^{n+2m+1}{}_{\mu_{\mathcal{G}}(e_{n+2m+1})}\right)$$

$\le 2S[\bar{1} + S + S^2 + \cdots . + S^{m-1}]\ D_{fs\,rb}\left(u^{n}{}_{\mu_{\mathcal{G}}(e_{n})} , u^{n+1}{}_{\mu_{\mathcal{G}}(e_{n+1})}\right) =$
$2S\left[\frac{S^{m-1}-\bar{1}}{S-\bar{1}}\right] D_{fs\,rb}\left(u^{n}{}_{\mu_{\mathcal{G}}(e_{n})} , u^{n+1}{}_{\mu_{\mathcal{G}}(e_{n+1})}\right)$ *from* (4,16) *we get*

$D_{fs\,rb}\left(u^{n}{}_{\mu_{\mathcal{G}}(e_{n})} , u^{n+P}{}_{\mu_{\mathcal{G}}(e_{n+P})}\right) = \bar{0}$. *Assume* p *is an even number, i.e.* $p = 2m$, *then apply the decreasing property to a* m *in* N.

$$D_{fs\,rb}\left(u^{n}{}_{\mu_{\mathcal{G}}(e_{n})} , u^{n+P}{}_{\mu_{\mathcal{G}}(e_{n+P})}\right) = D_{fs\,rb}\left(u^{n}{}_{\mu_{\mathcal{G}}(e_{n})} , u^{n+2m}{}_{\mu_{\mathcal{G}}(e_{n+2m})}\right)$$
$$\le S\left[D_{fs\,rb}\left(u^{n}{}_{\mu_{\mathcal{G}}(e_{n})} , u^{n+1}{}_{\mu_{\mathcal{G}}(e_{n+1})}\right) + D_{fs\,rb}\left(u^{n+1}{}_{\mu_{\mathcal{G}}(e_{n+1})} , u^{n+2}{}_{\mu_{\mathcal{G}}(e_{n+2})}\right) + D_{fs\,rb}\left(u^{n+2}{}_{\mu_{\mathcal{G}}(e_{n+2})} , u^{n+2m}{}_{\mu_{\mathcal{G}}(e_{n+2m})}\right)\right]$$
$$\le 2.S\, D_{fs\,rb}\left(u^{n}{}_{\mu_{\mathcal{G}}(e_{n})} , u^{n+1}{}_{\mu_{\mathcal{G}}(e_{n+1})}\right) + S^2\left[D_{fs\,rb}\left(u^{n+2}{}_{\mu_{\mathcal{G}}(e_{n+2})} , u^{n+3}{}_{\mu_{\mathcal{G}}(e_{n+3})}\right) + D_{fs\,rb}\left(u^{n+3}{}_{\mu_{\mathcal{G}}(e_{n+3})} , u^{n+4}{}_{\mu_{\mathcal{G}}(e_{n+4})}\right) + D_{fs\,rb}\left(u^{n+4}{}_{\mu_{\mathcal{G}}(e_{n+4})} , u^{n+2m}{}_{\mu_{\mathcal{G}}(e_{n+2m})}\right)\right]$$
$$\le 2.S\, D_{fs\,rb}\left(u^{n}{}_{\mu_{\mathcal{G}}(e_{n})} , u^{n+1}{}_{\mu_{\mathcal{G}}(e_{n+1})}\right) + 2.S^2 D_{fs\,rb}\left(u^{n+2}{}_{\mu_{\mathcal{G}}(e_{n+2})} , u^{n+3}{}_{\mu_{\mathcal{G}}(e_{n+3})}\right) + \cdots + 2.S^m D_{fs\,rb}\left(u^{n+2m-1}{}_{\mu_{\mathcal{G}}(e_{n+2m-1})} , u^{n+2m+1}{}_{\mu_{\mathcal{G}}(e_{n+2m+1})}\right)$$

$\le 2S[\bar{1} + S + S^2 + \cdots . + S^{m-1}]\ D_{fs\,rb}\left(u^{n}{}_{\mu_{\mathcal{G}}(e_{n})} , u^{n+1}{}_{\mu_{\mathcal{G}}(e_{n+1})}\right) =$
$2S\left[\frac{S^{m-1}-\bar{1}}{S-\bar{1}}\right] D_{fs\,rb}\left(u^{n}{}_{\mu_{\mathcal{G}}(e_{n})} , u^{n+1}{}_{\mu_{\mathcal{G}}(e_{n+1})}\right)$ *also from* (4.16) *we have* $D_{fs\,rb}\left(u^{n}{}_{\mu_{\mathcal{G}}(e_{n})} , u^{n+P}{}_{\mu_{\mathcal{G}}(e_{n+P})}\right) = \bar{0}$. *So, the* $\mathcal{F}_{ss}-$ *sequence* $\langle u^{n}{}_{\mu_{\mathcal{G}}(e_{n})}\rangle$ *in* $\left(\mathcal{F}_{ss}(C_A), D_{fs\,rb}\right)$ *is*

a $\mathcal{F}_{ss}-$ *Cauchy sequence. By the fact that* $\left(\mathcal{F}_{ss}(C_A), D_{fs\,rb}\right)$ *is complete, there is* $u_{\mu_{\mathcal{G}(a)}} \tilde{\tilde{\in}} \mathcal{F}_{ss}(C_A)$ *such that* $lim_{n\to\infty}\ u^n{}_{\mu_{\mathcal{G}(e_n)}} \cong u_{\mu_{\mathcal{G}(a)}}$

Next, we show that the $\mathcal{F}_{ss}-$ *point at the limit,* $u_{\mu_{\mathcal{G}(a)}}$, *is fixed at* T.

For $n \in N$,

$$\lim_{n\to\infty} D_{fs\,rb}\left(u^{n+1}{}_{\mu_{\mathcal{G}(e_{n+1})}}, T\left(u_{\mu_{\mathcal{G}(a)}}\right)\right) = \lim_{n\to\infty} D_{fs\,rb}\left(T\left(u^n{}_{\mu_{\mathcal{G}(e_n)}}\right), T\left(u_{\mu_{\mathcal{G}(a)}}\right)\right)$$

$$\leq \bar{\lambda} \lim_{n\to\infty} max\left[\begin{array}{c} D_{fs\,rb}\left(u^n{}_{\mu_{\mathcal{G}(e_n)}}, u_{\mu_{\mathcal{G}(a)}}\right), D_{fs\,rb}\left(u^n{}_{\mu_{\mathcal{G}(e_n)}}, T\left(u^n{}_{\mu_{\mathcal{G}(e_n)}}\right)\right) \\ , D_{fs\,rb}\left(u_{\mu_{\mathcal{G}(a)}}, T\left(u_{\mu_{\mathcal{G}(a)}}\right)\right), \\ \dfrac{D_{fs\,rb}\left(u^n{}_{\mu_{\mathcal{G}(e_n)}}, T\left(u^n{}_{\mu_{\mathcal{G}(e_n)}}\right)\right) + D_{fs\,rb}\left(u_{\mu_{\mathcal{G}(a)}}, T\left(u_{\mu_{\mathcal{G}(a)}}\right)\right)}{2} \end{array}\right]$$

$$\leq \bar{\lambda} \lim_{n\to\infty} max\left[\begin{array}{c} D_{fs\,rb}\left(u^n{}_{\mu_{\mathcal{G}(e_n)}}, u_{\mu_{\mathcal{G}(a)}}\right), D_{fs\,rb}\left(u^n{}_{\mu_{\mathcal{G}(e_n)}}, u^{n+1}{}_{\mu_{\mathcal{G}(e_{n+1})}}\right) \\ , D_{fs\,rb}\left(u_{\mu_{\mathcal{G}(a)}}, T\left(u_{\mu_{\mathcal{G}(a)}}\right)\right), \end{array}\right]$$

$\leq \bar{\lambda} \lim_{n\to\infty} D_{fs\,rb}\left(u_{\mu_{\mathcal{G}(a)}}, T\left(u_{\mu_{\mathcal{G}(a)}}\right)\right)$ *. which results,*

$\lim_{n\to\infty} D_{fs\,rb}\left(u_{\mu_{\mathcal{G}(a)}}, T\left(u_{\mu_{\mathcal{G}(a)}}\right)\right) = \bar{0}$, *implying that* $u_{\mu_{\mathcal{G}(a)}}$ *is* $\mathcal{F}_{ss}-$ *fixed point of* T. *Assume* $\mathcal{F}_{ss}-$ *point* $u_{\mu_{\mathcal{G}(b)}}$ *is another fixed point of* T.

By using (4.15)

$$D_{fs\,rb}\left(u_{\mu_{\mathcal{G}(a)}}, u_{\mu_{\mathcal{G}(b)}}\right) = D_{fs\,rb}\left(T\left(u_{\mu_{\mathcal{G}(a)}}\right), T\left(u_{\mu_{\mathcal{G}(b)}}\right)\right)$$

$$\leq \bar{\lambda}\, max\left[\begin{array}{c} D_{fs\,rb}\left(u_{\mu_{\mathcal{G}(a)}}, u_{\mu_{\mathcal{G}(b)}}\right), D_{fs\,rb}\left(u_{\mu_{\mathcal{G}(a)}}, T\left(u_{\mu_{\mathcal{G}(a)}}\right)\right) \\ , D_{fs\,rb}\left(u_{\mu_{\mathcal{G}(b)}}, T\left(u_{\mu_{\mathcal{G}(b)}}\right)\right), \\ \dfrac{D_{fs\,rb}\left(u_{\mu_{\mathcal{G}(a)}}, T\left(u_{\mu_{\mathcal{G}(a)}}\right)\right) + D_{fs\,rb}\left(u_{\mu_{\mathcal{G}(b)}}, T\left(u_{\mu_{\mathcal{G}(b)}}\right)\right)}{2} \end{array}\right]$$

$$\leq \bar{\lambda}\, max\left[\begin{array}{c} D_{fs\,rb}\left(u_{\mu_{\mathcal{G}(a)}}, u_{\mu_{\mathcal{G}(b)}}\right), D_{fs\,rb}\left(u_{\mu_{\mathcal{G}(a)}}, u_{\mu_{\mathcal{G}(a)}}\right) \\ , D_{fs\,rb}\left(u_{\mu_{\mathcal{G}(b)}}, u_{\mu_{\mathcal{G}(b)}}\right), \\ \dfrac{D_{fs\,rb}\left(u_{\mu_{\mathcal{G}(a)}}, u_{\mu_{\mathcal{G}(a)}}\right) + D_{fs\,rb}\left(u_{\mu_{\mathcal{G}(b)}}, u_{\mu_{\mathcal{G}(b)}}\right)}{2} \end{array}\right]$$

$\leq \bar{\lambda} D_{fs\,rb}\left(u_{\mu_{\mathcal{G}(a)}}, u_{\mu_{\mathcal{G}(b)}} \right)$. *This implies that* $D_{fs\,rb}\left(u_{\mu_{\mathcal{G}(a)}}, u_{\mu_{\mathcal{G}(b)}} \right) = \bar{0}$, *implying that* $u_{\mu_{\mathcal{G}(a)}} = u_{\mu_{\mathcal{G}(b)}}$. *As a result, obtaining* $\mathcal{F}_{ss}$ *−fixed point is is unique.* ∎.

CONCLUSION

We suggested a new $\mathcal{F}_{ss} -$ Rbms concept that builds on the concepts of many metric space modifications. fuzzy Soft set structure occupies underlying spaces, indicating a relationship between metric fixed point theory and fuzzy soft set theory. We established the acclaimed $\mathcal{F}_{ss} -$ Rbms. Banach fixed point theorem and presented an example to show our findings, as well as some fixed point results in the aforementioned space.

REFRENCE