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A Study of Boundedness in rectangular fuzzy n-normed spaces

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Abstract

Various forms of boundedness in rectangular fuzzy n-normed linear spaces of type $(X, N,*)$ are examined in this work, where $*$ represents an arbitrary t-norm. Some of these highly general notions of boundedness have no counterpart in the traditional topological metrizable linear spaces. We list the properties of these bounded sets and conduct a comparative analysis of the various forms of boundedness. Among these are a number of ideas on the symmetrical characteristics of the objects under study that come from the classical context and are relevant to the issues of this magazine. We define their consequences and provide instances to show how different these notions are from one another.

Keyword: fuzzy n-normed space, rectangular fuzzy n-normed space, fuzzy bounded set, fuzzy totally bounded

1. Introduction

In 1965, Zadeh[11] established the idea of fuzzy sets, which examined their properties and generalized sets. In 1984, Katsaras [5] introduced the concept of a fuzzy norm on a vector space for the first time. [3] given a Misiak [7] in 1989, developed the theory of n-normed space. In 1994, Cheng and Mordeson certain kind of fuzzy norm on a linear space, causing the accompanying fuzzy metric to be of the Kramosil and

Michalek kind. Rectangular metric space was first conceptualized in 2000 by Branciari [1]. wo concepts of boundedness are used here; the first was presented by Bag and Samanta [10] and the second by Sadeqi and Kia [6] in 2009. In 2018, N. Mlaiki, K.Abudaych, T. Abdeljawad and M.Abuloha [8] generalized the definition of a rectangular metric space by defining rectangular metric-like space. In 2021 [9], S. J. Mohammed and M. J. Mohammed introduced a rectangular fuzzy normed space.

2.Preliminaries

Definition (1.2) [13]:

The 3-tuple $(\text{Ɣ}, \mathcal{N}, *)$ is said to be a fuzzy $n-normed$ space (in short, F-n-NS), where Ɣ a vector space over the field F, $*$ be a continuous t-norm , $\mathcal{N}$ is a fuzzy set on $\text{Ɣ}^n \times (0,\infty)$ (i.e, $\mathcal{N}: \text{Ɣ} \times (0,\infty) \to [0,1]$ satisfying following: for each $\mathcal{v}_1, \mathcal{v}_2, \dots, \mathcal{v}_n, k \in \text{Ɣ}$, $\mathfrak{s}, t, \in \mathcal{R}$:

(N1) $\mathcal{N}(\mathcal{v}_1, \mathcal{v}_2, \dots, \mathcal{v}_n, \mathfrak{s}) = 0$, for every $\mathfrak{s} \in \mathcal{R}$ with $\mathfrak{s} \leq 0$,

(N2) $\mathcal{N}(\mathcal{v}_1, \mathcal{v}_2, \dots, \mathcal{v}_n, \mathfrak{s}) = 1$ iff $\mathcal{v}_1, \mathcal{v}_2, \dots, \mathcal{v}_n$ are linearly dependent.

(N3) $\mathcal{N}(\mathcal{v}_1, \mathcal{v}_2, \dots, \mathcal{v}_n, \mathfrak{s})$, is invariant under any permutation of, $\mathcal{v}_1, \mathcal{v}_2, \dots, \mathcal{v}_n$.

(N4) $\mathcal{N}(\alpha\mathcal{v}_1, \alpha\mathcal{v}_2, \dots, \alpha\mathcal{v}_n, \mathfrak{s}) = \mathcal{N}\left(\mathcal{v}_1, \mathcal{v}_2, \dots, \mathcal{v}_n, \frac{\mathfrak{s}}{|\alpha|}\right)$, if $\alpha \neq 0, \alpha \in F$

(N5) $\mathcal{N}(\mathcal{v}_1, \mathcal{v}_2, \dots, \mathcal{v}_n + k, t + \mathfrak{s}) \geq \mathcal{N}(\mathcal{v}_1, \mathcal{v}_2, \dots, \mathcal{v}_n, t) * \mathcal{N}(\mathcal{v}_1, \mathcal{v}_2, \dots \mathcal{v}_{n-1}, k, \mathfrak{s})$

$\lim_{\mathfrak{s}\to\infty} \mathcal{N}(\mathcal{v}_1, \mathcal{v}_2, \dots, \mathcal{v}_n, \mathfrak{s}) = 1$, (N6) $\mathcal{N}(\mathcal{v}_1, \mathcal{v}_2, \dots, \mathcal{v}_n, \mathfrak{s})$ is non decreasing function of $r \in R$, and

3. Fuzzy Bounded Sets

Definition (3.1):

suppose $\mathcal{Y}$ be a linear space with dimension $d \geq n,\ n \in \mathbb{N}$. A rectangular $n-norm$ on $\mathcal{Y}$ is a mapping $\|\cdot, \dots, \cdot\|$ on $\underbrace{\mathcal{Y} \times \cdots \times \mathcal{Y}}_{n}$, satisfying the following, for $v_1, v_2, \dots, v_n, , u, k \in \mathcal{Y}$

(1) $\|v_1, v_2, \dots, v_n\| = 0\ iff\ v_1, v_2, \dots, v_n$ are linearly dependent,

(2) $\|v_1, v_2, \dots, v_n\|$ is invariant under any permutation of $v_1, v_2, \dots, v_n$,

(3) $\|\alpha v_1, \alpha v_2, \dots, \alpha v_n\| = |\alpha| \|v_1, v_2, \dots, v_n\|$ for any $\alpha \in \mathbb{R}$,

(4) $\|v_1, v_2, \dots, v_{n-1}, v_n + u + k\| \leq \|v_1, v_2, \dots, v_{n-1}, v_n\| + \|v_1, v_2, \dots, v_{n-1}, u\|$

$+ \|v_1, v_2, \dots, v_{n-1}, k\|$

the pair $(\mathcal{Y}, \|\cdot, \dots, \cdot\|)$ is called a rectangular $n - normed$ space.

Definition (3.2):

The 3-tuple $(\mathcal{Y}, \mathcal{N}, *)$ is said to be a rectangular fuzzy $n - normed$ space, (in short, RF-n-NS); where $\mathcal{Y}$ a linear space over the field $F, *$ be a continuous t-norm , $\mathcal{N}$ is a fuzzy set on

(i.e $\mathcal{N}: \mathcal{Y}^n \times (0, \infty) \to [0,1]$ satisfies the following condition: for all $\mathcal{Y}^n \times (0, \infty)$

$, u, k \in \mathcal{Y}$ and $s, t, n \in R : v_1, v_2, \dots, v_n$

(N1) $\mathcal{N}(v_1, v_2, \dots, v_n, t) = 0$, for every $t \in R$ with $t \leq 0$,

(N2) $\mathcal{N}(v_1, v_2, \dots, v_n, t) = 1\ iff\ v_1, v_2, \dots, v_n$ are linearly dependent.

(N3) $\mathcal{N}(v_1, v_2, \dots, v_n, t)$, is invariant under any permutation of, $v_1, v_2, \dots, v_n$.

(N4) $\mathcal{N}(\alpha v_1, \alpha v_2, \dots, \alpha v_n, t) = \mathcal{N}\left(v_1, v_2, \dots, v_n, \frac{t}{|\alpha|}\right)$, if $\alpha \neq 0, \alpha \in F$

(N5) $\mathcal{N}(\vartheta_1, \vartheta_2, \dots, \vartheta_{n-1}, \vartheta_n + u + k\ , s + t + n) \geq \mathcal{N}(\vartheta_1, \vartheta_2, \dots, \vartheta_{n-1}, \vartheta_n, s) *$
$\mathcal{N}(\vartheta_1, \vartheta_2, \dots, \vartheta_{n-1}, u, t) * \mathcal{N}(\vartheta_1, \vartheta_2, \dots, \vartheta_{n-1}, k, n)$

(N6) $\mathcal{N}(\vartheta_1, \vartheta_2, \dots, \vartheta_n, t)$ is no decreasing function of $t \in R$ and

$\lim_{t\to\infty} \mathcal{N}(\vartheta_1, \vartheta_2, \dots, \vartheta_n, t) = 1$

Definition (3.3):

suppose $(\mathrm{Ɣ}, \mathcal{N}, *)$ be a RF-n-NS, then:

(i) A sequence $\{ƽ_n\}$ in $\mathcal{Y}$ is called convergent to $ƽ \in \mathrm{Ɣ}$ if for every $\partial \in (0,1)$, $t > 0$, there

is $n_\circ \in z^+$such that $\mathcal{N}\ (ƽ_1, ƽ_2, \dots, ƽ_{n-1}, ƽ_n - ƽ, t) > 1 - \partial$ for all $n \geq n_0$. (or equivalently,

$\lim_{t\to\infty} \mathcal{N}\ (\ ƽ_1, ƽ_2, \dots, ƽ_{n-1}, ƽ_n - ƽ, t) = 1)$

(ii) A sequence $\{ƽ_n\}$ in $\mathcal{Y}$ is called Cauchy if for every $\partial \in (0,1)$, $t > 0$, there is

$n_\circ \in z^+$such that, $\mathcal{N}(ƽ_1, ƽ_2, \dots, ƽ_{n-1}, ƽ_n - ƽ_m, t) > 1 - \partial$ for all $n, m \geq n_\circ$. (or

equivalently, $\lim_{t\to\infty} \mathcal{N}\ (\ ƽ_1, ƽ_2, \dots, ƽ_{n-1}, ƽ_n - ƽ_m, t) = 1)$

(iii) A rectangular fuzzy $n - normed$ space $(\mathrm{Ɣ}, \mathcal{N}, *)$ is called a complete, if every Cauchy

sequence is convergent

Definition (3.4):

Let $(\mathrm{Ɣ}, \mathcal{N}, *)$ be a RF-n-NS. The closed ball $B[\vartheta_1, \vartheta_2, \dots, \vartheta_n, r, t]$and the open ball l
$B(\vartheta_1, \vartheta_2, \dots, \vartheta_n, r, t)$ with center $\vartheta_1, \vartheta_2, \dots, \vartheta_n \in \mathrm{Ɣ}$ and radius r, $0 < r < 1, t > 0$ are defined as follows :

$B(\vartheta_1, \vartheta_2, \dots, \vartheta_n, r, t) = \{\ k \in \mathrm{Ɣ} : N(\vartheta_1, \vartheta_2, \dots, \vartheta_n - k\ , t) > 1 - r\}$
$.B[\vartheta_1, \vartheta_2, \dots, \vartheta_n, r, t] = \{\ k \in \mathrm{Ɣ} : N(\vartheta_1, \vartheta_2, \dots, \vartheta_n - k\ , t) \geq 1 - r\}$

Definition (3.5):

suppose $(\mathcal{Y}, \mathcal{N}, *)$ be a RF-n-NS, $E \subseteq \mathcal{Y}$, then:

(1) E is called open set, if for each $v \in E$, there is $t > 0$, $0 < r < 1$, such that $B(v_1, v_2, \dots, v_n, r, t) \subseteq E$.

(2) E is called closed if any sequence $\{v_n\}$ in E such that is convergent to v then $v \in E$.

Proposition (3.6):

Let $(\mathcal{Y}, \|\cdot, \dots, \cdot\|)$ *be a* RF − n − NS *and let* $x * y = x. y$. Let N be a

fuzzy set on $\mathcal{Y}^n \times (0, \infty)$ defined as follows $\mathcal{N}(v_1, v_2, \dots, v_n, r) = \frac{r}{r + \|v_1, v_2, \dots, v_n\|}$. Then N is a rectangular fuzzy n- norm induced by a rectangular n- normed space $(\mathcal{Y}, \|\cdot, \dots, \cdot\|)$ called ($\mathcal{N}$ *is called* standard rectangular fuzzy n-norm on X)

Proof:

(*N1*) for every $r \in \mathbb{R}$ with $r \leq 0$;

$\mathcal{N}(v_1, v_2, \dots, v_n, r) = 0$

(*N2*) for every $r \in \mathbb{R}$ with $r > 0$, we get $\mathcal{N}(v_1, v_2, \dots, v_n, r) = 1$

(*i*) *iff* $\frac{r}{r + \|v_1, v_2, \dots, v_n\|} = 1$,

(*ii*) *iff* $r = r + \|v_1, v_2, \dots, v_n\|$,

(*iii*) *iff* $\|v_1, v_2, \dots, v_n\| = 0$,

(*iv*) *iff* $v_1, v_2, \dots, v_n$ are linearly dependent.

(*N3*)

$$\mathcal{N}(v_1, v_2, \dots, v_n, r) = \frac{r}{r + \|v_1, v_2, \dots, v_n\|}$$
$$= \frac{r}{r + \|v_1, v_2, \dots, v_n, v_{n-1}\|} = \mathcal{N}(v_1, v_2, \dots, v_n, v_{n-1}, r) = \cdots$$

(*N4*) for every $\eta \in \mathbb{R}$ with > 0 , $\theta \in F, \theta \neq 0$;

$$\mathcal{N}\left(\theta v_1, \theta v_2, \dots, \theta v_n, \frac{\eta}{|\theta|}\right) = \frac{\eta\, /|\theta|}{(\eta\, /\, |\theta|) + ||v_1, v_2, \dots, v_n\, \|}$$

$$= \frac{\eta/|\theta|}{(\eta + |\theta|\; ||v_1, v_2, \dots, v_n||)/|\theta|}$$

$$= \frac{\eta}{\eta + |\theta|||v_1, v_2, \dots, v_n\, \|}$$

$$= \frac{\eta}{\eta + ||\theta v_1, \theta v_2, \dots, \theta v_n||} = N\, (\, \theta c_1, \theta c_2, \dots, \theta c_n, \eta).$$

(*N5*) for, $v_1, v_2, \dots, v_n, u, k \in \mathrm{Ɣ}\, , r, m, n > 0$

$$\mathcal{N}\, (v_1, v_2, \dots, v_n, r) * \mathcal{N}(v_1, v_2, \dots, v_{n-1}, u, m) * \mathcal{N}(v_1, v_2, \dots, v_{n-1}, k, n)$$

$$= \frac{r}{r + \| \, v_1, v_2, \dots, v_n \, \|} \cdot \frac{m}{m + \| \, v_1, v_2, \dots, v_{n-1}, u \, \|} \cdot \frac{n}{n + \| \, v_1, v_2, \dots, v_{n-1}, k \, \|}$$

$$= \frac{1}{1 + \frac{\| \, v_1, v_2, \dots, v_n \, \|}{r}} \cdot \frac{1}{1 + \frac{\| \, v_1, v_2, \dots, v_{n-1}, u \, \|}{m}} \cdot \frac{1}{1 + \frac{\| \, v_1, v_2, \dots, v_{n-1}, k \, \|}{n}}$$

$$\leq \frac{1}{1 + \frac{\| \, v_1, v_2, \dots, v_n \, \|}{r + m + n}} \cdot \frac{1}{1 + \frac{\| \, v_1, v_2, \dots, v_{n-1}, u \, \|}{r + m + n}} \cdot \frac{1}{1 + \frac{\| \, v_1, v_2, \dots, v_{n-1}, k \, \|}{r + m + n}}$$

$$\leq \frac{1}{1 + \frac{\| \, v_1, v_2, \dots, v_n \, \| + \| \, v_1, v_2, \dots, v_{n-1}, u \, \| + \| \, v_1, v_2, \dots, v_{n-1}, k \, \|}{r + m + n}}$$

$$= \frac{r + m + n}{r + m + n + \| \, v_1, v_2, \dots, v_n \, \| \; + \| \, v_1, v_2, \dots, v_{n-1}, u \, \| \; + \| \, v_1, v_2, \dots, v_{n-1}, k \, \|}$$

$$\leq \frac{r + m + n}{r \; + \; m + \; n + \| \, v_1, v_2, \dots, v_n, +u + k \, \|}$$

$$= \mathcal{N}\, (v_1, v_2, \dots, v_n, +u + k, r + m + n)$$

$$\mathcal{N}(v_1, v_2, \dots, v_n, r) * \mathcal{N}(v_1, v_2, \dots, v_{n-1}, u, m) * \mathcal{N}(v_1, v_2, \dots, v_{n-1}, k, n)$$

$\leq \mathcal{N}(v_1, v_2, \dots, v_n + u + k, r + m + n)$

$(N6)$ For all $r_1, r_2 \in \mathbb{R}$, if $r_1 < r_2 \leq 0$,

$$\mathcal{N}(v_1, v_2, \dots, v_n, r_1) = \mathcal{N}(v_1, v_2, \dots, v_n, r_2) = 0$$

Suppose $r_2 > r_1 > 0$, then

$$\frac{r_2}{r_2 + \|v_1, v_2, \dots, v_n\|} - \frac{r_1}{r_1 + \|v_1, v_2, \dots, v_n\|}$$
$$= \frac{\|v_1, v_2, \dots, v_n\|(r_2 - r_1)}{(r_2 + \|v_1, v_2, \dots, v_n\|)(r_1 + \|v_1, v_2, \dots, v_n\|)} \geq 0$$

for all $(v_1, v_2, \dots, v_n) \in X^n$, implies

$$\frac{r_2}{r_2 + \|v_1, v_2, \dots, v_n\|} \geq \frac{r_1}{r_1 + \|v_1, v_2, \dots, v_n\|}$$

which in turn implies $\mathcal{N}(v_1, v_2, \dots, v_n, r_2) \geq \mathcal{N}(v_1, v_2, \dots, v_n, r_1)$.

Thus $\mathcal{N}(v_1, v_2, \dots, v_n, r)$ is a non-decreasing function.

Also,

$$\lim_{r\to\infty} \mathcal{N}(v_1, v_2, \dots, v_n, r) = \lim_{r\to\infty} \frac{r}{r + \|v_1, v_2, \dots, v_n\|}$$
$$= \lim_{r\to\infty} \frac{r}{r(1 + (1/r)\|v_1, v_2, \dots, v_n\|)} = 1$$

Therefore $(\mathrm{Ɣ}, \mathcal{N}, *)$ is a rectangular fuzzy $n - normed$ space .

Definition (3.7):

Let $(X, N, *)$ be a RF-n-NS and $G \subseteq \mathrm{X}$. G is called bounded if

, $0 < r < 1$ such that $N(v_1, v_2, \dots, v_n, t) > 1 - r$, for all $\exists\, t > 0$

we will denote by T(X) the collection of all bounded subset of $X v_1, v_2, \dots, v_n \in G$

Definition (3.8):

Let $(X, N,*)$ be a RF-n-NS and A subset $Z \subseteq$ X *is* called a

closure of F if for any $\mathrm{v} \in Z$, there exists a sequence $\{\mathrm{v}_n\}$ in F such that

$= 1 \ \forall\, t > 0$, we denote the set Z by $\bar{F}\lim_{n\to\infty} N(\mathrm{v}_1, \mathrm{v}_2, \dots, \mathrm{v}_n - \mathrm{v}, t)$

Proposition (3.9):

Let $(X, N,*)$ be a RF-n-NS and G_1, G_2 be two bounded subsets of X.

Then $G_1 \cup G_2$ is bounded.

Proof.

Since G_1, G_2 are bounded subsets of X, $\exists\ \alpha_1, \alpha_2 \in (0,1), t_1, t_2 > 0$ such that

$1 - \alpha_1, (\forall)\mathrm{v}_1, \mathrm{v}_2, \dots, \mathrm{v}_n \in G_1$ and $N(\mathrm{v}_1, \mathrm{v}_2, \dots, \mathrm{v}_n, t_1) >$

. Let $t = \max\{t_1, t_2\}$ and$N(\mathrm{v}_1, \mathrm{v}_2, \dots, \mathrm{v}_n, t_2) > 1 - \alpha_2, (\forall)\mathrm{v}_1, \mathrm{v}_2, \dots, \mathrm{v}_n \in G_2$

$\alpha = \max\{\alpha_1, \alpha_2\}$. Let $\mathrm{v}_1, \mathrm{v}_2, \dots, \mathrm{v}_n \in G_1 \cup G_2$. If $\mathrm{v}_1, \mathrm{v}_2, \dots, \mathrm{v}_n \in G_1$, then

. $N(\mathrm{v}_1, \mathrm{v}_2, \dots, \mathrm{v}_n, t) \geq N(\mathrm{v}_1, \mathrm{v}_2, \dots, \mathrm{v}_n, t_1) > 1 - \alpha_1 \geq 1 - \alpha$

Similarly, if $\mathrm{v}_1, \mathrm{v}_2, \dots, \mathrm{v}_n \in G_2$, we obtain that

$N(\mathrm{v}_1, \mathrm{v}_2, \dots, \mathrm{v}_n, t) \geq N(\mathrm{v}_1, \mathrm{v}_2, \dots, \mathrm{v}_n, t_2) > 1 - \alpha_2 \geq 1 - \alpha$.

Thus $N(\mathrm{v}_1, \mathrm{v}_2, \dots, \mathrm{v}_n, t) > 1 - \alpha, (\forall)\mathrm{v}_1, \mathrm{v}_2, \dots, \mathrm{v}_n \in G_1 \cup G_2$.

Proposition (3.10):

Let $(X, N,*)$ be a RF-n-NS, where $*$ is almost strict. If G_1, G_2 are

two bounded subsets of X, then $G_1 + G_2$ is a bounded subset of X.

Proof.

Since G_1, G_2 are bounded subsets of X, $\exists\ \alpha_1, \alpha_2 \in (0,1), t_1, t_2 > 0$ such that

$1 - \alpha_1, (\forall)\mathrm{v}_1, \mathrm{v}_2, \dots, \mathrm{v}_n \in G_1$ and $N(\mathrm{v}_1, \mathrm{v}_2, \dots, \mathrm{v}_n, t_1) >$

. Let $\alpha \in (0,1)$ such that $N(\mathrm{v}_1, \mathrm{v}_2, \dots, \mathrm{v}_n, t_2) > 1 - \alpha_2, (\forall)\mathrm{v}_1, \mathrm{v}_2, \dots, \mathrm{v}_n \in G_2$

and $t = t_1 + t_2$. Let $z \in G_1 + G_2$. Then there exist $\alpha > 1 - (1 - \alpha_1) * (1 - \alpha_2)$

such that $z = \mathrm{v}_1, \mathrm{v}_2, \dots, \mathrm{v}_n + y$. We have that$\mathrm{v}_1, \mathrm{v}_2, \dots, \mathrm{v}_n \in G_1, y \in G_2$

$N(z, t) = N(\mathrm{v}_1, \mathrm{v}_2, \dots, \mathrm{v}_n + y, t_1 + t_2) \geq N(\mathrm{v}_1, \mathrm{v}_2, \dots, \mathrm{v}_n, t_1) * N(\mathrm{v}_1, \mathrm{v}_2, \cdots, y, t_2)$

$\geq (1 - \alpha_1) * (1 - \alpha_2) > 1 - \alpha.$

Proposition (3.11):

Let $(X, N, *)$ be a RF-n-NS and $G \in T(X)$. Then $\bar{G} \in T(X)$.

Proof.

As G is bounded we have that, $\exists\ \alpha_0 \in (0,1), t_0 > 0$ such that

$N(\mathrm{v}_1, \mathrm{v}_2, \dots, \mathrm{v}_n, t_0) > 1 - \alpha_0, (\forall)\mathrm{v}_1, \mathrm{v}_2, \dots, \mathrm{v}_n \in G$

Let $\alpha_1 \in (0,1)$ such that $(1 - \alpha_0) * (1 - \alpha_1) > 0$ and $\alpha \in (0,1)$ such that

$1 - \alpha < (1 - \alpha_0) * (1 - \alpha_1)$. Let $t_1 > 0$ and $\mathrm{v}_1, \mathrm{v}_2, \dots, \mathrm{v}_n \in \bar{G}$. Thus $(\exists)\{\mathrm{v}_n\} \subset G$ such

that $x_n \to x$. Hence $\lim_{n\to\infty} N(\mathrm{v}_1, \mathrm{v}_2, \dots, \mathrm{v}_n - \mathrm{v}, t_1) = 1$. Thus, $\exists n_0 \in \mathbb{N}$ such that

. Therefore, for $n \geq n_0$, we have that$N(\mathrm{v}_1, \mathrm{v}_2, \dots, \mathrm{v}_n - \mathrm{v}, t_1) > 1 - \alpha_1, (\forall)n \geq n_0$

$N(\mathrm{v}_1, \mathrm{v}_2, \dots, \mathrm{v}_n, t_0 + t_1) = N(\mathrm{v}_1, \mathrm{v}_2, \cdots, \mathrm{v} - \mathrm{v}_n + \mathrm{v}_n, t_0 + t_1)$

$\geq N(\mathrm{v}_1, \mathrm{v}_2, \cdots, \mathrm{v} - \mathrm{v}_n, t_1) * N(\mathrm{v}_1, \mathrm{v}_2, \dots, \mathrm{v}_n, t_0) \geq (1 - \alpha_1) * (1 - \alpha_0) > 1 - \alpha.$

Hence $\bar{G}$ is bounded.

Definition (3.12):

Let $(X, N, *)$ be a RF-n-NS. A subset G of $(X, N, *)$ is called fuzzy

Bounded (In short, FB) if $(\forall)\alpha \in (0,1), (\exists)t_\alpha > 0$ such that

$N(\mathrm{v}_1, \mathrm{v}_2, \ldots, \mathrm{v}_n, t_\alpha) > 1 - \alpha, (\forall)\mathrm{v}_1, \mathrm{v}_2, \ldots, \mathrm{v}_n \in G$.

The family of all fuzzy bounded subsets of X will be denoted by $FB(X)$.

Proposition (3.13) [2]:

Any continuous t-norm $*$ satisfies: $\forall\gamma \in (0,1), \exists\alpha, \beta \in (0,1)$ such that $\alpha * \beta = \gamma$

Proof.

Suppose $\gamma \in (0,1)$. Choose $\alpha > \gamma$. Let $f: [0,1] \to [0,1]$ defined by $f(y) = \alpha * y$. As $*$ is continuous, we have that f is continuous. As $f(0) = \alpha * 0 = 0$ and $f(1) = \alpha * 1 = \alpha$, for there exists $\beta \in (0,1)$ such that $f(\beta) = \gamma$, namely $\alpha * \beta = \gamma . \gamma \in (0, \alpha)$

Theorem (3.14):

Let $(X, N, *)$ be a RF-n-NS. A subset G of X is FB iff

$(\forall)\alpha \in (0,1), (\exists)t_\alpha > 0$ such that $N(\mathrm{v}_1, \mathrm{v}_2, \ldots, \mathrm{v}_n - y, t_\alpha) > 1 - \alpha$,

$(\forall)\mathrm{v}_1, \mathrm{v}_2, \ldots, \mathrm{v}_n, y \in G$

Proof:

Let⇒

. Then $\exists\ \beta \in (0,1)$ such that $(1 - \beta) * (1 - \beta) > 1 - \psi$. Since G is $\psi \in (0,1)$

FB, for $\beta \in (0,1), \exists t_\beta > 0$ such that

.$N\left(\mathrm{v}_1, \mathrm{v}_2, \ldots, \mathrm{v}_n, t_\beta\right) > 1 - \beta, (\forall)\mathrm{v}_1, \mathrm{v}_2, \ldots, \mathrm{v}_n \in G$

Let $v_1, v_2, \dots, v_n, y \in G$ and $t_\psi = 2t_\beta$. We have that

$$N(v_1, v_2, \dots, v_n - y, t_\psi) \geq N(v_1, v_2, \dots, v_n, t_\beta) * N(v_1, v_2, \cdots, y, t_\beta) \geq (1-\beta) * (1-\beta) > 1-\psi$$

Let $\psi \in (0,1)$. Using proposition (3.13) we obtain that there exist $\gamma, \delta \in (0,1)$ such that $\Leftarrow$

$(1-\gamma) * \delta.\ 1 - \frac{\psi}{2} =$

Let $v_0 \in G$ be fixed. As $\lim_{t\to\infty} N(v_1, v_2, \cdots, v_0, t) = 1$, we have that there exits $t_1 > 0$ such

that $N(v_1, v_2, \cdots, v_0, t_1) > \delta$. Based on our theory, for $\gamma \in (0,1)$, $\exists t_2 > 0$ such

that $N(v_1, v_2, \dots, v_n - v_0, t_2) > 1 - \gamma, (\forall) v_1, v_2, \dots, v_n \in G$. Let $t = t_1 + t_2$. Then, for all

, we have$v_1, v_2, \dots, v_n \in G$

$$N(v_1, v_2, \dots, v_n, t) \geq N(v_1, v_2, \dots, v_n - v_0, t_2) * N(v_1, v_2, \cdots, v_0, t_1) \geq (1-\gamma) * \delta = 1 - \frac{\psi}{2}$$

$$> 1 - \psi$$

<u>Proposition (3.15):</u>

Let $(X, N, *)$ be a RF-n-NS and $G \in FB(X)$. Then $\bar{G} \in FB(X)$.

<u>Proof.</u>

Since $G \in FB(X)$, $\exists \alpha_0 \in (0,1), t_0 > 0$ such that

$N(v_1, v_2, \dots, v_n - y, t_0) > 1 - \alpha_0, (\forall) v_1, v_2, \dots, v_n, y \in G$. Let $v_1, v_2, \dots, v_n, y \in \bar{G}$. Then,

such that $v_n \to v$ and $y_n \to y$. Let $\beta \in (0,1), \beta > \alpha_0$ and $s = 3t_0$. We have that$\exists \{v_n\}, \{y_n\} \subset G$

$$N(v_1, v_2, \dots, v_n - y, s) \geq N(v_1, v_2, \cdots, v - v_n, t_0) * N(v_1, v_2, \dots, v_n - y_n, t_0) * N(v_1, v_2, \cdots, y_n - y, t_0)$$
$$\geq N(v_1, v_2, \cdots, v - v_n, t_0) * (1 - \alpha_0) * N(v_1, v_2, \cdots, y_n - y, t_0)$$

For $n \to \infty$ we obtain that

$N(v_1, v_2, \dots, v_n - y, s) \geq 1 - \alpha_0 > 1 - \beta$.

Thus $\bar{G} \in FB(X)$.

Corollary (3.16):

Let $(X, N,*)$ be a RF-n-NS. Then:

1. If G, H are FB, then $G \cup H$ and $G + H$ are FB.
2. If G is FB, then $\bar{G}$ is FB.

Proposition (3.17):

Let $(X, N,*)$ be a RF-n-NS. If $G \subseteq X$ satisfies

$(\exists)\alpha_0 \in (0,1): \sup\{t \geq 0: N(v_1, v_2, \dots, v_n - y, t) \leq 1 - t\} < \alpha_0, (\forall)v_1, v_2, \dots, v_n, y \in G,$

then G is F-bounded.

Proof.

For $v_1, v_2, \dots, v_n, y \in G$, let

. By our hypothesis $d(x, y) = \sup\{t \geq 0:\ N(v_1, v_2, \dots, v_n - y, t) \leq 1 - t\}$

such that $d(x, y) < \alpha_0, (\forall)v_1, v_2, \dots, v_n, y \in G$. Thus $(\exists)\alpha_0 \in (0,1)$

. $N(v_1, v_2, \dots, v_n - y, \alpha_0) > 1 - \alpha_0, (\forall)v_1, v_2, \dots, v_n, y \in G$

Hence G is F-bounded.

Definition (3.18):

Let $(X,\ N,\ *)$ be a RF-n-NS. The set of all open balls with respect

to N is a topology on X, is called a metrizable topology on X, denoted by $(X, \mathcal{T}_N)$

Remark (3.19):

It can be seen that a subset G of a topological linear space X is said to be bounded if there is

$k > 0$ such that $G \subseteq kV$ for any neighborhood V of 0_X.

Theorem (3.20):

Let $(X, N, *)$ be a RF-n-NS. A subset G of X is FB iff

is bounded in $(X, \mathcal{T}_N).G$

Proof.

$(\Rightarrow)$ Suppose V be a neighborhood of 0_X. Then, $\exists\ \alpha \in (0,1), t > 0$

. Since G is FB, for $\alpha \in (0,1), (\exists)t_\alpha > 0 \Rightarrow B(0,0,\cdots,0,\alpha,t) \subseteq V$

$\Rightarrow N(\mathrm{v}_1, \mathrm{v}_2, \dots, \mathrm{v}_n, t_\alpha) > 1 - \alpha, (\forall)\mathrm{v}_1, \mathrm{v}_2, \dots, \mathrm{v}_n \in G$. Let $k = \frac{t_\alpha}{t}$. We have that

. Thus$N(\mathrm{v}_1, \mathrm{v}_2, \dots, \mathrm{v}_n, tk) = N(\mathrm{v}_1, \mathrm{v}_2, \dots, \mathrm{v}_n, t_\alpha) > 1 - \alpha, (\forall)\mathrm{v}_1, \mathrm{v}_2, \dots, \mathrm{v}_n \in G$

$G \subset B(0,0,\cdots,0,\alpha,tk) = kB(0,0,\cdots,0,\alpha,t) \subseteq kV$

Let $\alpha \in (0,1)$. Since $B(0,0,\cdots,0,\alpha,1)$ is a neighborhood of 0_X, $\exists\ k > 0$ $(\Leftarrow)$

$\Rightarrow G \subseteq kB(0,0,\cdots,0,\alpha,1) = B(0,0,\cdots,0,\alpha,k)$. Thus

$N(\mathrm{v}_1, \mathrm{v}_2, \dots, \mathrm{v}_n, k) > 1 - \alpha, (\forall)\mathrm{v}_1, \mathrm{v}_2, \dots, \mathrm{v}_n \in G.$

Hence G is F-bounded.

Definition (3.21):

Let $(X, N, *)$ be a RF-n-NS. A subset G of a $(X, N, *)$ is called

fuzzy totally bounded (In short, FTB) if

$(\forall)\alpha \in (0,1), (\exists)\{\mathrm{v}_1, \mathrm{v}_2, \dots, \mathrm{v}_n\} \subset X: G \subset \bigcup_{i=1}^{n} (\mathrm{v}_i + B(0,0,\cdots,0,\alpha,\alpha))$

The collection of all fuzzy totally bounded subsets of X will be denoted by FTB(X).

Theorem (3.22):

Let $(X, N,*)$ be a RF-n-NS. The statements that follow are equivalent:

(1) G is FTB;

(2) $(\forall)\alpha \in (0,1), (\exists)\{v_1, v_2, \dots, v_n\} \subset G: G \subset \bigcup_{i=1}^{n}(v_i + B(0,0,\cdots,0,\alpha,\alpha))$;

(3) $(\forall)\alpha \in (0,1), (\forall)t > 0, (\exists)\{v_1, v_2, \dots, v_n\} \subset G: G \subset \bigcup_{i=1}^{n}(v_i + B(0,0,\cdots,0,\alpha,t))$;

(4) $(\forall)\alpha \in (0,1), (\forall)t > 0, (\exists)\{v_1, v_2, \dots, v_n\} \subset X: G \subset \bigcup_{i=1}^{n}(v_i + B(0,0,\cdots,0,\alpha,t))$.

Proof.

(1) ⇒ (2). Let $\alpha \in (0,1)$. Then , $\exists\ n \in \mathbb{N}, n \geq 2$ such that

$$\left(1-\frac{\alpha}{n}\right) * \left(1-\frac{\alpha}{n}\right) > 1-\alpha.$$

Indeed, if we suppose that

$$\left(1-\frac{\alpha}{n}\right) * \left(1-\frac{\alpha}{n}\right) \leq 1-\alpha, (\forall)n \in \mathbb{N}, n \geq 2$$

by passing to the limit, for $n \to \infty$, we acquire that $1 * 1 \leq 1-\alpha$, which contradict it.

As G is FT-bounded,

$$(\exists)\{v_1, v_2, \cdots, v_m\} \subset X: G \subset \bigcup_{i=1}^{m}\left(v_i + B\left(0,0,\cdots,0,\frac{\alpha}{n},\frac{\alpha}{n}\right)\right)$$

Let $y_i \in G \cap \left(v_i + B\left(0,0,\cdots,0,\frac{\alpha}{n},\frac{\alpha}{n}\right)\right), i = \overline{1,m}$. We show that

$G \subset \bigcup_{i=1}^{m}(y_i + B(0,0,\cdots,0,\alpha,\alpha))$. Let $v \in G$. Then , $\exists k \in \{1,\cdots,m\}$ such that $v_1, v_2, \dots, v_n \in v_k + B\left(0,0,\cdots,0,\frac{\alpha}{n},\frac{\alpha}{n}\right)$, namely $N\left(v_1, v_2, \cdots, v - v_k, \frac{\alpha}{n}\right) > 1-\frac{\alpha}{n}$.

We have that,

$N(\mathrm{v}_1, \mathrm{v}_2, \ldots, \mathrm{v}_n - y_k, \alpha) \geq N\left(\mathrm{v}_1, \mathrm{v}_2, \ldots, \mathrm{v}_n - y_k, \frac{\alpha}{n} + \frac{\alpha}{n}\right) \geq$

$N\left(\mathrm{v}_1, \mathrm{v}_2, \cdots, \mathrm{v} - \mathrm{v}_k, \frac{\alpha}{n}\right) * N\left(\mathrm{v}_1, \mathrm{v}_2, \cdots, \mathrm{v}_k - y_k, \frac{\alpha}{n}\right) \geq \left(1 - \frac{\alpha}{n}\right) * \left(1 - \frac{\alpha}{n}\right) > 1 - \alpha.$

Thus $\mathrm{v}_1, \mathrm{v}_2, \ldots, \mathrm{v}_n \in y_k + B(0,0,\cdots,0,\alpha,\alpha)$.

. Let $\alpha \in (0,1)$, $t > 0$. Let $\beta = \min\{\alpha, t\}$. By assumption,(2) $\Rightarrow$ (3)

$(\exists)\{\mathrm{v}_1, \mathrm{v}_2, \ldots, \mathrm{v}_n\} \subset G: G \subset \bigcup_{i=1}^{n} \left(\mathrm{v}_i + B(0,0,\cdots,0,\beta,\beta)\right) \subset \bigcup_{i=1}^{n} \left(e_i + B(0,0,\cdots,0,\alpha,t)\right)$

. Clearly, it is.(3) $\Rightarrow$ (4)

. Let $\alpha \in (0,1)$. For $t = \alpha$, by assumption, $(\exists)\{\mathrm{v}_1, \mathrm{v}_2, \ldots, \mathrm{v}_n\} \subset X$ such that (4) $\Rightarrow$ (1)

.$G \subset \bigcup_{i=1}^{n} (\mathrm{v}_i + B(0,0,\cdots,0,\alpha,\alpha))$

Proposition (3.23):

Let $(X, N, *)$ be a RF-n-NS. If G, H are FTB subsets of X, then $G + H$ and $G \cup H$ are FTB.

Proof.

Let $\alpha \in (0,1)$ and $t > 0$. Then , $\exists \alpha_1, \alpha_2 \in (0,1)$ such that

$(1 - \alpha_1) * (1 - \alpha_2) > 1 - \alpha$. Let $t_1 = t_2 = \frac{t}{2}$. Then

$B(0,0,\cdots,0,\alpha_1,t_1) + B(0,0,\cdots,0,\alpha_2,t_2) \subset B(0,0,\cdots,0,\alpha,t)$.

Indeed, if $\mathrm{v}_1, \mathrm{v}_2, \ldots, \mathrm{v}_n \in B(0,0,\cdots,0,\alpha_1,t_1)$ and $y_1, y_2, \cdots, y_n \in B(0,0,\cdots,0,\alpha_2,t_2)$, then

and $N\left(y_1, y_2, \cdots, y_n, \frac{t}{2}\right) > 1 - \alpha_2$. Thus$N\left(\mathrm{v}_1, \mathrm{v}_2, \ldots, \mathrm{v}_n, \frac{t}{2}\right) > 1 - \alpha_1$

$$N(\mathrm{v}_1, \mathrm{v}_2, \ldots, \mathrm{v}_n + y_1, y_2, \cdots, y_n, t) \geq N\left(\mathrm{v}_1, \mathrm{v}_2, \ldots, \mathrm{v}_n, \frac{t}{2}\right) * N\left(y_1, y_2, \cdots, y_n, \frac{t}{2}\right) \geq (1 - \alpha_1) * (1 - \alpha_2) > 1 - \alpha$$

Hence $\mathrm{v}_1, \mathrm{v}_2, \ldots, \mathrm{v}_n + y_1, y_2, \cdots, y_n \in B(0,0,\cdots,0,\alpha,t)$. If G, H are FT-bounded, then , $\exists\{v_1, v_2, \dots, v_n\} \subset G$ and $\{y_1, y_2, \cdots, y_m\} \subset H$ such that

$G \subset \bigcup_{i=1}^{n}\left(v_i + B(0,0,\cdots,0,\alpha_1,t_1)\right)$ and $H \subset \bigcup_{k=1}^{m}\left(y_k + B(0,0,\cdots,0,\alpha_2,t_2)\right)$. Therefore

$G + H \subset \bigcup_{i=1}^{n} \bigcup_{k=1}^{m} \left(v_i + y_k + B(0,0,\cdots,0,\alpha_1,t_1) + B(0,0,\cdots,0,\alpha_2,t_2)\right) \subset \bigcup_{i=1}^{n} \bigcup_{k=1}^{m} (v_i + y_k + B(0,0,\cdots,0,\alpha,t))$

Hence $G + H$ is FT-bounded.

Now, let $\alpha \in (0,1)$. As G, H are FT-bounded, $\exists\{v_1, v_2, \dots, v_n, y_1, y_2, \cdots, y_m\} \subset X$ such that

and $G \subset \bigcup_{i=1}^{n}(v_i + B(0,0,\cdots,0,\alpha,\alpha))$

. Thus$H \subset \bigcup_{k=1}^{m}(y_k + B(0,0,\cdots,0,\alpha,\alpha))$

$G \cup H \subset \left(\bigcup_{i=1}^{n} (v_i + B(0,0,\cdots,0,\alpha,\alpha))\right) \cup \left(\bigcup_{k=1}^{m} (y_k + B(0,0,\cdots,0,\alpha,\alpha))\right)$

Hence $G \cup H$ is FT-bounded.

Proposition (3.24):

Let $\Omega, \eta \in (0,1)$ such that $\eta < \Omega$. Then $\overline{B(0,0,\cdots,0,\eta,\eta)} \subset B(0,0,\cdots,0,\Omega,\Omega)$.

Proof.

If $v_1, v_2, \dots, v_n \in \overline{B(0,0,\cdots,0,\eta,\eta)}$, then there exists $\{e_n\} \subset B(0,0,\cdots,0,\eta,\eta)$ such that $v_n \to v$, namely $N(v_1, v_2, \dots, v_n, \eta) > 1 - \eta$ and $\lim_{n\to\infty} N(v_1, v_2, \dots, v_n - v, t) = 1, (\forall)t > 0$.

Let $\gamma \in (0,1) \Rightarrow (1-\eta) * (1-\gamma) > 1 - \Omega$. The existence of γ results by the

continuity of the mapping $g: [0,1] \to [0,1], g(y) = (1-\eta) * y$. Indeed, for

, as $g(0) = 0, g(1) = 1 - \eta$ and $0 < 1 - \Omega_1 < 1 - \eta, \exists\gamma \in (0,1)$ $\Omega_1 \in (0,1)$: $\eta < \alpha_1 < \Omega$

such that $g(1-\gamma) = 1 - \Omega_1$, namely $(1-\eta) * (1-\gamma) = 1 - \Omega_1 > 1 - \Omega$.

Finally, for $t > 0 \Rightarrow \Omega = \eta + t$, as $\lim_{n\to\infty} N(v_1, v_2, \dots, v_n - v, t) = 1$, $\exists n_0 \in \mathbb{N}^*$

such that $N(v_1, v_2, \cdots, v - v_n, t) > 1 - \gamma, (\forall)n \geq n_0$. Thus

$$N(v_1, v_2, \dots, v_n, \Omega) = N(v_1, v_2, \cdots, v - v_n + v_n, \eta + t) \geq N(v_1, v_2, \cdots, v - v_n, t) * N(v_1, v_2, \dots, v_n, \eta)$$
$$\geq (1-\gamma) * (1-\eta) > 1 - \Omega$$

Hence $v_1, v_2, \dots, v_n \in B(0,0, \cdots, 0, \Omega, \Omega)$.

Proposition (3.25):

Let $(X, N, *)$ be a RF-n-NS. If G is FTB, then $\bar{G}$ is FTB.

Proof.

Let $\alpha \in (0,1)$. Let $\beta < \alpha$. As G is FTB, $(\exists)\{v_1, v_2, \dots, v_n\} \subset X$ such that

. Thus, by Proposition (3.24), it follows$G \subset \bigcup_{i=1}^{n} (v_i + B(0,0, \cdots, 0, \beta, \beta))$

$\bar{G} \subset \bigcup_{i=1}^{n} \left(v_i + \overline{B(0,0, \cdots, 0, \beta, \beta)}\right) \subset \bigcup_{i=1}^{n} \left(v_i + B(0,0, \cdots, 0, \alpha, \alpha)\right)$

Hence $\bar{G}$ is FT-bounded.

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