Journal of Education for Pure Science- University of Thi-Qar
Vol.14, No. 2(2024)
Website: jceps.utq.edu.iq *Email: jceps@eps.utq.edu.iq*

DOI: https://doi.org/10.32792/jeps.v14i2.438

Bifurcation Analysis of an eco-epidemiological model involving prey refuge, fear impact and hunting cooperation

Alaa khadim Mohammed[1] and Salam Jasim Majeed [2,*]

[1,2] Department of Mathematics,College of computer science and mathematics, University of Thi-Qar, Thi-Qar, Iraq

[*] Corresponding email: alaa.khadim.mohammed@utq.edu.iq; sjmajeed@utq.edu.iq

Received 1/2/2024, Accepted 28/2/2024, Published 1/6/2024

Abstract:

This paper centers on an eco-epidemiological predator-prey model that accounts for hunting cooperation among predators and prey shelter and fear in afflicted prey. The objective is to investigate the effects of parameter factors on the model's bifurcation behavior. Theoretical part of this study demonstrate that a transcritical bifurcation can result from infection rate and refuge rate. Furthermore, fear rate can cause a Hopf-bifurcation to arise close to the positive equilibrium point. The presence of local bifurcations close to the non-trivial equilibrium points is verified by numerical analysis, which also guarantees the veracity of the theoretical results.

Keywords: prey-predator, eco-epidemiological, fear effect, prey refuge, Bifurcation.

1-Introduction

The dynamic of eco-epidemiological models is one of the main theme in mathematical biology, it considers the dynamics of infectious disease spreading among the ecosystem, notably prey-predator models with infectious diseases. Many studies shown that prey-predator relations have a significant impact on the way the ecosystems are organized, even though they considering predation as the alone source of interaction between the predator and its prey [1-4]. Recently in some theoretical and experimental studies prey individuals have been noted to alter their typical foraging behavior due the psychological strain of being discovered and slain by predators [5-8]. Through these and other studies, researchers have shown how crucial indirect effects like fear are in determining the dynamics between preys and predators.

In the last few years, many prey-predator mathematical models have been developed to investigate the consequences of fear effect. In [9-12] ecological models with fear effect are investigated, while, fear effect in eco-epidemiological models are thoughtful in [13-17]. Various biological deductions have been reached based on the different assumptions included in these mathematical models. Researchers in [18-19] discovered that the impact of fear can minify the numbers of various species, both prey and predators. Additionally, this fear-induced effect may initiate a process where diseases within the prey or predator population have a tendency to go extinct.

In most ecological systems, predators species are living in social groups for a variety of reasons, the most crucial being mating, more effective hunting and team attacks on large prey or other predators, in addition to increased protection against other predators. In ecosystems the widespread and important of population cooperation is shown in many different taxa, including birds, aquatic organisms, and carnivores [20-22]. Through the use of mathematical, ecological, and eco-epidemiological models, lots of studies have been investigated the implications of intra-species cooperation. For instance, cooperative hunting can change how predator-prey models behave, and may resulting in intricate patterns with several periodic cycles [15-17, 23-26]. So, including the function of cooperation in ecological and eco-epidemiological systems is essential to study and analysis species interactions and population dynamics. Preys' disease and the collaborative hunting efforts of predators can result in heightened fear among prey and an increased likelihood of being captured by predators. Therefore, the presence of prey refuge plays a crucial role in the field of mathematical biology, as it can be viewed as a form of anti-predator behavior that helps safeguard against the extinction of prey populations [1,2,6,8,9,26, 27]. In last years, some research papers have proposed models that explore the dynamics between preys and predators, specifically focusing on the presence of diseases within prey species. These studies have also investigated the impact of fear on the growth of susceptible prey and its role in reducing interactions between susceptible and infected individuals [13-17]. While authors in [14, 15, 17] assume the predator eating both healthy and ill preys, and hence they used the cost of fear on the healthy preys growth and their contact with ill preys. In [16] the authors assume the predator eating only ill preys and they used the cost of fear only on the growth of healthy preys. While in [13] the authors used the predation assumption as in [16] but the impact of fear as in [14, 15, 17].

In dynamical systems, a bifurcation arises when a slight, continuous adjustment to the parameter values known as bifurcation (critical) values of a system results in an abrupt alteration in its behavior, either qualitatively or topologically. Typically, during a bifurcation, the stability characteristics of local equilibria, periodic orbits, or other invariant sets undergo modifications. Local bifurcations, which involve changes in stability in the saddle-node, transcritical, pitchfork, and Hopf bifurcations of the system, are restricted to the vicinity of a periodic solution or a fixed point of the system [28-30]. The mathematical analysis of models in [13-17] showed the existence of different types of local bifurcation due to the effect of fear or other parameters of their eco-epidmological models.

Our work is motivated by the studies mentioned above (particularly [13-17]). In this paper, we investigate a prey-predator model in which predator subject to hunting cooperation, while diseased prey subject to the anti-predator behaviors such as the fear impact and prey refuge. In this study we assume the predators exclusively consume infected preys, and so we incorporate fear as a factor influencing only the contact between infected and susceptible individuals. Also, we consider the saturated incident as a function of disease transmission. The bifurcation theory and numerical results of this study can reveal how the interactions between disease transition, fear effect, prey refuge and hunting cooperation influence the dynamic structure of the system.The rest of the paper is organized as follows. In section 2, we formulate an eco-epidomoloigical mathematical model including the above factors. In section 3, we perform the equilibrium and their local stability conditions of the model. In section 4, we perform the bifurcation analysis. In section 5, we illustrate our analytical findings numerically with discussion. A brief conclusion is finding at the end section.

2- Mathematical model formulation

In this study, we make our assumption that predators exclusively consume infected prey (as in [13, 16]), but we think this leads to incorporate fear as a factor influencing only the contact between infected and susceptible individuals. Additionally, we introduce the concept of infected prey seeking refuge and exhibiting anti-predator behavior as a means of defending against predator hunting cooperation.

In this section, to construct our model, we begin by considering the principle assumptions: the populations are composed into prey species $N(T)$ and predator species $Q(T)$. Prey species also divide into susceptible prey $R(T)$ and infected prey $P(T)$, i.e $N = R + P$. Infected population $P(T)$ is unable to recover or build immunity, and the disease does not spread to predators via eating or any other methods. According to Holling-two functional response predator population $Q(T)$ exclusively consuming infected prey, where the disease infection renders the prey weak and vulnerable. Further, only infected prey has prey refuge and fear effect, while predator has hunting cooperation. Furthermore, the disease spreads occurs when infected prey comes into contact with susceptible prey, based on saturated incident rate. Therefore, from the aforementioned assumptions, the following eco-epidemiological model is derived.

$$\frac{dR}{dT} = rR\left(1 - \frac{R+P}{K}\right) - \frac{\beta RP}{(1+mP)(1+K_1Q)} - d_1R$$
$$\frac{dP}{dT} = \frac{\beta RP}{(1+mP)(1+K_1Q)} - \frac{(a+Q)(1-\theta)PQ}{C+(1-\theta)P} - d_2P \tag{1}$$

$$\frac{dQ}{dT} = \frac{e(a+Q)(1-\theta)PQ}{C+(1-\theta)P} - d_3Q$$

with initial $R(0) > 0, P(0) > 0, Q(0) > 0$. In this system, r is the susceptible prey logistic growth rate and K is the total prey environmental carrying capacity. β is the infection force and α is the predation rate. The terms $\frac{1}{1+mP}$ and $\frac{1}{1+K_1Q}$ are modeling the saturated incident rate and the fear effect, where m and K_1 represent saturation and fear factors, respectively. θ is infected prey refuge and e is conversion efficiency of Q on P. The death rates of susceptible prey, infected prey, and predator are given by d_1, d_2, and d_3, respectively. Now the next dimensionless variables are applied in the system (1):

$x_s = \frac{R}{K}$, $x_i = \frac{P}{K}$, $y = \frac{Q}{K}$ and $t = rT$.

Then (after some simplification) system (1) takes the form:

$$\begin{aligned}
\frac{dx_s}{dt} &= x_s(1 - x_s - x_i) - \frac{\alpha x_s x_i}{(1+Mx_s)(1+k_2y)} - D_1x_s = x_sf_1(x_s, x_i, y) \\
\frac{dx_i}{dt} &= \frac{\alpha x_s x_i}{(1+Mx_s)(1+k_2y)} - \frac{\gamma(A+y)(1-\theta)x_iy}{\eta+(1-\theta)x_i} - D_2x_i = x_if_2(x_s, x_i, y) \\
\frac{dy}{dt} &= \frac{\gamma_1(A+y)(1-\theta)x_iy}{\eta+(1-\theta)x_i} - D_3y = yf_3(x_s, x_i, y)
\end{aligned} \tag{2}$$

where, $k_2 = KK_1$, $\alpha = \frac{\beta K}{r}$, $M = mK, D_1 = \frac{d_1}{r}$, $\gamma = \frac{K}{r}$, $\eta = \frac{C}{K}$,

$A = \frac{a}{K}$, $D_2 = \frac{d_2}{r}$, $D_3 = \frac{d_3}{r}$, $\gamma_1 = \frac{eK}{r}$.

3- Equilibrium Points and Local Stability

System (2) possesses four equilibria that are non-negative. The conditions for their existence and local stability will be stated in this section. The points of equilibrium are as follows:

1. The trivial equilibrium :$\mathcal{E}_0$ $(0, 0, 0)$.
2. The axial equilibrium $\mathcal{E}_1$ $(\hat{x}_s, 0, 0)$, where $\hat{x}_s = 1 - D_1$.
3. Thepredator-freeboundary equilibrium$\mathcal{E}_2$ $(\bar{x}_s, \bar{x}_i, 0)$,

$\bar{x}_s = \frac{D_2}{\alpha}(1 + M\bar{x}_i)$, $\bar{x}_i = \frac{-\zeta_2 + \sqrt{\zeta_2^2 - 4\zeta_1\zeta_3}}{2\zeta_1}$, where, $\zeta_1 = M(D_2M + \alpha)$, $\zeta_2 = \alpha^2 + \frac{\zeta_1}{M} + M\zeta_3$, $\zeta_3 = D_2 - \alpha(1 - D_1)$.

4. Interior equilibrium: $\mathcal{E}_3$ (x_s^*, x_i^*, y^*).

The Jacobian matrix of Sys.(2) which evaluated at any arbitrary point (x_s, x_i, y) is expressed as follows:

$$J(x_s, x_i, y) = (\omega_{ij})_{3\times 3} \tag{3}$$

where,

$\omega_{11} = 1 - 2x_s - x_i - \frac{\alpha x_i}{(1+Mx_s)(1+k_2 y)} - D_1,\ \omega_{12} = -x_s - \frac{\alpha x_i}{(1+Mx_s)^2(1+k_2 y)}$,

$\omega_{13} = \frac{\alpha k_2 x_s x_i}{(1+Mx_s)(1+k_2 y)^2},\ \omega_{21} = \frac{\alpha x_i}{(1+Mx_s)(1+k_2 y)}, \omega_{22} = \frac{\alpha x_i}{(1+Mx_s)(1+k_2 y)} - \frac{\gamma\eta(A+y)(1-\theta)y}{(\eta+(1-\theta)x_i)^2} - D_2,$

$\omega_{23} = \frac{-\alpha k_2 x_s x_i}{(1+Mx_s)(1+k_2 y)^2} - \frac{\gamma(A+2y)(1-\theta)x_i}{\eta+(1-\theta)x_i}$

$\omega_{31} = 0,\ \omega_{32} = \frac{\gamma_1\eta(A+y)(1-\theta)y}{(\eta+(1-\theta)x_i)^2},\ \omega_{33} = \frac{\gamma_1(A+2y)(1-\theta)x_i}{\eta+(1-\theta)x_i} - D_3$.

The existence and local stability conditions of these equilibria may be summarized as follows:

The trivial equilibrium $\mathcal{E}_0\ (0,0,0)$is exist trivially, and unstable saddle node point due that the Jacobin matrix (3) at $\mathcal{E}_0$ has the eigenvalues: $\lambda 1 = 1 - D_1$, $\lambda_2 = -D_2$, and $\lambda_3 = -D_3$.

The axial equilibrium $\mathcal{E}_1\ (\hat{x}_s, 0, 0)$is exist when $D_1 < 1$, and local asymptotically stably provided that: $\widehat{x_s} < \frac{D_2}{\alpha}$.

The predator-free boundary equilibrium $\mathcal{E}_2\ (\bar{x}_s, \bar{x}_i, 0)$ exists if $D_2 < \alpha(1 - D_1)$, and local asymptotically stable if $\bar{x}_i < \frac{D_3\eta}{(1-\theta)[\gamma_1 A - D_3]}$.

Interior equilibrium: $\mathcal{E}_3\ ({x_s}^*, {x_i}^*, y^*)$ if exist then its local asymptotical stable if the following conditions hold: $\vartheta_{33} < \min\{-\vartheta_{11}, -\vartheta_{22}\}$ and $q_3 > 0$, where

$q_3 = \vartheta_{11}\vartheta_{23}\vartheta_{32} + \vartheta_{12}\vartheta_{21}\vartheta_{33} - \vartheta_{11}\vartheta_{22}\vartheta_{33} - \vartheta_{21}\vartheta_{13}\vartheta_{23},$

$\vartheta_{11} = -x_s^* < 0\ ,\ \ \vartheta_{12} = -x_s^* - \frac{\alpha x_s^*}{(1+Mx_i^*)^2(1+k_2 y^*)} < 0,\ \ \vartheta_{13} = \frac{\alpha k_2 x_i^* x_s^*}{(1+Mx_i^*)(1+k_2 y^*)^2} > 0,$

$\vartheta_{21} = \frac{\alpha x_i^*}{(1+Mx_i^*)(1+k_2 y^*)} > 0,\ \ ,\ \vartheta_{22} = \frac{\alpha x_s^*}{(1+Mx_i^*)^2(1+k_2 y^*)} - \frac{\eta\gamma(A+y^*)(1-\theta)y^*}{(\eta+(1-\theta)x_i^*)^2} - D_2 < 0,$

$\vartheta_{23} = \frac{-\alpha x_s^* x_i^* k_2}{(1+Mx_i^*)(1+k_2 y^*)^2} - \frac{\gamma(1-\theta)(A+2y^*)x_i^*}{\eta+(1-\theta)x_i^*} < 0\ \ ,\ \vartheta_{32} = \frac{\eta\gamma_1(1-\theta)(A+y^*)y^*}{(\eta+(1-\theta)x_i^*)^2} > 0,$

$\vartheta_{33} = \frac{\gamma_1(1-\theta)x_i^* y^*}{\eta+(1-\theta)x_i^*} < 0.$

4-Bifurcation Analysis

In a dynamic system, bifurcation happens when a control parameter affects the system's state and causes the system to branch out to a different state at a crucial control parameter value. Stated otherwise, a bifurcation is a moment at which behavior begins to diverge into distinctly different categories. Usually, there is a rapid shift as opposed to a slow, steady evolution. The relationship between the multiplicity of solutions and the control parameter is the focus of bifurcation theory. Bifurcations are categorized based on how an equilibrium solution's stability varies. Determining the existence and stability of different branches of the solution, such as equilibrium points and periodic orbits, is the aim of bifurcation theory. For a local bifurcation to appears, it is essential but not sufficient that the equilibrium point be non-hyperbolic. Because the parameters are not fixed values and always changing based on the conditions of the environment in which the system's organisms reside, it is imperative to investigate the bifurcation of the system (2).

In this section, Sotomayor's bifurcation theorem was utilized to ascertain whether local bifurcation may occur close to the system's (2) equilibrium points when the parameter crosses a particular value that turns the equilibrium point into a non-hyperbolic point. For simplicity, rewrite system (2) as follow in vector form:

$$\frac{dx}{dt} = F(X), X = (x_s, x_i, y), F = (x_s f_1, x_i f_2, y f_3) \tag{4}$$

Let $U = (u_1, u_2, u_3)^T$ be any vector, then the second directional derivative of the system (2) may be written as:

$$D^2F(X)(U,U) = [\mathfrak{D}_{i1}]_{3\times1} \tag{5}$$

where,

$$\mathfrak{D}_{11} = -2{u_1}^2 - 2\left(\frac{(1+k_2y)(1+Mx_i)^2+\alpha}{(1+k_2y)(1+Mx_i)^2}\right)u_1u_2 + \left(\frac{2\alpha Mx_s}{(1+k_2y)(1+Mx_i)^3}\right){u_2}^2 + \left(\frac{2\alpha k_2x_i}{(1+k_2y)(1+Mx_i)^2}\right)u_1u_3$$
$$+ \left(\frac{2\alpha k_2x_s}{(1+k_2y)^2(1+Mx_i)^2}\right)u_2u_3 + \left(\frac{2\alpha k_2^2x_sx_i}{(1+k_2y)^3(1+Mx_i)}\right){u_3}^2,$$

$$\mathfrak{D}_{21} = \left(\frac{2\alpha}{(1+k_2y)(1+Mx_i)^2}\right)u_1u_2 + 2\left(\frac{\alpha Mx_s}{(1+k_2y)(1+Mx_i)^3} + \frac{2\eta\gamma y(A+y)(1-\theta)^2}{(\eta+(1-\theta)x_i)^3}\right){u_2}^2 - \left(\frac{2\alpha k_2x_i}{(1+k_2y)^2(1+Mx_i)}\right)u_1u_3 -$$
$$2\left(\frac{\alpha k_2x_s}{(1+k_2y)(1+Mx_i)^3} - \frac{\eta\gamma(1-\theta)(A+2y)}{(\eta+(1-\theta)x_i)^2}\right)u_2u_3 - 2\left(\frac{\alpha k_2^2x_sx_i}{(1+k_2y)(1+Mx_i)^3} + \frac{\gamma(1-\theta)x_i}{\eta+(1-\theta)x_i}\right){u_3}^2,$$

$$\mathfrak{D}_{31} = \frac{-2\eta\gamma_1y(A+y)(1-\theta)^2}{(\eta+(1-\theta)x_i)^3}{u_2}^2 + \frac{2\eta\gamma_1(1-\theta)(A+y)}{(\eta+(1-\theta)x_i)^2}u_2u_3 + \frac{2\gamma_1(1-\theta)x_i}{\eta+(1-\theta)x_i}{u_3}^2.$$

Theorem 1: Let $\hat{\alpha}$ equal to the value of $\frac{D_2}{1-D_1}$. In this case, system (2) at $\mathcal{E}_1 = (\hat{x}, 0, 0)$, when α equal $\hat{\alpha}$, offers a transcritical bifurcation.

Proof : The Jacobian matrix (3) at α equal $\hat{\alpha}$ can be written as follow:

$$J_{\mathcal{E}_1} = J(\mathcal{E}_1, \hat{\alpha}) = \begin{pmatrix} -\hat{x}_s & -(1+\hat{\alpha})\hat{x}_s & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -D_3 \end{pmatrix}$$

The triangle matrix $J_{\mathcal{E}_1}$ has three eigenvalues expressed as follows:

$$\lambda_1 = -\hat{x}_s < 0, \qquad \lambda_2 = 0, \qquad \lambda_3 = -D_3 < 0.$$

Hence $\mathcal{E}_1$ is a non-hyprbolic point.

Now, considering the eigenvalue $\lambda_2 = 0$, let $\mathfrak{U}_1 = (u_{11}, u_{12}, u_{13})^T$ be the eigenvector with respect to the matrix $J_{\mathcal{E}_1}$. In this situation, $J_{\mathcal{E}_1}\mathfrak{U}_1 = 0$, which leads to $\mathfrak{U}_1 = (-(1+\hat{\alpha})u_{12}, u_{12}, 0)^T$, where $u_{12} \neq 0$ be any real number. Additionally, let $\Theta_1 = (\theta_{11}, \theta_{12}, \theta_{13})^T$ be the eigenvector of the transpose of matrix $J_{\mathcal{E}_1}^T$. Consequently, $J_{\mathcal{E}_1}^T\Theta_1 = 0$, which implies that $\Theta_1 = (0, \theta_{12}, 0)^T$ where $\theta_{12} \neq 0$ be any real number. Now, according to the Sotomayor's theorem, one can deduce that:

$$\frac{\partial F}{\partial \alpha} = F_\alpha(X,\alpha) = \begin{pmatrix} \frac{-x_s x_i}{(1+k_2 y)(1+Mx_i)} \\ \frac{x_s x_i}{(1+k_2 y)(1+Mx_i)} \\ 0 \end{pmatrix} \quad (6)$$

and hence

$\frac{\partial F}{\partial \alpha}|_{\substack{\alpha=\hat{\alpha} \\ X=\mathcal{E}_1}} = F_\alpha(\mathcal{E}_1,\hat{\alpha}) = (0,0,0)^T.$

Because of that, $\Theta_1^T F_\alpha(\mathcal{E}_1,\hat{\alpha}) = 0$. Therefore, system (2) has no saddle node bifurcation.

Moreover, given that:

$$DF_\alpha(\mathcal{E}_1,\hat{\alpha}) = \begin{pmatrix} 0 & -\widehat{x_s} & 0 \\ 0 & \widehat{x_s} & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$DF_\alpha(\mathcal{E}_1,\hat{\alpha})\mathfrak{U}_1 = (-\widehat{x_s}u_{12}, \widehat{x_s}\, u_{12}, 0)^T. \quad (7)$$

Consequently, $\Theta_1^T DF_\alpha(\mathcal{E}_1,\hat{\alpha})\mathfrak{U}_1 = \widehat{x_s}\, u_{12}\theta_{12} \neq 0$, further, through using equation (5), may deduce that:

$$D^2F(\mathcal{E}_1,\hat{\alpha})(\mathfrak{U}_1,\mathfrak{U}_1) = [\widehat{\mathfrak{D}}_{i1}]_{3\times 1} \quad (8)$$

where,

$\widehat{\mathfrak{D}}_{11} = 2u_{12}^2(M\hat{\alpha}\,\widehat{x_s} + 2(1+\hat{\alpha})^2)$, $\widehat{\mathfrak{D}}_{21} = 2\hat{\alpha}\, u_{12}^2(M\,\widehat{x_s} - (1+\hat{\alpha}))$, $\widehat{\mathfrak{D}}_{31}=0$.

Thus, it is simple to confirm that:

$$\Theta_1^T D^2F(\mathcal{E}_1,\hat{\alpha})(\mathfrak{U}_1,\mathfrak{U}_1) = 2\hat{\alpha}\, u_{12}^2(M\,\widehat{x_s} - (1+\hat{\alpha})) \neq 0$$

Hence, the proof complete. ■

Theorem 2: Let $D_3 < \gamma_1 A$, $\gamma_1 < \gamma$ are satisfied, then the system (2) at $\mathcal{E}_2$ undergoes a transcortical bifurcation when the parameter θ passes through the value $\bar{\theta} = 1 - \frac{D_3\eta}{\bar{x}_i(\gamma_1 A - D_3)}$, as long as the following condition met

$$\bar{x}_s^2 < \frac{D_2^2\bar{x}_i^2\gamma_2}{D_3[\gamma_2+\gamma]} \quad (9)$$

where $\gamma_2 = k\bar{x}_i\gamma_1$.

Proof. : Then Jacobian matrix(3) $J_{\mathcal{E}_2}$ at θ equal $\bar{\theta}$ may be written as follows

$$J_{\mathcal{E}_2} = J(\mathcal{E}_2, \bar{\theta}) = \begin{pmatrix} \omega_{11} & \omega_{12} & \omega_{13} \\ \omega_{21} & \omega_{22} & \bar{\omega}_{23} \\ 0 & 0 & 0 \end{pmatrix},$$

where $\bar{\omega}_{23} = \omega_{23}$ at θ equal $\bar{\theta}$. The matrix $J_{\mathcal{E}_2}$ has three eigenvalues given by $\bar{\lambda}_1 = \bar{\omega}_{33} = 0$, and the roots of quadrate polynomial $\lambda^2 - \hbar_1\lambda - \hbar_2 = 0$. Therefore, $\mathcal{E}_2$ is a non-hyprbolic point.

Let $\mathfrak{U}_2 = (u_{21}, u_{22}, u_{23})^T$ be the eigenvecctor corresponding to eigenvalue $\bar{\lambda}_1 = 0$. Thus, $J_{\mathcal{E}_2}\mathfrak{U}_2 = 0$, gives that $\mathfrak{U}_2 = (\Lambda_1 u_{23}, \Lambda_2 u_{23}, u_{23})^T$, where $u_{23} \neq 0$ be any real number, and $\Lambda_1 = -\{\frac{\Lambda_2\omega_{12}+\omega_{13}}{\omega_{11}}\}$, while $\Lambda_2 = \frac{\omega_{21}\omega_{13}-\omega_{11}\bar{\omega}_{23}}{\omega_{22}\omega_{11}-\omega_{21}\omega_{12}}$. Clearly, $\Lambda_1 < 0$ and $\Lambda_2 > 0$, provided that (9).

Further, let $\Theta_2 = (\theta_{21}, \theta_{22}, \theta_{23})^T$ represents the eigenvector corresponding to the eigenvalue $\bar{\lambda}_1 = 0$ of the matrix $J_{\mathcal{E}_2}^T$. Thus, $J_{\mathcal{E}_2}^T \Theta_2 = 0$ gives that $\Theta_2 = (0, 0, \theta_{23})^T$ where $\theta_{23} \neq 0$ be any real number. Now, Applying Sotomayor's theorem, leads to

$$\frac{\partial F}{\partial \theta} = F_\theta(X, \theta) = \begin{pmatrix} 0 \\ \frac{\eta\gamma(A+y)x_i y}{(\eta+(1-\theta)x_i)^2} \\ \frac{-\eta\gamma_1(A+y)x_i y}{(\eta+(1-\theta)x_i)^2} \end{pmatrix} \tag{10}$$

and

$$F_\theta(\mathcal{E}_2, \bar{\theta}) = (0, 0, 0)^T$$

Then we find, $\Theta_2^T F_\theta(\mathcal{E}_2, \bar{\theta}) = 0$, therefore, the system (2) has no saddle node bifurcation.

Moreover, with the help of (10), one can have

$$DF_\theta(X, \theta) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{\eta\gamma(A+y)x_i y(\eta-(1-\theta)x_i)}{\eta+(1-\theta)x_i} & \frac{\eta\gamma(A+2y)x_i}{(\eta+(1-\theta)x_i)^2} \\ 0 & -\frac{\eta\gamma_1(A+y)x_i y(\eta-(1-\theta)x_i)}{\eta+(1-\theta)x_i} & -\frac{\eta\gamma_1(A+2y)x_i}{(\eta+(1-\theta)x_i)^2} \end{pmatrix}$$

Hence, it follows

$$DF_\theta(\mathcal{E}_2, \bar{\theta})\mathfrak{U}_2 = \left(0, 0, \frac{\eta A\bar{x}_i(\gamma-\gamma_1)}{(\eta+(1-\theta)\bar{x}_i)^2} u_{23}\right)^T, \tag{11}$$

and so,

$$\Theta_2^T DF_\theta(\mathcal{E}_2, \bar{\theta})\mathfrak{U}_2 = \frac{\eta A\bar{x}_i(\gamma-\gamma_1)}{(\eta+(1-\theta)\bar{x}_i)^2} \neq 0.$$

Moreover, using equation (5), it follows that

$$D^2F(\mathcal{E}_2, \bar{\theta})(\mathfrak{U}_1, \mathfrak{U}_1) = [\bar{\mathfrak{D}}_{i1}]_{3\times 1}, \tag{12}$$

where,

$$\bar{\mathfrak{D}}_{11} = -2(\Lambda_1 u_{23})^2 - 2\left(\frac{(1+M\bar{x}_i)^2+\alpha}{(1+M\bar{x}_i)^2}\right)\Lambda_1\Lambda_2 u_{23}^2 + \left(\frac{2\alpha M\bar{x}_s}{(1+M\bar{x}_i)^3}\right)(\Lambda_2 u_{23})^2$$
$$+\left(\frac{2\alpha k_2\bar{x}_i}{(1+M\bar{x}_i)^2}\right)\Lambda_1 u_{23}^2 + \left(\frac{2\alpha k_2^2\bar{x}_s}{(1+M\bar{x}_i)^2}\right)\Lambda_2 u_{23}^2 + \left(\frac{2\alpha k_2^2\bar{x}_s\bar{x}_i}{(1+M\bar{x}_i)}\right)u_{23}^2$$

$$\bar{\mathfrak{D}}_{21} = \left(\frac{2\alpha}{(1+M\bar{x}_i)^2}\right)\Lambda_1\Lambda_2 u_{23}^2 + 2\left(\frac{\alpha M\bar{x}_s}{(1+M\bar{x}_i)^3}\right)(\Lambda_2 u_{23})^2 - \left(\frac{2\alpha k_2\bar{x}_i}{(1+M\bar{x}_i)}\right)\Lambda_1 u_{23}$$
$$-2\left(\frac{\alpha k_2\bar{x}_s}{(1+M\bar{x}_i)^2} - \frac{\eta\gamma(1-\theta)A}{(\eta+(1-\theta)\bar{x}_i)^2}\right)\Lambda_2 u_{23}^2 - \left(\frac{2\alpha k_2\bar{x}_s\bar{x}_i}{(1+M\bar{x}_i)^2} + \frac{\gamma(1-\theta)\bar{x}_i}{\eta+(1-\theta)\bar{x}_i}\right)u_{23}^2$$

$$\bar{\mathfrak{D}}_{31} = \frac{2\eta\gamma_1(1-\theta)A}{(\eta+(1-\theta)\bar{x}_i)^2}\Lambda_2 u_{23}^2 + \frac{2\gamma_1(1-\theta)\bar{x}_i}{\eta+(1-\theta)\bar{x}_i}u_{23}^2$$

Accordingly, may obtained that:

$$\Theta_2^T D^2F(\mathcal{E}_2, \bar{\theta})(\mathfrak{U}_1, \mathfrak{U}_1) = \frac{2\eta\gamma_1(1-\theta)}{(\eta+(1-\theta)\bar{x}_i)}\left(\frac{\eta A\Lambda_2}{(\eta+(1-\theta)\bar{x}_i)} + \bar{x}_i\right)u_{23}^2.$$

Therefore, under the condition (9), $\Theta_2^T D^2 F(\mathcal{E}_2,\bar{\theta})(\mathfrak{U}_1,\mathfrak{U}_1) \neq 0$, and the proof is finished. ■

Theorem 3: A Hopf bifurcation of sys.(2) arises in the vicinity of $\mathcal{E}_3$ with regard to k_2 if

$$-(q_2 - 3\varpi^2)\left(\frac{dq_3}{dk_2} - \varpi^2\frac{dq_1}{dk_2}\right) + 2q_1\varpi^2\frac{dq_2}{dk_2} \neq 0 \quad (13)$$

Proof. :

According to bifurcation theory, it is observed that a dynamical system (2) experiences a Hopf bifurcation when the characteristic equation of (5) at $\mathcal{E}_3$ has three roots, one has a negative real part and other two are purely imaginary, further $Re(\frac{d\lambda}{dk_2}) \neq 0$ at the bifurcation value k_2^*.

Assume the characteristic equation given by

$$\lambda^3 + q_1\lambda^2 + q_2\lambda + q_3 = 0, \quad (14)$$

where,

$$q_1 = -(\vartheta_{11} + \vartheta_{22} + \vartheta_{33}),$$
$$q_2 = \vartheta_{11}\vartheta_{22} + \vartheta_{11}\vartheta_{33} + \vartheta_{22}\vartheta_{33} - \vartheta_{32}\vartheta_{23} - \vartheta_{21}\vartheta_{12},$$
$$q_3 = \vartheta_{11}\vartheta_{23}\vartheta_{32} + \vartheta_{12}\vartheta_{21}\vartheta_{33} - \vartheta_{11}\vartheta_{22}\vartheta_{33} - \vartheta_{21}\vartheta_{13}\vartheta_{23},$$

and,

$$\vartheta_{11} = -x_S^* < 0\ ,\ \vartheta_{12} = -x_S^* - \frac{\alpha x_S^*}{(1+Mx_i^*)^2(1+k_2y^*)} < 0,\ \ \vartheta_{13} = \frac{\alpha k_2 x_i^* x_S^*}{(1+Mx_i^*)(1+k_2y^*)^2} > 0,$$

$$\vartheta_{21} = \frac{\alpha x_i^*}{(1+Mx_i^*)(1+k_2y^*)} > 0,\ \vartheta_{22} = \frac{\alpha x_S^*}{(1+Mx_i^*)^2(1+k_2y^*)} - \frac{\eta\gamma(A+y^*)(1-\theta)y^*}{(\eta+(1-\theta)x_i^*)^2} - D_2 < 0,$$

$$\vartheta_{23} = \frac{-\alpha x_S^* x_i^* k_2}{(1+Mx_i^*)(1+k_2y^*)^2} - \frac{\gamma(1-\theta)(A+2y^*)x_i^*}{\eta+(1-\theta)x_i^*} < 0,\ \ \vartheta_{31} = 0\ ,\ \ \vartheta_{32} = \frac{\eta\gamma_1(1-\theta)(A+y^*)y^*}{(\eta+(1-\theta)x_i^*)^2} > 0$$

$$\vartheta_{33} = \frac{\gamma_1(1-\theta)x_i^*y^*}{\eta+(1-\theta)x_i^*} < 0\ .$$

Furthermore, let $\triangle(k_2^*) = q_1q_2 - q_3 = 0$, then Eq.(14) take the form

$$(\lambda + q_1)(\lambda^2 + q_2) = 0. \quad (15)$$

Direct computation gives that

$$\lambda_1 = -q_1,\ \lambda_{2,3} = \pm i\varpi = \pm i\sqrt{q_2}.$$

Now consider the derivative of Eq.(14) with respect to k_2 we state

$$\frac{d\lambda}{dk_2} = -\frac{\lambda^2\frac{dq_1}{dk_2} + \lambda\frac{dq_2}{dk_2} + \frac{dq_3}{dk_2}}{[3\lambda^2 + 2q_1\lambda + q_2]}$$

Moreover, for $\lambda = i\varpi$, may have that

$$\left(\frac{d\lambda}{dk_2}\right)_{\lambda=i\varpi} = -\frac{(i\varpi)^2\frac{dq_1}{dk_2} + (i\varpi)\frac{dq_2}{dk_2} + \frac{dq_3}{dk_2}}{3(i\varpi)^2 + 2q_1(i\varpi) + q_2}$$

$$= -\frac{(\frac{dq_3}{dk_2} - \varpi^2 \frac{dq_1}{dk_2}) + (i\varpi)\frac{dq_2}{dk_2}}{(q_2 - 3\varpi^2) + 2q_1 i\varpi}$$

Consequently,

$$(\frac{d(Re\lambda)}{dk_2})_{\lambda = i\varpi} = \frac{-(q_2 - 3\varpi^2)(\frac{dq_3}{dk_2} - \varpi^2 \frac{dq_1}{dk_2}) + 2q_1\varpi^2 \frac{dq_2}{dk_2}}{(q_2 - 3\varpi^2)^2 + 4{q_1}^2\varpi^2}$$

Therefore, if (13) is violated, one may obtain $e(\frac{(d\lambda)}{dk_2})_{\lambda = i\varpi} \neq 0$. ■

5- Numerical Analysis and Discussion

Some numerical simulations are offered in this part to back up our analytical and mathematical findings. These simulations also reveal the systems fascinating, complicated behavior. In this section, we will use a hypothetical collections of data in several examples, and numerical solutions are run using Matlab (8.1) software to back up our earlier findings.

We consider the following hypothetical data as parametric values of model (2):

$$\begin{gathered} k_2 = 6,\ \ \alpha = 0.5,\ \ M = 0.5, D_1 = 0.05, D_2 = 0.105, D_3 = 0.05, \\ \gamma = 1,\ \ \gamma_1 = 0.75,\ \ A = 0.15,\ \ \eta = 0.2,\ \ \theta = 0.615. \end{gathered} \tag{16}$$

Example 1. In this example, to verify numerically the accuracy of the analytical results of theorem 1, we suppose $\alpha \in (0.05, 0.3)$ and other parameters values of model (2) as given in the data set (16).

Figure 1 shows that sys.(2)'s dynamic is steady at $\mathcal{E}_1 = (0.95, 0, 0)$ for $\alpha \leq 0.11$, but at the bifurcation value of turns out to no longer be a hyperbolic point. As a consequence of this, the model's dynamics is shown to loses the stability at $\mathcal{E}_1$ when $\alpha > 0.1103$, and this supporting the claim that made in ($\mathcal{E}_1$ is locally asymptotically stably provided that: $\hat{x_s} < \frac{D_2}{\alpha}$. The graphical presentation of phase diagram, for $\alpha = 0.1,\ 0.2, 0.3$, that shown in Fig.2 may confirmed this results

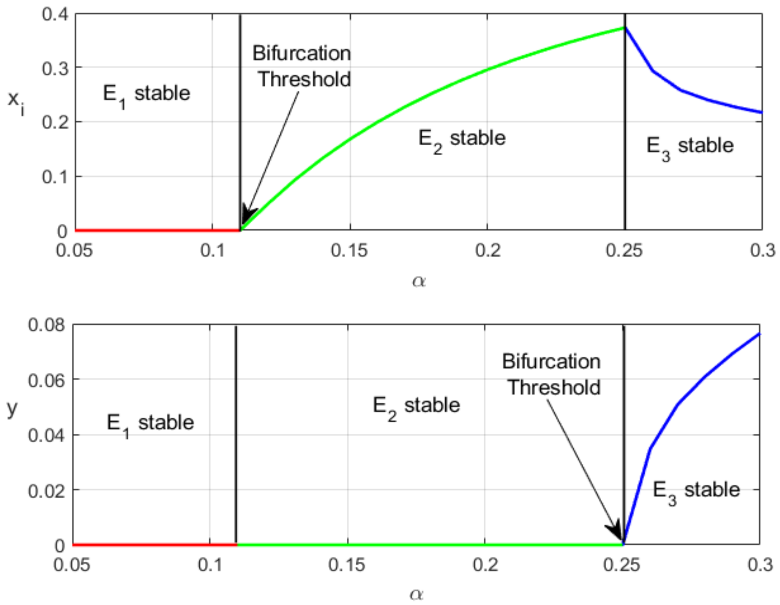

Figure 1: Bifurcation diagram illustrate that for $\alpha \leq 0.1103$, the equilibrium solution $\mathcal{E}_1$ is stable, and unstable for $\alpha > 0.1103$.

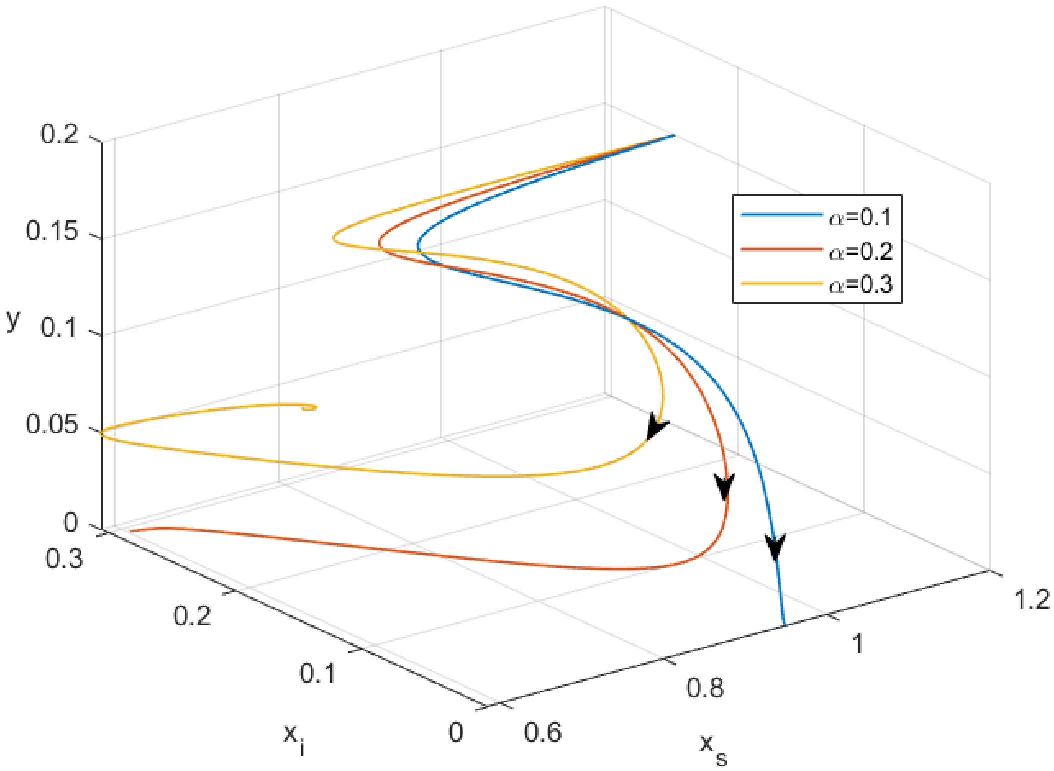

Figure 2: Phase portrait of system (2) for the value set (16) with different values of α. System (2) around equilibrium solution $\mathcal{E}_1$ it is asymptotically stable for $\alpha \leq 0.1103$, and loses this stability for $\alpha > 0.1103$.

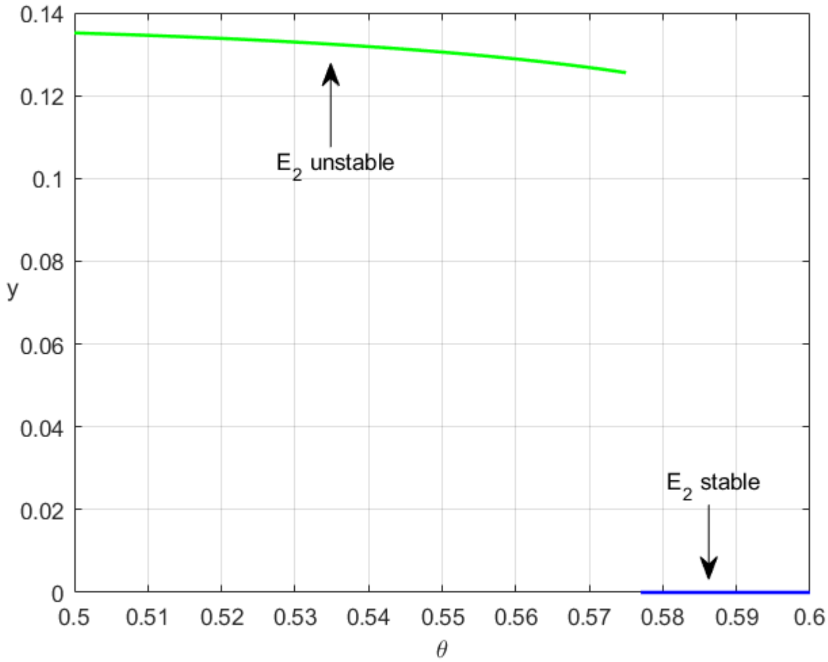

Figure 3: Bifurcation diagram illustrate that for $\theta \leq 0.577$, system (2) loses it is stability near $\mathcal{E}_2$.

Example 2. In this example, we take the parameters value as in (16) and $\gamma_1 = 0.49$ with infected prey refuge $\theta \in (0.5, 0.6)$ to verify the analytical results of theorem 2.

Figure 3 shows that for $\theta \geq 0.577$ susceptible prey, infected prey, and predator are stable around the equilibrium solution $\mathcal{E}_2$, while decrease the amount of infected prey refuge $\theta < 0.577$ create instability of the predator. Phase diagram, for $\theta = 0.5, 0.6$ is shown in Fig. 4. illustrate that the amount of infected prey refuge has a great impact to change the stability of $\mathcal{E}_2$.

Example 3. This example, we use k_2 as a bifurcation parameter while other parameters as in (16) to verify the analytical results of theorem 3.

Here can show that when k_2 take the value not more than (16), the equilibrium solution $\mathcal{E}_3$is unstable and the system undergoes a Hopf bifurcation around $\mathcal{E}_3$ as plotted in Fig. 5a for k_2 = 1.15, and when increasing the value of k_2 more than (16) by increasing the value of fear level, the equilibrium solution $\mathcal{E}_3$is stable after periodic dynamics (see Fig. 5(b-c)). This periodic oscillation can be controlled with more value of fear level $k_2$as seen in Fig. 5(d).

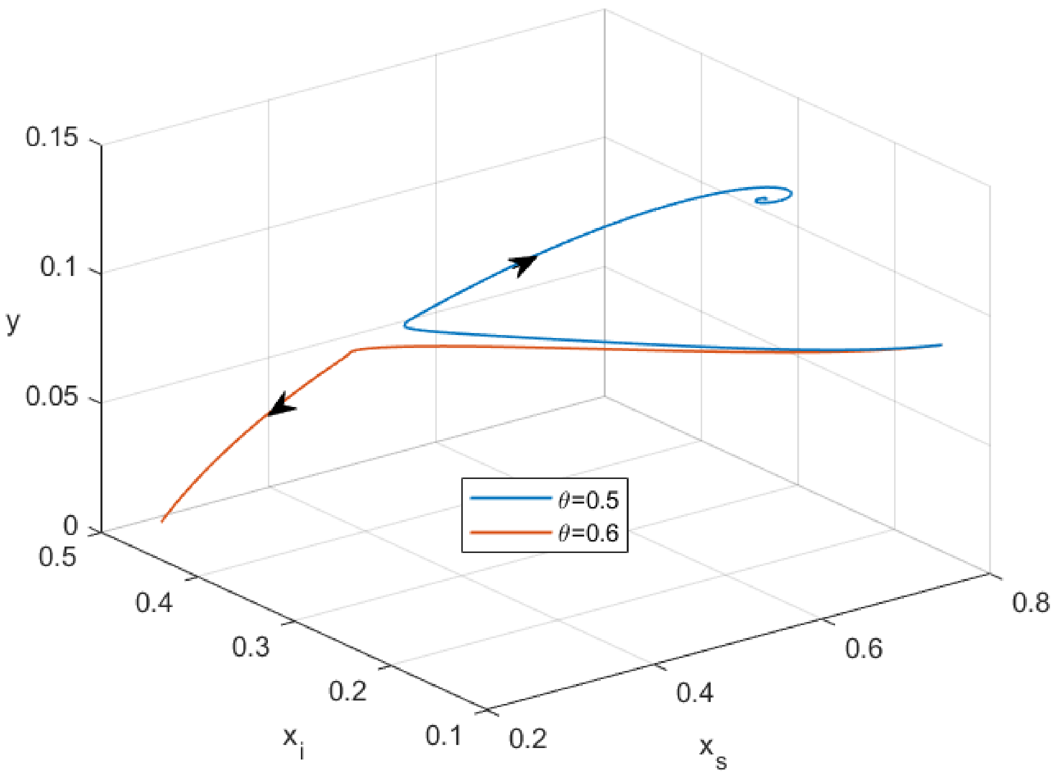

Figure 4: Phase portrait of system (2) for the value set (16) with different values of θ. System (2) around equilibrium solution $\mathcal{E}_2$it is asymptotically stable for $\theta = 0.6 \geq 0.577$, and loses this stability for $\theta = 0.5 < 0.577$.

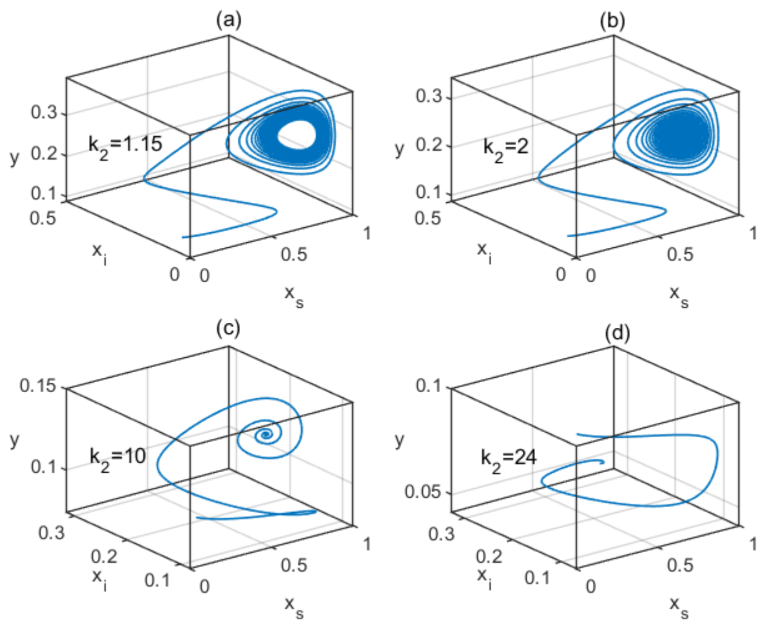

Figure 5: Phase portrait illustrate the effect of k_2 on the stability of system (2).

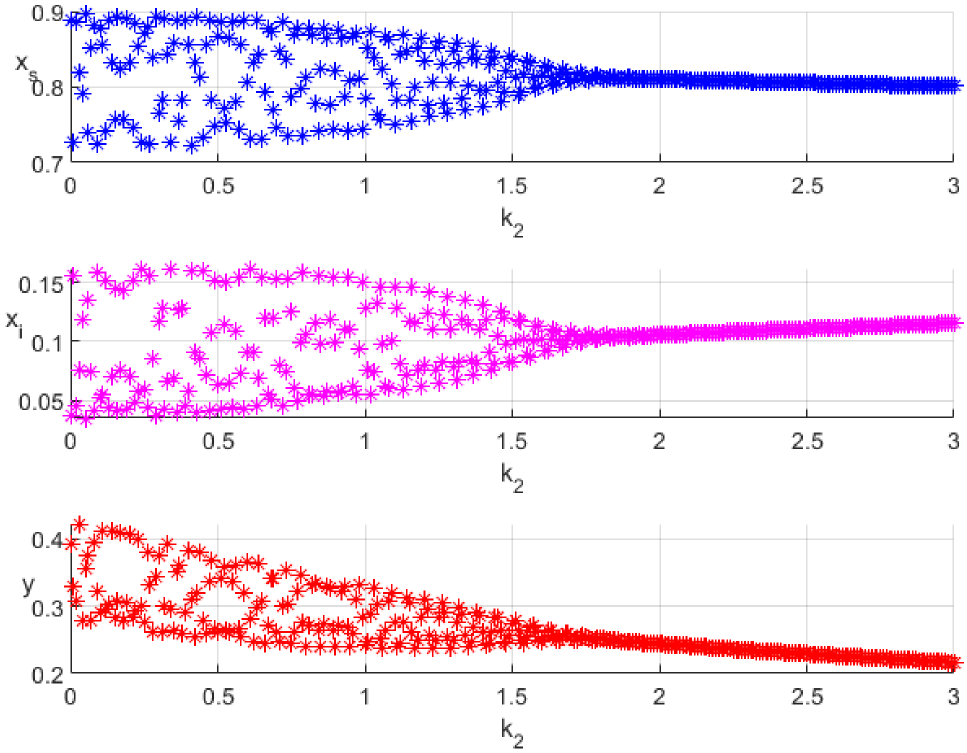

Figure 6: Bifurcation diagram illustrate that for $k_2 \leq 1.77$, system (2) undergoes Hopf bifurcation around $\mathcal{E}_3$.

6-Conclusions

This paper involved the study of the impact of infected prey refuge and fear effect on the dynamic of eco-epidemiological system of the prey-predator in the present of hunting cooperating. The proposed mathematical model contains at most four equilibrium points. The local bifurcation analysis near these equilibria are studies theoretically. To confirm the analytical findings and understand the impact of parameters on the dynamic of the system (2), numerical simulation was used. Figures (1-2) show that the infection force parameter α has an extinction effect on the infected prey and predator species. Figures (3-

4) show that the infected prey refuge parameter θ has the same effect on the predator species. Finally, as shown in Figure (5-6), fear level rate k_2 has a beneficial effect on the overall coexistence of the system since it is an instability effect at the start, but when it exceeds a certain level, it has a stability effect and the system switches for cyclic dynamic to stable oscillations and then to stable steady state.

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