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The Equivalence Between (QTA) of Lie Groups and Hom- Space with Tencer Product

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Abstract:

The primary purpose of this research is to work out a new action of Lie group through dual representation. In our paper we mention the basic definitions, we'll discuss the study of activity for Lie group upon Hom-space utilizing equivalence relationship between tensor product and Hom. Their measures will be studied on a structure consisting of four and five vector spaces. In the end we obtain new generalizations using action of dual representation for Lie group Ǥ.

Key words:

Lie group, representation of Lie group, dual representation of Lie group, tensor product for representation of Lie group.

1. Introduction:

Define Ǥ to be Lie group. It is finite dimensional manifold which being as well a group, the structure will be the multiplication Ǥ × Ǥ → Ǥ and a ttaching of an inverse function Ǥ → Ǥ , $g \to g^{-1}$ are smooth maps. [6]. In [1] Hall B.C. composed a book of Lie group and explained the algebras. And we will use double representation for Lie group, because it works the group′s action on some vector space. Schu'r, Lemma presented the concept of action for Lie algebra upon the linear maps, from Z_2 into Z_1, referred via $Hom(Z_2, Z_1)$, such that $Hom(Z_2, Z_1) \cong Z_2^* \otimes Z_1$ [1]. Also, the interest in the present work is to give representations by inter wine dual of these actions, and explain the action′s structure via a diagram. And then generalizing them. In this paper we symbolize for drawings quadrilaterals and pentagons by (QTA) and (PTA), respectively, see [8]. and pentagons by (QTA) and (PTA), respectively. see [8]. In 2o16 , H.I.Lefta and T.H.Majeed " Action of

Reductive Lie Groups on Hom-Space and Tensor Product of Five Representation" , 2018, A.K.Radhi and T.H.Majeed " Certain Types of Complex Lie Group Action " 2021, W.S.Gan " Lie Groups and Lie Algebra" .And M. is field . And M. is field

2. Basic Concepts:

In this section gives the main definitions of group action and group representation.

Lemma (2.1): [3]

Assume that w_1' & w_2' are representations for Lie algebra g affects the finite dimensional spaces Z_2 & Z_1, correspondingly. It defines the T-action of g upon $Hom_F(Z_2, Z_1), w'$: g $\to$ gL $(Hom_F(Z_2, Z_1))$ for all $v \in$ g, $h \in Hom_F(Z_2, Z_1)$,

$w_1'(v)h - hw_2'(v)$ and $Hom(Z_2, Z_1) = Z_2^* \otimes Z_1$ as the equivalence of rep.

Definition (2.2): [9]

Let Lie group Ǥ, be finite dim. real (complex) representations of Ǥ being a homomorphism of Lie group, w: Ǥ $\to$ Ǥ$L(n, R) = (n \geq 1)$. In general, a homomorphism for the Lie group is w: Ǥ $\to$ Ǥ$L(Z)$ where Z has the characteristics of a real (complex) vector space and has a finite number of dimensions a $Z \geq 1$.

w_1 and w_2 being the representation on $(w_1 \otimes w_2)(v, r) = w_1(v) \otimes w_2(r)$ for all $v \in$ Ǥ and $r \in H$.

Definition (2.3): [8]

Let $w_i, i = 1,2, \dots, m$ are representations of Lie group Ǥ affects the vector spaces $Z_i, i = 1,2, \dots, m$ then the direct sum of w_i, bring the representation defined by: $\{ w_1 \oplus w_2 \dots \oplus w_m(v)\}(Z_1, Z_2, \dots, Z_m) = w_1(v)Z_1, w_2(v)Z_2, \dots, w_m(s)Z_m$ for all $r \in$ Ǥ $, Z_1, Z_2, \dots, Z_m \in Z_1 \times Z_2 \times \dots \times Z_m$.

Definition (2.4): [2]

Let both Ǥ and H groups of liars, let w_1 is a rendition of Ǥ effects the space Z_1 and let w_2 is a rendition of H effects the space Z_2, then the tensor product of $(w_1 \otimes w_2)$(v,r) = $w_1(v) \otimes w_2(r)$ .For all v ∊ Ǥ and r ∊ H.

Definition (2.5): [5]

Let Z_1 & Z_2 being space of real (complex) vectors of finite dimensions, after that, a tensor product of Z_1 & Z_2 is a vector space Z , together with a bilinear map

$I: Z_1 \times Z_2 \to Z(Z_1 \otimes Z_2)$ having this quality: if φ is every bilinear map of $Z_1 \times Z_2$ into a vector space $\underline{Z}$, then there exists a sole linear map $\underline{\varphi}$ of Z into $\underline{Z}$, the next diagram commutes, and so forth.

$$\begin{tikzcd} Z_1 \times Z_2 \arrow[r, "I"] \arrow[d, "\varphi"'] & Z(Z_1 \otimes Z_2) \arrow[dl, "\exists!\,\bar{\varphi}"] \\ \bar{Z} & \end{tikzcd}$$

Figure(1)

Definition (2.6): [4]

Assume Ǥ is Lie group as well as w is representation of Ǥ effects the vector space Z. After that, the model of dual w to w is an expression of Ǥ effects the Z provided via : $w^*(v) = [w(v^{-1})]^{tr}$ Dual representation is also known as the contragredient representation.

Example (2.7) :

Let $w = S^1 \to So(2, ₵)$, where $S^1 = e^{i\vartheta} = cos\, cos\, \vartheta + i\ sin\, sin\, \vartheta$ such that

$S^1 = \{(cos\, \vartheta, sin\, \vartheta\,), 0 \leq \vartheta \leq 2\pi\}$, and

$w(cos\, cos\, \vartheta\ , sin\, sin\, \vartheta\,) = (cos\, cos\, \vartheta\ \ -sin\, sin\, \vartheta\ \ sin\, sin\, \vartheta\ \ cos\, cos\, \vartheta\ )\ , w\left(e^{i\vartheta}\right) = (cos\, cos\, \vartheta\ \ -sin\, sin\, \vartheta\ \ sin\, sin\, \vartheta\ \ cos\, cos\, \vartheta\ ),$

w is representation of Lie group S^1. Then

$v = e^{i\vartheta}, w\left(e^{i\vartheta}\right) = (cos\, cos\, \vartheta\ \ -sin\, sin\, \vartheta\ \ sin\, sin\, \vartheta\ \ cos\, cos\, \vartheta\ ), v^{-1} = cos\, cos\, \vartheta - i\ sin\, sin\, \vartheta$

$w(v)^{-1} = (cos\, cos\, \vartheta\ \ -sin\, sin\, \vartheta\ \ sin\, sin\, \vartheta\ \ cos\, cos\, \vartheta\ )$, then $[w(v)^{-1}]^{tr} = (cos\, cos\, \vartheta\ - sin\, sin\, \vartheta\ \ sin\, sin\, \vartheta\ \ cos\, cos\, \vartheta\ )$.

3. The equivalence between the Lie group (QAT) and the Hom-space, Tensor product:

The action (QAT) , (PTA) of Ǥ upon Hom-space and upon tensor product will be studied in the present section.

Proposition (3.1):

Let w_1, w_2, w_3, w_4 being the four representations for the Lie group affects the vector spaces Z_1, Z_2, Z_3, Z_4, correspondingly, put $Hom_m(Hom\ (Z_1^*, Z_2)\ , Hom(Z_3, Z_4^*)$ be M-vector space of all linear mappings from Z_4^* to Z_1^* and from $Hom\ (Z_1^*, Z_2)$ to $Hom(Z_3, Z_4^*)$. Then the (QAT) of the Lie group on

$Hom_m(Hom\ (Z_1^*, Z_2)\ , Hom(Z_3, Z_4^*))$.

Proof:

Define ψ: Ǥ $\rightarrow$ ǤL $(Hom_m(Hom\ (Z_1^*, Z_2)\ , Hom(Z_3, Z_4^*))$ such that:

$\psi(v)\hbar = w_1(v) \circ \hbar_1 \circ w_4(v)$, for all $v \in$ Ǥ, $\hbar \in Hom(Z_1, Z_4)$.

The following diagram can be used to show that the action of Lie group Ǥ on

$Hom(Z_1, Z_4)$ is as follows $\psi(v)\hbar = w_2(v) \circ \hbar_2 \circ w_3(v)$.

$$\begin{tikzcd}
Z_1^* \arrow[r, "\acute{h}"] \arrow[d, "w_4(v)"'] \arrow[dr, shift left, "w_3(v)\acute{h}"] \arrow[dr, shift right, "\acute{h}w_4^*(v)"'] & Z_2 \arrow[d, "w_3(v)"] \\
Z_3 \arrow[r, "\acute{h}"'] & Z_4^*
\end{tikzcd}$$

Figure (2)

Where ψ: Ǥ $\rightarrow$ ǤL $(Hom_m(Hom\ (Z_1^*, Z_2)\ , Hom(Z_3, Z_4^*))$ induced by representation, such that $\psi(v)\hbar = [(w_4(v)^{-1} \circ \hbar_2 \circ w_3(v) \circ \hbar_3 \circ w_1(v)^{-1})]$, for all

$v \in$ Ǥ and $\hbar_i: Z_i \rightarrow M$ thus w^* is representation from Ǥ to Hom-space

$$\begin{tikzcd}
Z_4^* \arrow[r, leftrightarrow, "\acute{h}_1"] \arrow[d, leftrightarrow, "(\psi_4(v))^{-1}"'] & Z_3 \arrow[r, leftrightarrow, "\acute{h}_2"] \arrow[d, leftrightarrow] & Z_2 \arrow[r, leftrightarrow, "\acute{h}_3"] \arrow[d, leftrightarrow] & Z_1^* \arrow[d, leftrightarrow] \\
Z_4 \arrow[r, leftrightarrow, "\acute{h}_1"] & Z_3 \arrow[r, leftrightarrow, "\acute{h}_2"] & Z_2 \arrow[r, leftrightarrow, "\acute{h}_3"] & Z_1
\end{tikzcd}$$

Figure (3)

Since $(Hom_m(Hom\,(Z_1^*, Z_2)\,, Hom(Z_3, Z_4^*)) \cong ((Z_1^*, Z_2)^* \otimes (Z_3, Z_4^*)^*) \cong \big((Z_1 \otimes Z_2^*) \otimes (Z_3^*, Z_4)\big)$, since $Hom_m(Z_2, Z_1) \cong (Z_2^* \otimes Z_1)$, so we construct the action of Ǥ on the product. It is the bilinear map, therefore via utilizing By virtue of the tensor product and the universal quality of this tensor product, one obtains a special linear map:

$$\begin{tikzcd}
(Z_1 \times Z_2^*) \times (Z_3^* \times Z_4) \arrow[rr, "canianical"'] \arrow[dr, "\text{bilinear map}"'] & & (Z_1 \otimes Z_2^*) \otimes (Z_3^* \otimes Z_4) \arrow[dl, "\text{Linear map}"] \\
& Hom_m(\mathrm{Hom}\,(Z_1^*, Z_2)\,, Hom(Z_3, Z_4^*)) &
\end{tikzcd}$$

Figure (4)

And we will explain related between the (QTA) of the Lie group on $Hom_m(Hom\,(Z_1^*, Z_2)\,, Hom(Z_3, Z_4^*))$ and (QTA) of the Lie group on $\big((Z_1 \otimes Z_2^*) \otimes (Z_3^*, Z_4)\big)$ up to the representation given:

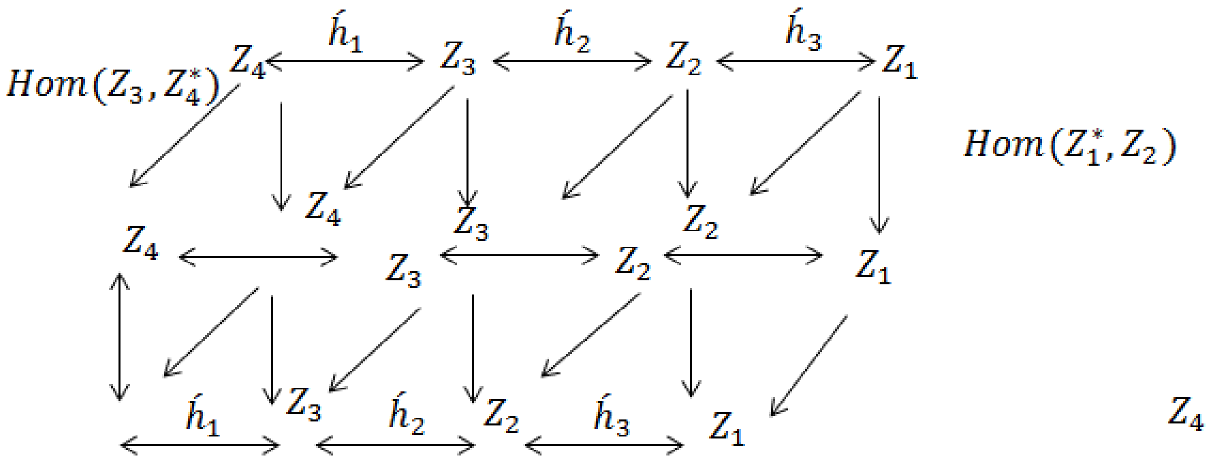

Figure (5)

Proposition (3.2):

Let w_1, w_2, w_3, w_4 being the four representations for the Lie group effects the vector spaces Z_1, Z_2, Z_3, Z_4, correspondingly, put $Hom_m((Z_4, Hom\ (Z_3, Z_2^*))\ , Z_1^*)$ be M- vector space of all linear mappings from Z_3 to Z_2^* and from $Hom(Z_3, Z_2^*)$ to Z_1^* and Z_4 to Hom $(Hom\ (Z_3, Z_2^*))\ , Z_1^*)$ Then the (QAT) of the Lie group on

$Hom_m((Z_4, Hom\ (Z_3, Z_2^*))\ , Z_1^*)$.

Proof:

Define : $G \to GL\ Hom_m((Z_4, Hom\ (Z_3, Z_2^*))\ , Z_1^*)$. By pro(3.1) such that:

$\left[\left(w_1(v)^{-1} \circ \acute{h}_1 \circ \left(w_2(v)^{-1} \circ \acute{h}_2 \circ w_3(v)\right)\right)\right] \circ \acute{h}_3 \circ w_4(v)$. For all $v \in G$, $h_i : Z_i \to M$. Where w^* is representation from G to Hom-space.

Since $\left(Hom_m\left(Z_4, Hom(Z_3, Z_2^*)\right), Z_1^*)\right) \cong (Z_4^* \otimes (Z_3 \otimes Z_2)^* \otimes Z_1)$.

The following diagram illustrates representation from G to Hom-space

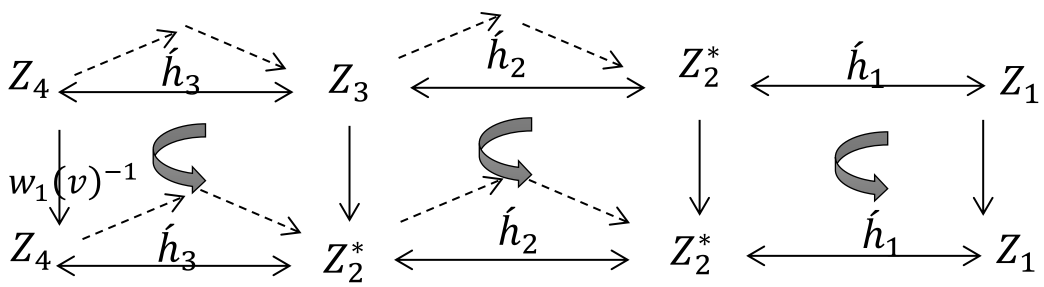

Figure (6)

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Therefore, one gets a unique linear map by utilizing the tensor product and universal property.

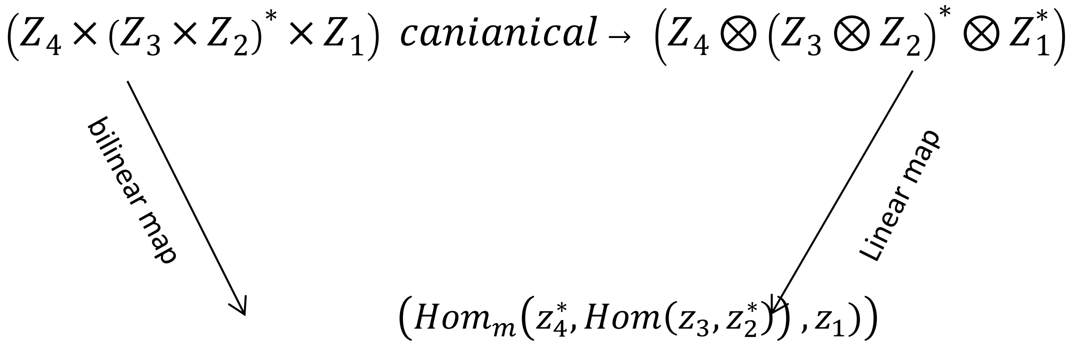

Figure (7)

The following representation illustrates the relationship between (QTA) on Hom-space and (QTA) on tensor product, given as:

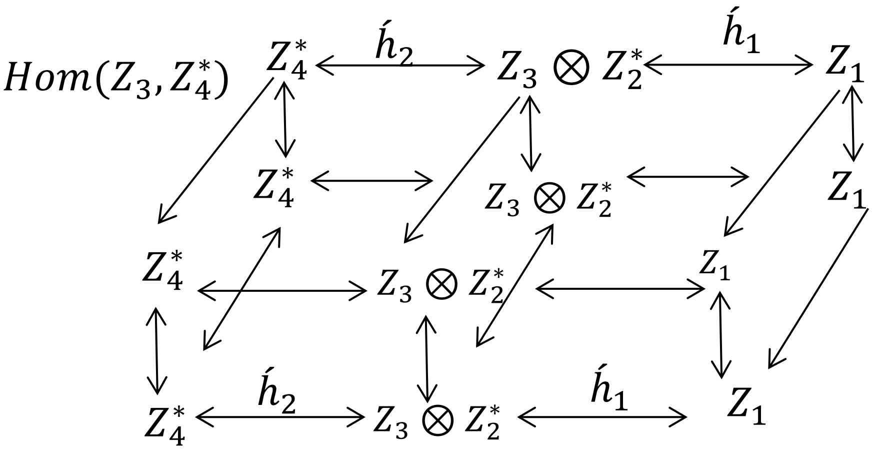

Figure (8)

Proposition (3.3) :

Let w_1, w_2, w_3, w_4, w_5, be five representations of Lie group affects the vector spaces Z_1, Z_2, Z_3, Z_4, Z_5, correspondingly. Put $Hom_m(Z_5, Hom\ (Hom(Z_4, Z_3^*), Z_2^* \otimes Z_1^*))$ be M-vector space of all linear mappings from Z_4^* to Z_3 as well as $Hom(Z_4, Z_3^*)$ to $Z_2^* \otimes Z_1^*$, from Z_5 to $Hom(Hom(Z_4, Z_3^*), Z_2^* \otimes Z_1^*)$.

Then the (QTA) of the Lie group on $Hom_m(Z_5, Hom(Hom\ (Z_4, Z_3^*)), Z_2^* \otimes Z_1^*)$.

Proof:

Define $\sigma: \text{G} \to \text{G}L\ (Hom(Z_5, Hom(Hom\ (Z_4, Z_3^*), Z_2^* \otimes Z_1^*)$ induced by

Representation, such that $\sigma(v)h = [(w_1(v)^{-1} \oplus w_2(v)^{-1})]$

$$\sigma(v)h = \left[\left(w_1(v)^{-1} \oplus w_2(v)^{-1}\right) \circ h_1 \circ \left(w_3(v)^{-1} \circ h_2 \circ w_4(v)\right)\right]$$

$\circ\, h_3 \circ w_5(v)$. For all $v \in \text{G}$ and $h_i: Z_i \to M$, where w^* is representation from G to Hom-space.

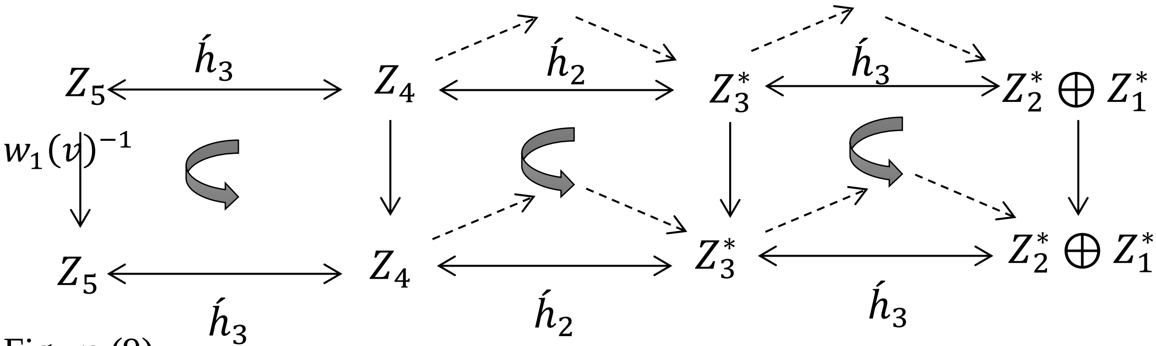

Figure (9)

Since

$\left(Hom_m\left(Z_5, Hom(Z_4, Z_3^*)\right), Z_2^* \oplus Z_1^*)\right) \cong (Z_5^* \otimes (Z_4 \otimes Z_3)^* \otimes Z_2 \oplus Z_1))$.

Therefore , one gets a sole linear map by utilizing the tensor product and universal property.

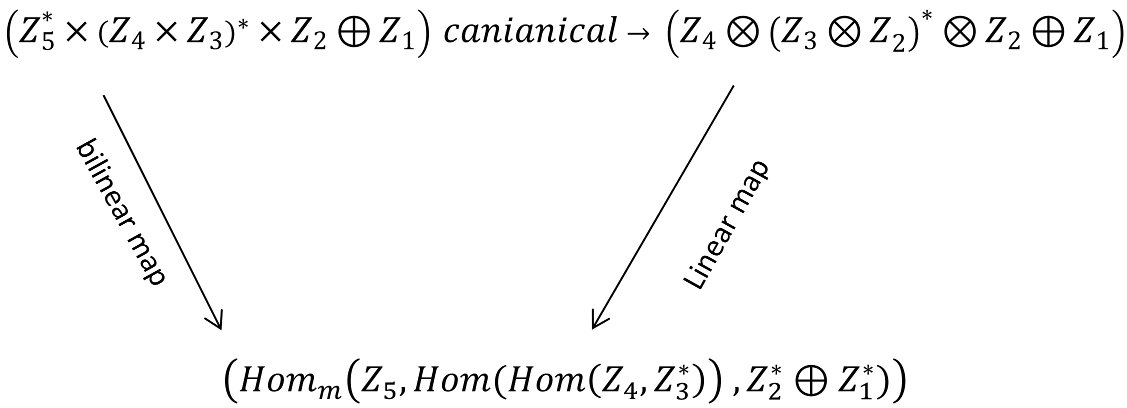

Figure (10)

The following representation illustrates the relationship between (PTA) on Hom-space and (PTA) on tensor product, given:

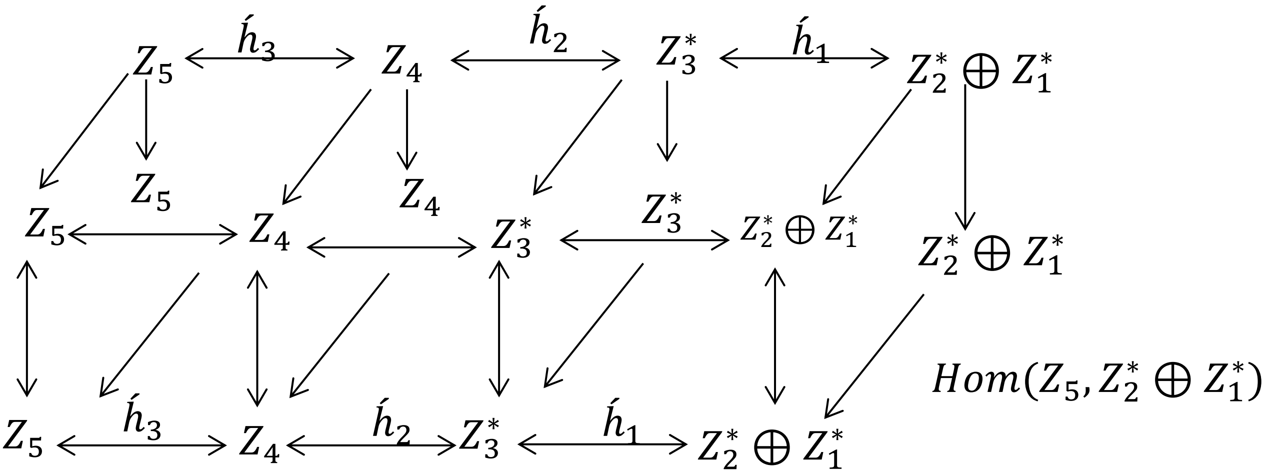

Figure (11)

Proposition (3.4) :

Let $Z_1, Z_2, Z_3, Z_4, Z_5,$ being the vector spaces, Z_i^* Is the dual of vectors Z_i, and $i = 1,2,3,4,5$, then the following assertions are:

$$Hom_m\left(Z_5, Hom\big(Z_4^*, Hom(Z_3^*, Z_2) \oplus Hom(Z_3^*, Z_1)\big)\right)^*. \qquad .\mathrm{I}$$

$$Hom_m\left(Hom(Z_1^*, Z_3^{**}) \oplus Hom\big((Z_2^*, Z_3^{**}), Hom(Z_4^{**}, Z_5^*)\big)\right). \qquad .\mathrm{II}$$

$$Hom_m\left(Hom(Z_1^*, Z_3) \oplus Hom\big((Z_2^*, Z_3), Hom(Z_4, Z_5, M)\big)\right). \qquad .\mathrm{III}$$

$$Hom_m\left(Hom(Z_1^*, Z_3) \oplus Hom\big(Z_2^*, (Z_3, M), Hom(Z_4, Z_5^*)\big)\right). \qquad .\mathrm{IV}$$

$$Hom_m\left(Hom(Z_1^*, Z_3) \oplus Hom\big(Z_2^*, (Z_3, M), Hom(Z_4, Z_5^*)\big)\right). \qquad .\mathrm{V}$$

$$\left(Hom_m\left(Z_5, Hom\left(Z_4^*, Hom(Z_3^*, Z_2\right) \oplus Hom\left(Z_3^*, Z_1\right)\right)\right) n^{\frown(****)} \qquad .\mathrm{VI}$$

$$= \{Hom_m\big(Z_5, Hom(Z_4^*, Hom(Z_3^*, Z_2) \oplus Hom(Z_3^*, Z_1)\big) \; if\ n\ is\ an\ even\ number.\ Hom_m\big(Z_5^*, Hom(Z_4, Hom(Z_2^*, Z_3) \oplus Hom(Z_1^*, Z_3)\big) \; if\ n\ is\ an\ odd\ number.\ \}$$

Proof:

$I \cong II$ to show that:

$\left(Hom_m(Z_5, Hom(Z_4^*, Hom(Z_3^*, Z_2) \oplus Hom(Z_3^*, Z_1))\right)^* \cong Hom_m\left(Hom(Z_1^*, Z_3^*) \oplus Hom\big((Z_2^*, Z_3^*), Hom(Z_4^{**}, Z_5^*)\big)\right).$

Let $\hbar_4 \in Hom(Z_5, Z_4^*)$, where $\hbar_4 : Z_5 \to Z_4^*$,

And $h_4^* \in \left(Hom(Z_5, Z_4^*)\right)^*$;

$\hbar_3 \in Hom(Z_4, Z_3)$, where $\hbar_3 : Z_4 \to Z_3$; and $h_3^* \in \left(Hom(Z_4, Z_3)\right)^*$; and

$h_3^* \in \left(Hom(Z_4, Z_3)\right)^*$;

$\hbar_2 \times \hbar_1 \in Hom(Z_3, Z_2 \times Z_1)$, where $\hbar_2 \times \hbar_1 : Z_3 \to Z_2 \times Z_1$; and

$$h_2^* \times h_1^* \in \left(Hom(Z_3, Z_2 \times Z_1)\right)^*$$

And there exists an intertwining map:

$$Hom_m\left(Z_5, Hom\big(Z_4^*, Hom(Z_3, Z_2) \oplus Hom(Z_3, Z_1)\big)\right)^* \to Hom_m\big(Hom(Hom(Hom(Z_3, Z_2)^* \oplus Hom(Z_3, Z_2)^*), Z_4^*)^*, (Z_5^*)\big)$$

Such that

$\pi(w^*(v))(a) = w^*(v)\pi(a)$, for all $v \in w^*$ and $a \in Z_1^* \times Z_2^*$, π is an invertible map.

$II \cong III$ to show that:

$Hom_m\left(Z_5, Hom\big(Z_4^*, Hom(Z_3^*, Z_2) \oplus Hom(Z_3^*, Z_1)\big)\right)^* \cong Hom_m\left(Hom(Z_1^*, Z_3) \oplus Hom\big(Z_2^*, Z_3), Hom(Z_4, Z_5, M)\big)\right).$

Since Z_5^* can be written as $Hom(Z_5, M)$, by proof (I). thus:

$Hom_m\left(Hom(Z_1^*, Z_3) \oplus Hom\big(Z_2^*, Z_3), Hom(Z_4, Z_5^*)\big)\right) = Hom_m\left(Hom(Z_1^*, Z_3) \oplus Hom\big(Z_2^*, Z_3), Hom(Z_4, (Z_5, M))\big)\right)$ By the same method, we have the other parts.

Corollary (3.5) :

Let $GL(l_i, G) \cong GL\left(w_5^* \otimes \left(w_4 \otimes \left((w_3 \otimes w_2^*) \oplus (w_3 \otimes w_1)\right)\right)\right)$

Where $i = 1,2,3,4,5$ are a matrix representations, after that the (PTA) for Lie group of Ǥ on

$\left(w_5^* \otimes \left(w_4 \otimes \left((w_3 \otimes w_2^*) \oplus (w_3 \otimes w_1)\right)\right)\right)^*$ is

$w^*(v) =$
$\left(\left(w_5(v)\right)^{tr} \otimes (w_4(v)^{-1})^{tr} \otimes ((w_4(v)^{-1})^{tr}) \otimes \left(w_2(v)\right)^{tr}\right) \oplus ((w_3(v)^{-1})^{tr} \otimes$
$(w_1(v)^{-1})^{tr}))$), for all ∈ Ǥ .

Proof:

$$w^*(v) = \left(w_5(v)^{-1} \otimes \left(w_4(v) \otimes (w_3(v) \otimes w_2(v)^{-1})\right) \oplus \left(w_3(v) \otimes w_1(v)\right)\right)^* =$$

$\left(\left(w_5(v)\right)^{tr} \otimes (w_4(v)^{-1})^{tr} \otimes \left((w_4(v)^{-1})^{tr}\right) \otimes (w_2(v))^{tr}\right) \oplus \left((w_3(v)^{-1})^{tr} \otimes\right.$
$\left.(w_1(v)^{-1})^{tr}\right))$), for all ∈ Ǥ .

And $w^*(vt) = (w(vt)^{tr})$

$= (w(t)^{-1}w(v)^{-1})$

$= (w(v)^{-1})^{tr}(w(t)^{-1})^{tr}$

$= w^*(v)w^*(t)$, for all ∈ Ǥ .

Hence the T-action is a dual matrix dual representation.

4. Conclusion:

. In this paper, we have provided on overview of the Lie group, Lie algebra, representation of the Lie group, and tensor product have been defined and associated with dual model by new structures consisting of tetramers and pentagons vector spaces by actions on $Hom_m\left((Z_4^*, Hom(Z_3, Z_2^*)), Z_1\right)$ In the proposals. Then we generalized it.

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