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Theta Generalized Semi Weak Separation Axioms

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Abstract:

We study new types of generalized closed set called (θ-generalized – semi closed set) by using semi-theta closure operator. We examine its relation with both semi-θ-closed and generalized – semi closed sets. We also define θ (generalized- semi, generalized-semi)-R_0 and R_1 spaces and characterize some of their properties. Moreover, we introduced the notion of θ – generalized – semi – kernel of a subset U of X to investigate the behaviors of theta (generalized- semi, generalized-semi)-R_0 and R_1 spaces

1- Introduction

Separation properties represent fundamental and captivating concepts within the realm of topology. In 2023, Ahmed Hussein [1] introduced the notion of semi-open sets, which emerged as a significant development. Building upon this foundation, Marwa Munther Hassan, Al-Hachami, A. K.in (20٢٢) employed semi-open sets to formulate and explore novel separation axioms referred to as semi-separation axioms. In a similar vein, Jassim Saadoun Shuwaie et al. [3] in 2022 introduced the concept of a semi-R_0 space. Progressing further, in 2021, Ali. Alkhazragy Faik. Mayah and Ali Kalaf. H. Al-Hachami[4] extended the concept of closed sets to semi-generalized closed sets, utilizing the notion of semi-closures. Notably, Cueva M. C. in 2000 [5] defined a novel category of topological spaces termed semi-$T_1/_2$ spaces, characterized by a correspondence between the classes of semi-closed sets and semi-generalized closed sets. This concept denotes spaces where each semi-T_1 space is also a semi-$T_1/_2$ space, and each semi-$T_1/_2$ space is a semi-T_0 space, although the converse implications do not hold true.

Al-Hachami, A. K. [6] marked the inception of investigating feebly open sets in 1978. The objective of this current research is to provide certain delineations of weak separation axioms.

2- Fundamental and Basic Concepts

Definition 2.1: Consider a subset A of the topological space (X, $\mathcal{T}$). A is termed a feeble open set if there exists an open set U such that U is a subset of A and A is enclosed by the closure of U, denoted as$U \subset A \subset \overline{U}^s$.

A set qualifies as feeble closed when its complement is feeble open. The intersection of all feeble closed sets encompassing A yields the feeble closure of A, denoted as ($\overline{A}^{f}$).

Proposition 2.1: For a subset A of the topological space (X, $\mathcal{T}$), A is feeble open if and only if A is contained within the closure of its interior in a strong sense, i.e., $A \subset \overline{U^o}^{s}$.

Proof: Assuming A is feeble open, an open set U exists satisfying $U \subset A \subset \overline{U}^{s}$. Since $A^o \subset A$, this implies $A^o \subset A \subseteq$ A $\subseteq \overline{U}^{s}$. Consequently, $\overline{U}^{s} \subset \overline{A^o}^{s} \subset \overline{A}^{s} \subset \overline{U}^{s}$.Then $A \subset \overline{U}^{s} \subset \overline{A^o}^{s}$, then we get $A \subset \overline{A^o}^{s}$

Conversely: Let $A \subset \overline{A^o}^{s}$ then $A^o \subset A \subset \overline{A^o}^{s}$,(i.e.$A$ is feebly open set)

Remark 2.1: "Every open set is also feeble open."

Proposition 2.2: Each feeble open set is categorized as a semi-open set.

Lemma 2.1: Given a subset A of the topological space (X, $\mathcal{T}$), the closure of A in a strong sense is equivalent to the union of A and the interior of its closure, i.e then $\overline{A}^{s} = A \cup \left(\overline{A}\right)^{o}$.

Proof

Since $scl(A)$ is semi-closed, we have $\left(\overline{\overline{A}^{s}}\right)^{o}$.

Therefore, $\left(\overline{A}\right)^{o} \subset \overline{A}^{s}$, and hence $A \cup \left(\overline{A}\right)^{o} \subset \overline{A}^{s}$.

To established the opposite inclusion we observe that

$$\left(\overline{A \cup \left(\overline{A}\right)^{o}}\right)^{o} = \left(\overline{A \cup \left(\overline{A}\right)^{o}}\right)^{o} \subset \left(\overline{A}\right) \cup \left(\overline{\left(\overline{A}\right)^{o}}\right)^{o} = \overline{A} \cup \left(\overline{A}\right)^{o} = \overline{A}.$$

Thus $\left(\overline{A \cup \left(\overline{A}\right)^{o}}\right)^{o} \subset \left(\overline{A}\right)^{o} \subset A \cup \left(\overline{A}\right)^{o}$.

Hence $A \cup \left(\overline{A}\right)^{o}$is semi-closed, and so $\overline{A}^{s} \subset A \cup \left(\overline{A}\right)^{o}$.

Proposition 2.3: For a subset A of the topological space (X, $\mathcal{T}$), A is feeble open if and only if A is an element of the collection $\mathcal{T}^{\propto}$, which consists of sets of the form $A \in \mathcal{T}^{\propto}(\overline{A})^{o}$

Proof: If A is feebly open, there is an open set U such that $U \subset A \subset \overline{U}^{s}$. By lemma 2.1, $\overline{U}^{s} = U \cup \left(\overline{U}\right)^{o} = \left(\overline{U}\right)^{o}$. Hence $U \subset A \subset \left(\overline{U}\right)^{o}$, and consequently $\left(\overline{A^o}\right)^{o} = \left(\overline{U}\right)^{o}$. Thus we have that $A \subset \left(\overline{A^o}\right)^{o}$, so that $A \in \mathcal{T}^{\propto}$.

“Conversely, if $A \in \mathcal{T}^{\propto}$, we have that $A^o \subset A \subset \left(\overline{A^o}\right)^o$. So if $U = A^o$ we have $U \subset A \subset \left(\overline{U}\right)^o$, and **lemma 2.1** implies $U \subset A \subset \overline{U}^s$, so that A is feebly open set”.

Lemma 2.2: “Let x be a point of a topological space $(X,\mathcal{T})$. Then either $\{x\}$ is nowhere dense or $\{x\} \subset \left(\overline{\{x\}}\right)^o = \overline{\{x\}}^s$”.

Proof: Suppose that $\{x\}$ is not nowhere dense. Then $\left(\overline{\{x\}}\right)^o \neq \emptyset$, and so $x \in \left(\overline{\{x\}}\right)^o$. Lemma 2.1 implies that $scl\{x\} = \{x\} \cup \left(\overline{\{x\}}\right)^o$

Thus $\{x\} \subset \left(\overline{\{x\}}\right)^o = \overline{\{x\}}^s$.

Proposition 2.4: “Let A be a subset oftopological space $(X,\mathcal{T})$, and U is open in X, then $A \cap U$ is feebly open in”X and U.

Proposition 2.5: The union of family feebly open set are feebly open.

Proof: Let $A_\propto$ be feebly open $\forall \propto \in \Lambda$, then there exist open set $U_\propto$ such that $U_\propto \subset A_\propto \subset \overline{(U_\propto)}^s$ $\forall \propto \in \Lambda$, then $\cup_{\propto\in\Lambda} U_\propto \subset \cup_{\propto\in\Lambda} A_\propto \subset \cup_{\propto\in\Lambda} \overline{(U_\propto)}^s$. Since $\cup_{\propto\in\Lambda} \overline{(U_\propto)}^s = \overline{(\cup_{\propto\in\Lambda} U_\propto)}^s$.Then $\cup_{\propto\in\Lambda} A_\propto$ is feebly open

Remark 2.2: A set B is feebly closed if and only if $\overline{B}^f = B$.

3. Feebly R_0 Properties

“The collection of all semi-open set is denoted by $(SO(X,\mathcal{T}))$.

The collection of all feebly open set is denoted by $(FO(X,\mathcal{T}))$

Definition 3.1: A topological space$(X,\mathcal{T})$ is semi-R_0 if for each semi-open set $U, x \in$ H implies that$\overline{\{x\}}^s \subset U$

Definition 3.2: A topological space$(X,\mathcal{T})$ is feebly R_0 if for each feebly open set $H, x \in$ H implies that$\overline{\{x\}}^f \subset H$

Theorem 3.1: “A topological space$(X,\mathcal{T})$ is feebly R_0 if and only if for every feebly closed set F and $x \notin F$, there exist a feebly open set U such that”$F \subset U, x \notin U$.

Definition 3.2: Within a given topological space (X, $\mathcal{T}$), the feeble kernel of the point x is characterized as the set $fker\{x\} = \left\{y : x \in \overline{\{y\}}^f\right\}$.

Theorem 3.2: "In a topological space $(X,\mathcal{T})$, for $y,x \in X, fker\{x\} \neq fker\{y\}$. If and only if" $\overline{\{x\}}^f \neq \overline{\{y\}}^f$.

Proof: Necessity. We assume that there exist a point z of X such that $z \in \overline{\{x\}}^f$ but $z \notin \overline{\{y\}}^f$. Therefore $x \in \overline{\{z\}}^f$ and $y \notin \overline{\{z\}}^f$. Consequently there is a feebly open set which contains y but not x. and so, $y \notin \overline{\{x\}}^f$. Thus $fker\{x\} \neq fker\{y\}$.

Sufficiency. Let p is a point of X such that $p \in \overline{\{x\}}^f$ and $p \notin \overline{\{y\}}^f$. And so there is a feebly open set which contains x but not y. therefore, $y \notin \overline{\{x\}}^f$. This competes the proof.

Theorem 3.3: "Let $(X,\mathcal{T})$ be a topological space, then the following conditions are equivalent:

(a) X if feebly R_0.
(b) for every $x \in X, \overline{\{x\}}^f \subset fker\{x\}$.
(c) If F is feebly closed in X, then $F = \cap \{G: G\ is\ feebly\ open, F \subset G\}$.
(d) If G is feebly open in X, then $G = \cup \{F: F\ is\ feebly\ closed, F \subset G\}$.
(e) For any nonempty set A and feebly open set G in X such that $A \cap G \neq \emptyset$, there exist a feebly closed set F for which $F \subset G$ and $A \cap F \neq \emptyset$.
(f) For any feebly closed set F in X and $x \notin F, \overline{\{x\}}^f \cap F = \emptyset$."

Proof

(a) $\Longrightarrow$ (b): Let $y \in \overline{\{x\}}^f$. Let G be any feebly open set such that $x \in G$. Now by (a), $y \in G$. This gives that $x \in \overline{\{y\}}^f$. Therefore $y \in fker\{x\}$.

(b) $\Longrightarrow$ (c): suppose that x does not belong to the feebly closed set F. and so, $X \sim F$ is feebly open and contains x. Let $y \in \overline{\{x\}}^f$. Then by (b), $x \in \overline{\{y\}}^f$. Therefore, every feebly open set which contains x contains y. Hence, $\overline{\{y\}}^f \subset X \sim F$. Now $X \sim \overline{\{x\}}^f$ is a feebly open set containing F to which x does not belong. Consequently, x does not belong to the intersection of all the feebly open set which contain F. thus (c) hold.

(c) $\Longrightarrow$ (d): Evident.

set which contain F. thus (c) hold.

(c) $\Longrightarrow$ (d): Evident.

(d) $\Longrightarrow$ (e): "Let G be feebly open and A is non-empty such that $A \cap G \neq \phi$. Let $x \in A \cap G$. By (d) there exist a feebly closed set F such that $x \in F \subset G$.clearly, $A \cap F \neq \phi$"

(e) $\Longrightarrow$ (f): “Let F be a feebly closed set and $x \notin F$ Then $X \sim F$ is feebly open and $\{x\} \cap (X \sim F) \neq \phi$ By (e), there exists a feebly closed set H such that”$H \cap \{x\} \neq \phi$ and $H \subset X \sim F$. Therefore $\overline{\{x\}}^{f} \subset X \sim F$. Consequently, $F \cap \overline{\{x\}}^{f} = \phi$.

(f) $\Longrightarrow$ (a): By **theorem 3.1**.

Theorem 3.4: If for any point x of a feebly R_0 space X, $\overline{\{x\}}^{f} \cap fker\{x\} = \{x\}$ then $\overline{\{x\}}^{f} = \{x\}$.

The proof follows from **theorem 3.4(b)**

Proposition 3.1: Let $(X, \mathcal{T})$ be a topological space, then $X$is feeblyR_0 if and only if $(X, T^{\propto})$ is R_0.

Proposition 3.2:(i) If $(X, \mathcal{T})$ is feebly R_0, then it is semi- R_0.

(ii) If $(X, \mathcal{T})$ is R_0, then it is feeblyR_0.

Proof: (i) we first show that R_0 implies semi-R_0 .

Suppose $U \in SO(X, \mathcal{T})$ and $x \in U$. There is an open set V such that $V \subset U \subset \overline{V}$.

Let $x \in V$. Since $(X, \mathcal{T})$ is R_0 , $\overline{\{x\}} \subset V$ and hence $\overline{\{x\}}^{s} \subset V \subset U$.

Let $U - V \subset \overline{V} - V$. Then $\left(\overline{\{x\}}\right)^{o} \phi$. And $\overline{\{x\}}^{s} \subset U$. Then $(X, \mathcal{T})$ is semi-R_0.

Now suppose that $(X, \mathcal{T})$ is feebly R_0. By **Proposition 3.1**, $(X, T^{\propto})$ is R_0 and so is semi-R_0 by the argument above. But $SO(X, T^{\propto}) = SO(X, \mathcal{T})$, so that $(X, \mathcal{T})$ is semi-R_0.

(ii) let $x \in V \in T^{\propto}$. By Lemma 2.2,$\{x\}$ is nowhere dense or $\{x\} \subset \left(\overline{\{x\}} = \overline{\{x\}}^{s}\right)^{o}$

If $\{x\}$ is nowhere dense, $T^{\propto}\overline{\{x\}} = \{x\} \subset V$.

If $\{x\} \subset \left(\overline{\{x\}} = \overline{\{x\}}^{s}\right)^{o}$ then $\mathcal{T}\overline{\{x\}} \subset \left(\overline{\{x\}}\right)^{o}$ since $(X, \mathcal{T})$ is R_0.

But $T^{\propto}\overline{\{x\}} \subset \mathcal{T}\overline{\{x\}}$, so that $T^{\propto}\overline{\{x\}} \subset \overline{\{x\}}^{s}$. By part (i) $(X, \mathcal{T})$ is semi-R_0, and $V \in SO(X, \mathcal{T})$ implies $\overline{\{x\}}^{s} \subset V$. Hence $T^{\propto}\overline{\{x\}} \subset V$. Thus $(X, T^{\propto})$ is R_0, so that by **Proposition3.1**$(X, \mathcal{T})$ is feebly R_0.

4. Feebly T_0 Properties

Definition 4.1: A topological space (X, T) is said to be feeblyT_0 if, for any distinct pair of points x, there exists a feeble open set within (X, T) that includes one of the points but not the other.

Proposition 4.1: If (X,T) isT_0 space, then it is also feebly T_0

The proof is based on Proposition 2.1.

Additionally, we present an example to demonstrate that the converse of Proposition 4.1 does not hold true

Example 4.1: Let $X = \{a, b, c\}, \mathcal{T} = \{\emptyset, X, \{a\}\}$be a topology on X.

The feebly open sets in X are $T^f = \{\emptyset, X, \{a\}, \{a, b\}, \{a, c\}\}$

It is clear (X, T^f) is feebly T_0 but not T_0 space.

Theorem 4.1: Every open subspace of an feebly T_0 space is feebly T_0.

Proof: Let Y be an open subspace of X and $x, y \in Y$ such that $x \neq y$.

Since X is feebly T_0, then there exist feebly open, U in X, such that $x \in U and\ y \notin U\ or\ x \notin U\ and\ y \in U$, since Y open subset in X, then $Y \cap U$ is feebly in Y (**Proposition 2.4**). then $x \in Y \cap U$ and $y \notin Y \cap U$ or $x \notin Y \cap U$ and $y \in Y \cap U$. Then Y is feebly T_0.

Proposition 4.2: Let $(X, \mathcal{T})$ be a topological space, then $X$is feeblyT_0 if and only if $\overline{\{x\}}^f \neq \overline{\{y\}}^f$(such that x, y are two distinct point in X).

Proof: Suppose that $x, y \in X$ such that $x \neq y$, and $\overline{\{x\}}^f \neq \overline{\{y\}}^f$. Let $z \in X$ such that $z \in \overline{\{x\}}^f$and $z \notin \overline{\{y\}}^f$,therefor $x \notin fcl\{y\}$. If $x \in fcl\{y\}$ that is mean $\overline{\{x\}}^f \subset \overline{\{y\}}^f$, this is contradiction. Then $x \in \left(\overline{\{y\}}^f\right)^c and\ y \notin \left(\overline{\{y\}}^f\right)^c$ such that $\left(\overline{\{y\}}^f\right)^c$ is feebly open. Then X is feebly T_0.

Conversely let X be feebly T_0 space such that $x, y \in X$. Then there exist a feebly open set, G such that $x \in G$ and $y \notin G$ or $x \notin G$ and $y \in G$. Let $x \in G$ and $y \notin G$ then G^c is feebly closed and $x \notin G^c$ and $y \in G^c$. Since $\overline{\{y\}}^f$ is the smallest feebly closed set contain y therefore $\overline{\{y\}}^f \subset G^c$ and $x \notin \overline{\{y\}}^f$. Then $\overline{\{x\}}^f \neq \overline{\{y\}}^f$.

Remark 4.1: The axioms of feebly T_0space and feebly R_0spaceare independent.

5. Feebly $T_{1/3}$ Properties

Definition 5.1: A subset A of a topological space$(X, \mathcal{T})$ is called Ψ-closed set if, $\overline{A}^f \subset O$ hold whenever $A \subset O$ and O is fg-open of $(X, \mathcal{T})$.

Theorem 5.1: Let A be a subset of topological space $(X, \mathcal{T})$, then

1) A is Ψ-closed if and only if $\overline{A}^f - (A)$ does not contain any non-empty fg-closed set.
2) If A is Ψ-closed and $A \subset B \subset \overline{A}^f$, then B is Ψ-closed.

Definition 5.2: "A topological space $(X,\mathcal{T})$ is called feebly $T_{1/3}$ space if every Ψ-closed set in $(X,\mathcal{T})$ is feebly closed.

Theorem 5.2: Every feebly $T_{1/2}$ space is feebly $T_{1/3}$ space.

The converse of **Theorem 5.2** is false as it can be seen from the following example.

Example 5.1: Let $X = \{1,2,3\} and\ \mathcal{T} = \{\phi, X, \{1\}, \{2,3\}\}$. $(X,\mathcal{T})$is feebly $T_{1/3}$ space but not a feebly $T_{1/2}$ space, because $\{2\}$ is a fg-closed set but not feebly closed set of $(X,\mathcal{T})$.

Theorem 5.3: for a topological space $(X,\mathcal{T})$, the following conditions are equivalent:

(i) $(X,\mathcal{T})$ is a feebly $T_{1/3}$ space.
(ii) Every singleton of X is either fg-closed or feebly open.
(iii) Every singleton of X is either fg-closed or open.

Proof: (i)$\Longrightarrow$ (ii) Let $x \in X$ and suppose that $\{x\}$ is not fg-closed of $(X,\mathcal{T})$. Then $X - \{x\}$ is fg-open set. So, X is the only fg-open set containing $X - \{x\}$. Hence $X - \{x\}$ is Ψ-closed set. Since $(X,\mathcal{T})$ is feebly $T_{1/3}$ space, then $X - \{x\}$ is a feebly closed set or equivalently $\{x\}$ is feebly open set.

(ii)$\Longrightarrow$ (i) Let A be a Ψ-closed set. Clearly"$A \subset \overline{A}^f$. let $x \in X$. By assumption, {x} is either fg-closed or feebly open.

Case(1) Suppose $\{x\}$ is fg-closed. By **Theorem 5.1,** $\overline{A}^f A$ does not contain any non-empty fg-closed set. Since $x \in \overline{A}^f$, then $x \in A$.

Case(2) Suppose {x} is a feebly open set. Since $x \in \overline{A}^f$, then $\{x\} \cap A \neq \phi$. So $x \in A$. Thus in any case, $\overline{A}^f \subset A$.

Therefore $A = fcl(A)$ or equivalently A is feebly closed set of $(X,\mathcal{T})$. Hence $(X,\mathcal{T})$ is an feebly $T_{1/3}$ space.

(ii)$\Longrightarrow$ (iii).This conclusion stems from the observation that a singleton set is feebly open if and only if it is open.

6. Feebly $T_{1/2}$ Properties

Definition 6.1: "A subset A of a topological space$(X,\mathcal{T})$ is called to be feebly generalized closed set (written in short as fg-closed) if, $\overline{A}^f \subset O$ hold whenever $A \subset O$ and O is feebly open. A subset B of $(X,\mathcal{T})$ is called a fg-open set of $(X,\mathcal{T})$ if, B^c is fg-closed in $(X,\mathcal{T})$. Every feebly closed is fg-closed but the converse is not true.

Definition 6.2:"A topological space $(X,\mathcal{T})$ is called feebly $T_{1/2}$ if every fg-closed set in $(X,\mathcal{T})$ is feebly closed in $(X,\mathcal{T})$"

Theorem 6.1: Let $(X,\mathcal{T})$ be a topological space". Then, every feebly $T_{1/2}$ space is feebly T_0 space.

Proof

If $(X,\mathcal{T})$ be a feebly $T_{1/2}$ space which is not a feebly T_0 space then there exit $x, y, x \neq y$ such that $\overline{\{x\}}^f = \overline{\{y\}}^f$ (by **Proposition 4.2**).Let $A = \overline{\{x\}}^f \cap \{x\}^c$. It will be shown that A is fg-closed but not feebly closed. Let O be any feebly open set containing x. Since $x \in \overline{\{y\}}^f, \{y\} \cap O \neq \emptyset, i.e. y \in O$. Now $\{y\} \subset \overline{\{y\}}^f \cap O$ and this shows in succession,$\{y\} \subset \overline{\{x\}}^f \cap O, \{x\}^c \cap \{y\} \subset \overline{\{x\}}^f \cap O \cap \{x\}^c, \{y\} \subset \overline{\{x\}}^f \cap O \cap \{x\}^c, \{y\} \subset O \cap A \neq \emptyset$.This implies $x \in \overline{A}^f$. But $x \notin A$. Consequently, $x \in fD(A)(=$ set of all feebly $-$ limit point of $A)$. Therefore $fD(A) \not\subset A$ and then A is not feebly closed.

Now suppose $A \subset G$ where $G \in FO(X,\mathcal{T})$. To show that $\overline{A}^f \subset G$, it suffices to prove that $\overline{\{x\}}^f \subset G$. But $\overline{\{x\}}^f \cap \{x\}^c = A \subset G$ and $fD(\{x\}) \subset \{x\}^c$. Then $fD(\{x\}) \subset G$ and thus it needs only to show that $x \in G$. If possible, let $x \in G^c$. Then $y \in \overline{\{x\}}^f \subset G^c$. Hence $y \in \overline{\{x\}}^f \cap \{x\}^c = A \subset G$. Thus $y \in G \cap G^c$, a contradiction. Therefore,$\overline{\{x\}}^f \subset G$ so that A is fg-closed. Therefore $(X,\mathcal{T})$ is not feebly $T_{1/2}$.

Theorem 6.2: A topological space is feebly $T_{1/2}$ if and only if for each $x \in X$, either $\{x\}$ is feebly open or $\{x\}$ is feebly closed.

Proof

Necessity: suppose X is feebly $T_{1/2}$ and for some $x \in X, \{x\}$ is not feebly closed. Since X is the only feebly open of $\{x\}^c$ and $\{x\}^c$ is fg-closed and thus feebly closed. Hence $\{x\}$ is feebly open.

Sufficiency: Let $A \subset X$ be fg-closed with $x \in \overline{A}^f$. If {x} is feebly open, $\{x\} \cap A \neq \emptyset$.

Otherwise $\{x\}$ is feebly closed and $A \cap \{x\} = A \cap \overline{\{x\}}^f \neq \emptyset$. In either case $x \in A$ and so A is feebly closed **(Remark 2.2).**

Corollary 6.1: "$X$is feebly $T_{1/2}$if and only if every subset of X is the intersection of all feebly open sets and all feebly closed sets containing it.

Proof: Necessity: Let X be feebly $T_{1/2}$ with $B \subset X$ arbitrary. Then $B = \cap \{\{x\}^c, x \notin B\}$, an intersection of feebly open and feebly closed by **Theorem 6.2.** the result follows".

Sufficiency: "For each $x \in X, \{x\}^c$is the intersection of all feebly open sets and all feebly closed sets containing it. Thus $\{x\}^c$ is either feebly open or feebly closed and X is feebly $T_{1/2}$.

Theorem 6.3: "The property of being a feebly $T_{1/2}$ space is hereditary, i.e. every subspace of a feebly $T_{1/2}$ space is also a feebly $T_{1/2}$"

Proof: Let Y be a subspace of a feebly $T_{1/2}$ space X. let $y \in Y \subset X$. Then $\{y\} \in FO(X, \mathcal{T})$ or $\{y\} \in FC(X, \mathcal{T})$(=family of all feebly closed sets in the topological space $(X, \mathcal{T})$. Therefore by ([5] Theorem 6) $\{y\}$ is either feebly open in Y or feebly closed in Y. By **Theorem 6.2**, Y is feebly $T_{1/2}$."

7. Feebly $T_{3/4}$ Properties

Definition 7.1:"A topological space $(X, \mathcal{T})$ is called feebly $T_{3/4}$space if every fg-closed subset of X is feebly closed.

Lemma 7.1: Let $A \subset (X, \mathcal{T})$ be every fg-closed. Then $\overline{A}^f - A$ does not contain a non-empty closed set.

Lemma 7.2: In any space a singleton is feebly open if and only if it is regular open.

Theorem 7.1: For a topological space $(X, \mathcal{T})$ the following conditions are equivalent:

(1) X is a feebly $T_{3/4}$space
(2) Every singleton $\{x\}$ is feebly open or closed.
(3) Every singleton $\{x\}$ is regular open or closed."

Corollary 7.1: Every feebly $T_{3/4}$ space is a feebly $T_{1/2}$ space. But the converse is not always true

Example 7.1: the Sierpinski space is an easy example of a space, which is feebly $T_{1/2}$ but not feebly $T_{3/4}$ space.

8. Feebly T_1 Properties

Definition 8.1: "A topological space $(X, \mathcal{T})$ is feebly T_1 space if for each pair of distinct point x, y in X there is a feebly open set containing x but not y."

It is evident that T_1 space implies feebly T_1 space. In general the convers is not always true. Consider the following example:

Example 8.1: Let $X = \{a, b, c, d\}$ and $\mathcal{T} = \{X, \phi, \{a\}, \{b, c\}, \{a, b, c\}\}$, be the topology on X. The feebly open on $X$are :$X, \phi, \{a\}, \{a, d\}, \{b, c\}, \{a, b, c\}$. Then the space $(X, \mathcal{T})$ is feebly T_1 but not T_1space.

Proposition 8.1: "Every feebly T_1 space is semi-T_1 space, but the convers is not always true".

Example 8.2:"Let $X = \{a, b, c\}$ and $\mathcal{T} = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$, be the topology on X. The feebly open on $X$are :$X, \phi, \{a\}, \{a, d\}, \{b, c\}, \{a, b, c\}$, and the semi-open on X are" : $X, \phi, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}$. It is clear that$(X, \mathcal{T})$ is semi- T_1 but not feebly T_1space.

Proposition 8.3: "Every feebly T_1 space is feebly-T_0 space".

Proof:"Let $(X, \mathcal{T})$ is feebly T_1space. It suffices to show that a set which is not feebly closed also not a fg-closed set To this end, suppose $A \subset X$ and A is not feebly closed". Let $x \in \overline{A}^f - A$. Then $\{x\} \subset \overline{A}^f - A$. Since X is feebly T_1space, $\{x\}$ is a feebly closed set.

1) A is not fg-closed.

“The convers of **Proposition 8.3** is not always true”.

Example 8.2:“Let $X = \{a, b, c\}$ and $\mathcal{T} = \{X, \phi, \{a\}\}$, be the topology on”$X$. The feebly open on $X$are :$X, \phi, \{a\}, \{b, c\}, \{a, b, c\}$. It is clear that $(X, \mathcal{T})$ is feebly T_0 but not feebly T_1space.

Theorem 8.1: “For a topological space $(X, \mathcal{T})$ the following conditions are equivalent”:

(i) Each singleton is feebly closed.
(ii) X is a feebly T_1space.

Proof: (i) $\Longrightarrow$ (ii). “Let X be feebly T_1space and $y \in X$. To prove $\{y\}$ is feebly closed set. Let $x \in \{y\}^c$then $x \neq y$. Since X feebly T_1space, then there exist feebly open set in X, U such that $x \in U \; and \; y \notin U$. Then $x \in U \subset \{y\}^c = \cup \{U_x : x \in \{x\}^c$. Then $\{a\}^c$feebly open set by **Proposition 2.4.** Then $\{y\}$ feebly closed set.

(i) $\Longrightarrow$ (ii). “Let $\{z\}$ feebly closed set $\forall z \in X$. Let $x, y \in X$ such that $x \neq y$. Then $x, y \in \{y\}^c$ and $\{y\}$ is feebly closed. Therefore $\{y\}^c$ feebly open set containing x but not y, and $\{x\}^c$ feebly open set containing y but not x. then X is feebly T_1space

Theorem 8.2: “A topological space $(X, \mathcal{T})$ is feebly T_1space if and only if it is feebly T_0space and feebly R_0space.

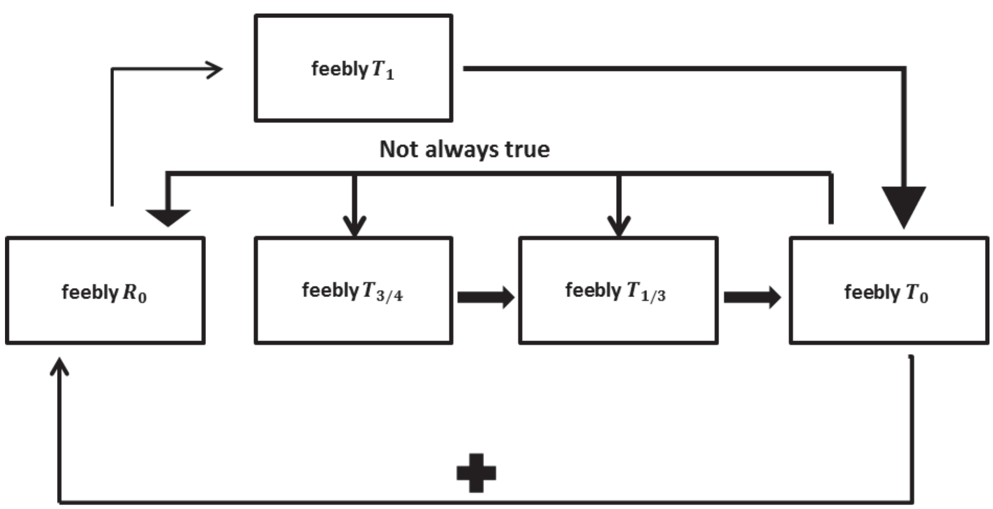

Diagram1: shows the relationship between some separation axioms.

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