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An Analytical Technique to Obtain Approximate Solutions of Nonlinear Fractional PDEs

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Abstract:

In this work we obtain analytical approximate solutions for the two dimensional nonlinear PDEs with Liouville-Caputo fractional derivative. Numerical simulations of alternative models are presented for evaluating the effectiveness of these representations. Different source terms are considered in the fractional differential equations. The classical behaviors are recovered when the fractional order α is equal to 1.

Keywords: Fractional PDEs; Caputo fractional operator; approximate solution.

1-Introduction

Fractional calculus is primarily concerned with fractional integration and differentiation operations. It is an outstanding approach to situations where existing local operators are incapable of producing effective results, as it has been observed that the fractional order models are better matched with 9536 the real data than the classical integer-order derivatives [1]. The theory of fractional-order calculus was initially studied and further explored in the 18th and 19th centuries. One of the distinct features of fractional derivatives is their capacity to provide a pertinent and practical choice to model important physical problems. Many physical applications are not correctly modeled using the local differential operators. Therefore, the theory of the fractional-order derivative has attracted the attention of applied mathematicians to use fractional differential equations (FDEs) as a powerful tool in various areas, particularly in the fields of physics and engineering [2]. Fractional-order differential equations hold a strong foothold in some major domains, particularly in control theory [3], diffusion problems [4], signal processing [5], dynamics [6] and bio-engineering [7]. In addition, fractional-order models applied in microgrids are used in wireless networks [8]. Similarly, in fractional calculus, fractional-order models provide unprecedented significance in studying the dynamics of biological systems [9]. Kilbas et al. addressed the theory of fractional differential equations and their applications [10], therefore, I found many numerical and approximate methods to solve FDEs [11-49].

Fractional derivative operators (FDOs) are significantly relevant to real data analysis, which has drawn great attention from various mathematicians and modelers in the applied sciences. A variety of fractional operators are widely used in the literature, although few of them are comparatively more common, including Riemann-Liouville, Hadamard, Weyl, [51], Caputo [50], and Jumarie [52]. The kernel of the most commonly used fractional operators namely Caputo and Riemann- Liouville contains singularity, and hence, they may not always be able to express the non-locality of real-world situations properly.

2- Preliminaries

Definition: [50] The Caputo derivative of fractional order v of a function $\varphi(\mu)$ is defined as :-

$$D^{v}\varphi(\mu) = I^{m-v}D^{m}\varphi(\mu)$$

$$= \frac{1}{\Gamma(m-v)}\int_{0}^{\mu}(\mu-\tau)^{m-v-1}\varphi^{(m)}(\tau)d\tau\,, m-1 < v < m$$

The following are the basic properties of the operator D^{v}:-

1. $D^{v}k = 0$, where k is a constant.
2. $D^{v}I^{v}\varphi(\mu) = \varphi(\mu)$,
3. $D^{v}\mu^{\sigma} = \frac{\Gamma(\sigma+1)}{\Gamma(\sigma-v+1)}\,\mu^{\sigma-v}$,
4. $D^{v}D^{\sigma}\varphi(\mu) = D^{v+\sigma}\varphi(\mu)$

Definition 2.4: [50-52] The Mittag-Leffler function $E_{v}(z)$ with $v > 0$ is defined as:-

$$E_{v}(z) = \sum_{m=0}^{\infty}\frac{z^{v}}{\Gamma(mv+1)}$$

Definition 2.5: The Elzaki transform is defined over the set of functions

$$A=\left\{\varphi(\tau):\exists\mu\,,k_1,k_2>0\,,|\varphi(\tau)|<\mu\,e^{\frac{|\tau|}{k_j}}\,,\tau\in(-1)^j\times[0,\infty)\right\}.$$

by the following formula:

$$E[\varphi(\tau)]=T(s)=s\int_0^\infty e^{\frac{-\tau}{s}}\varphi(\tau)d\tau,s\in[k_1,k_2]$$

Some Elzaki transform Properties:-

1. $E[1]=s^2$
2. $E[\tau^v]=\Gamma(v+1)\,s^{v+2}$

Definition 2.6: The Elzaki transform of the Caputo fractional derivative is given by:

$$E[D_\tau^v\,\varphi(\mu,\tau)]=\frac{E[\varphi(\mu,\tau)]}{S^v}-\sum_{k=0}^{m-1}S^{2-v+k}\,\varphi^{(k)}(\mu,0),\;\; m-1<v<m$$

3- Analysis of EVIM

Consider the following fractional PDE:

$$^cD_t^\alpha\varphi(\mu,\tau)+R[\varphi(\mu,\tau)]+N[\varphi(\mu,\tau)]=g(\mu,\tau),t>0,\;n-1<\alpha\leqslant n$$

Taking ET, we have

$$E\{\,^cD_t^\alpha\varphi(\mu,\tau)+R[\varphi(\mu,\tau)]+N[\varphi(\mu,\tau)]\}=E\{g(\mu,\tau)\},$$

where

$$E\{\,^cD_t^\alpha\varphi(\mu,\tau)\}=\frac{T_n(w)}{w^\alpha}-\sum_{k=0}^{n-1}w^{2-\alpha+k}\varphi^{(k)}(\mu,0)$$

$$\frac{T_n(w)}{w^\alpha}-\sum_{k=0}^{n-1}w^{2-\alpha+k}\varphi^{(k)}(\mu,0)=E\{g(\mu,\tau)\}-E\{R[\varphi(\mu,\tau)]+N[\varphi(\mu,\tau)]\}$$

The iteration formula is

$$T_{n+1}(w) = T_n(w) + \lambda(w)\left[\frac{T_n\,(w)}{w^{\alpha}} - \sum_{k=0}^{n-1} w^{2-\alpha+k}\,\varphi^{(k)}(\mu,0) + E\{\varphi_n(\mu,\tau) + N\big(\varphi_n(\mu,\tau)\big)\ - g(\mu,\tau)\}\right],$$

where $\lambda(w)$ Lagrange multiplier.

We impose the condition $\frac{sT_{n+1}}{sT_n} = 0$, we have

$$1 + \frac{\lambda(w)}{w^{\alpha}} = 0$$

$$\rightarrow \frac{\lambda(w)}{w^{\alpha}} = -1$$

$$\rightarrow \lambda(w) = -w^{\alpha}$$

By applying Elzaki inverse and put $\lambda(w) = -w^{\alpha}$, we get

$$\varphi_{n+1} = E^{-1}\left(w^{\alpha}\sum_{k=0}^{n-1} w^{2-\alpha+k}\varphi^{(k)}(\mu,0) + w^{\alpha}E\{R(\varphi_n(\mu,\tau) + N(\varphi_n(\mu,\tau) - g(\mu,\tau)\right)$$

$$= E^{-1}\left(w^{\alpha}\sum_{k=0}^{n-1} w^{2-\alpha+k}\varphi^{(k)}(\mu,0)\right) + E^{-1}(w^{\alpha}E\{R(\varphi_n(x,t) +\ N(\varphi_n(x,t) - g(x,t))$$

The solution is given by

$$\varphi(\mu,\tau) = \lim_{n\rightarrow\infty}\varphi_n$$

4- Applications of EVIM

Example 1: Consider the fractional 2D partial differential equation:

$${}^{c}D_{\tau}^{v}\ \varphi(\mu,\zeta,\tau) - \varphi_{\mu\mu}^{2} - \varphi_{\zeta\zeta}^{2} - \varphi\left(\frac{8}{9}\varphi + 1\right) = 0,$$

where $0 < v \leq 1$ and subject to the initial condition

$$\varphi(\mu,\zeta,0) = e^{\frac{1}{3}(\mu+\zeta)}.$$

By taking Elzaki transform:

$$\frac{E[\varphi_{n+1}(\mu,\zeta,\tau)]}{S^v} - S^{2-v}\varphi(\mu,\zeta,0) - E\left[\frac{\partial^2\varphi_n^2}{\partial\mu^2} + \frac{\partial^2\varphi_n^2}{\partial\zeta^2} - \frac{8}{9}\varphi_n^2 - \varphi_n\right] = 0$$

$$E[\varphi_{n+1}(\mu,\zeta,\tau)] = S^2\varphi(\mu,\zeta,0) + S^v E\left[\frac{\partial^2\varphi_n^2}{\partial\mu^2} + \frac{\partial^2\varphi_n^2}{\partial\zeta^2} - \frac{8}{9}\varphi_n^2 - \varphi_n\right]$$

$$\varphi_{n+1}(\mu,\zeta,\tau) = \varphi(\mu,\zeta,0) + E^{-1}\left[S^v E\left[\frac{\partial^2\varphi_n^2}{\partial\mu^2} + \frac{\partial^2\varphi_n^2}{\partial\zeta^2} - \frac{8}{9}\varphi_n^2 - \varphi_n\right]\right]$$

The initial iteration $\varphi_0(\mu,\zeta,\tau)$ is given as follows:

$$\varphi_0(\mu,\zeta,\tau) = \varphi(\mu,\zeta,0) = e^{\frac{1}{3}(\mu+\zeta)}.$$

Now, we get the first approximation namely:

$$\varphi_1(\mu,\zeta,\tau) = e^{\frac{1}{3}(\mu+\zeta)} + E^{-1}\left[S^v E\left[\frac{4}{9}\, e^{\frac{2}{3}(\mu+\zeta)} + \frac{4}{9}\, e^{\frac{2}{3}(\mu+\zeta)} - \frac{8}{9}\, e^{\frac{2}{3}(\mu+\zeta)} - e^{\frac{1}{3}(\mu+\zeta)}\right]\right]$$

$$= e^{\frac{1}{3}(\mu+\zeta)} + E^{-1}\left[-S^{v+2}\, e^{\frac{1}{3}(\mu+\zeta)}\right]$$

$$= e^{\frac{1}{3}(\mu+\zeta)} - \frac{\tau^v}{\Gamma_{(v+1)}}\, e^{\frac{1}{3}(\mu+\zeta)}.$$

$$= \left(1 - \frac{\tau^v}{\Gamma_{(v+1)}}\right) e^{\frac{1}{3}(\mu+\zeta)}.$$

The second approximate reads as follows:

$$\varphi_2(\mu,\zeta,\tau) = \varphi(\mu,\zeta,0) + E^{-1}\left[S^v E\left[\frac{\partial^2\varphi_1^2}{\partial\mu^2} + \frac{\partial^2\varphi_1^2}{\partial\zeta^2} - \frac{8}{9}\varphi_1^2 - \varphi_1\right]\right]$$

$$= e^{\frac{1}{3}(\mu+\zeta)} + E^{-1}\left[S^v E\left[\begin{array}{l}\frac{4}{9}\left(1 - \frac{\tau^v}{\Gamma_{(v+1)}}\right)^2 e^{\frac{2}{3}(\mu+\zeta)} + \frac{4}{9}\left(1 - \frac{\tau^v}{\Gamma_{(v+1)}}\right)^2 e^{\frac{2}{3}(\mu+\zeta)} \\ -\frac{8}{9}\left(1 - \frac{\tau^v}{\Gamma_{(v+1)}}\right)^2 e^{\frac{2}{3}(\mu+\zeta)} - e^{\frac{1}{3}(\mu+\zeta)} + \frac{\tau^v}{\Gamma_{(v+1)}} e^{\frac{1}{3}(\mu+\zeta)}\end{array}\right]\right]$$

$$= e^{\frac{1}{3}(\mu+\zeta)} + E^{-1}\left[-S^{v+2}\, e^{\frac{1}{3}(\mu+\zeta)} + S^{2v+2}\, e^{\frac{1}{3}(\mu+\zeta)}\right]$$

$$= e^{\frac{1}{3}(\mu+\zeta)} - \frac{\tau^v}{\Gamma_{(v+1)}} e^{\frac{1}{3}(\mu+\zeta)} + \frac{\tau^{2v}}{\Gamma_{(2v+1)}} e^{\frac{1}{3}(\mu+\zeta)}.$$

$$\vdots$$

Then, we have:

$$\varphi(\mu,\zeta,\tau) = \lim_{n\to\infty} \varphi_n(\mu,\zeta,\tau)$$

$$= e^{\frac{1}{3}(\mu+\zeta)} \left(1 - \frac{\tau^{v}}{\Gamma_{(v+1)}} + \frac{\tau^{2v}}{\Gamma_{(2v+1)}} - \cdots\right)$$

$$= e^{\frac{1}{3}(\mu+\zeta)}\, E_v(-\tau^{v}).$$

When $v = 1$

$$\varphi(\mu,\zeta,\tau) = e^{\frac{1}{3}(\mu+\zeta)} \left(1 - \tau + \frac{\tau^2}{2!} - \cdots\right)$$

$$= e^{\frac{1}{3}(\mu+\zeta)-\tau},$$

which is an exact solution to the standard form biological population equation. The Figure 1 show the graphs of the approximate and the exact solutions among different values of τ and v when μ and ζ are fixed for the biological population equation in the Caputo fractional operator.

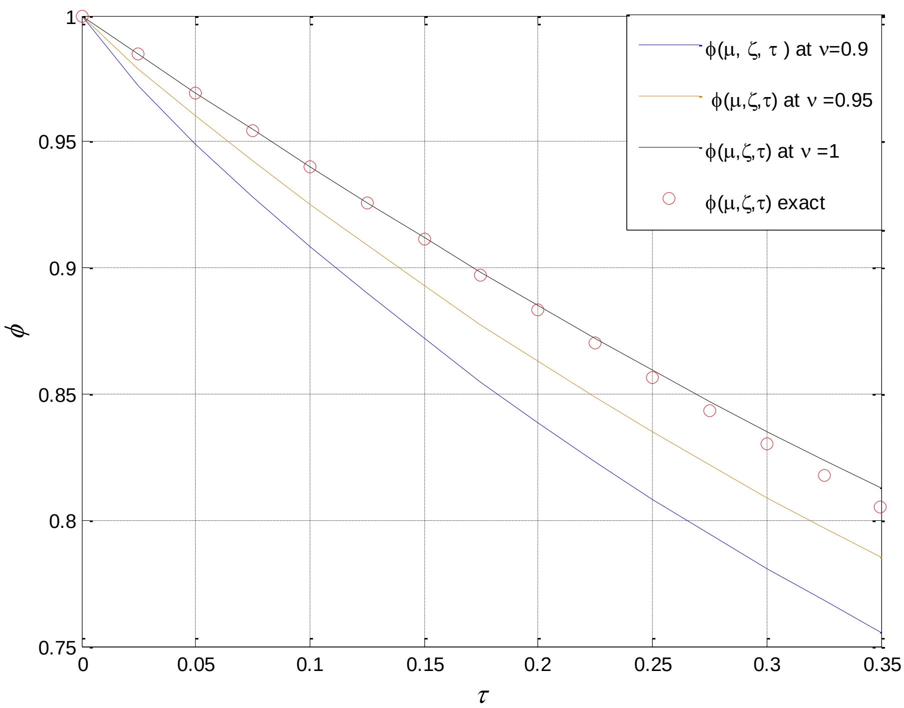

Figure 1. The approximate and the exact solutions among different values of τ and v when μ and ζ are fixed.

5-Conclusion

The integration of ODEs using the novel fifth-order DIRKTO5 and fourth-order DIRKTO4 methods with three stages has been discussed in this paper. In comparison to the implicit RK methods currently used in the scientific literature, numerical findings demonstrate that the suggested approaches are much more effective in terms of the number of function evaluations while solving the generic 4th-order ODEs.

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