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A new approximation solution for the Fractional Order Biological Population Model

Ghazwan Ali Abdul Hussein[1], Djelloul Ziane[2]

[1]Department of Mathematics, College of Education for Pure Science, University of Thi-Qar, Nasiriyah, Iraq.
[2]Laboratory of Mathematics and its applications (LAMAP), University of Oran1, Oran, Algeria.

[*] Corresponding email: ghazwanali84@gmail.com

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Abstract

The homotopy permutation approach is used in this article to solve the fractional-order biological population model (FOBPM). The fractional derivative is defined using the Atangana-Baleanu operator (ABO). The suggested technique provides of several solutions for FOBPM. The HPM technique is regarded as one of the finest analytical processes for solving fractional-order, nonlinear PDEs, notably the FOBPM, Since it requires fewer computations and has a greater rate of convergence than other analytical approaches.

Keywords: Fractional-order biological population model; homotopy permutation method; Atangana-Baleanu operator.

1. Introduction

Fractional calculus, three centuries older than regular calculus, is rarely employed in research and engineering. This subject is unusual because fractional derivatives and integrals are not restricted to a certain location or amount. This accounts for both historical and non-local distributed consequences. In other words, perhaps this topic better portrays the truth of nature! Making this subject accessible to scientists and engineers improves our grasp of fundamental nature. It's plausible that nature understands fractional calculus, making interacting with it easier .

Population biological models, which are important for understanding ecological dynamics, epidemiology, and evolutionary biology, also benefit from the use of fractional calculus. These models use fractional derivatives to describe the troublesome interplay between memory results, nonlocal interactions, and spatial heterogeneity, offering a more realistic depiction of population dynamics. Furthermore, the Atangana-Baleanu operator, which is distinguished by its non-unique and self-similar behavior, provides a unique path to improving the accuracy of biological population models, notably in contingencies connected to long-term dispersion and recall outcome[8-12]. There are many analytical and numerical methods to solve FPDEs[13-63]. The paper is organized as follows: Section 2 discusses key principles in FC. Section 3 describes the algorithm for the approach used. Section 4 provides various instances that demonstrate the efficacy of the suggested strategy. Finally, Section 5 includes the closing remarks. Now we will introduce the definition of Atangana-Baleanu fractional derivative and Atangana-Baleanu fractional integral.

2. Preliminaries

Definition 1. [10]: The Atangana-Baleanu fractional derivative (ABFD) of order γ is defined as follows:

$$ {}_{a}^{AB}D_{\boldsymbol{t}}^{\gamma}\varphi(t) = \frac{1}{1-\gamma}\int_{a}^{\boldsymbol{t}} E_{\gamma}\left(\frac{-\gamma(t-\xi)^{\gamma}}{\gamma-1}\right)\varphi'(\xi)d\xi $$

Definition 2. [10]: The Atangana-Baleanu fractional integral (ABFI) of order γ is defined as follows:

$$ {}_{a}^{AB}{}_{\boldsymbol{t}}^{\gamma}\varphi(t) = (1-\gamma)\varphi(t) + \frac{\gamma}{\Gamma(\gamma)}\int_{a}^{\boldsymbol{t}} (t-\xi)^{\gamma-1}\varphi(\xi)d\xi $$

A few properties of (2) are defined as follows

1- ${}_{a}^{AB}I_{\boldsymbol{t}}\,{}_{a}^{AB}D_{\boldsymbol{t}}^{\gamma}\varphi(t) = \varphi(t) - \varphi(0).$

2- ${}_{a}^{AB}I_{\boldsymbol{t}}^{\gamma}c = \frac{c}{M(\gamma)}\left(1-\gamma+\frac{\boldsymbol{t}^{\gamma}}{\Gamma(\gamma)}\right).$

3- ${}_{a}^{AB}I_{\boldsymbol{t}}^{\gamma}t^{k} = \frac{\boldsymbol{t}^{k}}{M(\gamma)}\left(1-\gamma+\frac{\gamma\Gamma(k+1)\boldsymbol{t}^{\gamma}}{\Gamma(\gamma+k+1)}\right).$

3. Analysis of Homotopy Permutation Method

Let us consider a generalized non-linear biological population equation of the form:

$$ {}^{AB}D_{t}^{\gamma}\varphi(x,y,t) = (\varphi^{2})_{xx} + (\varphi^{2})_{yy} + h\varphi^{a}(1-r\varphi^{b}),\ 0<\gamma\leqslant 1 \tag{3.1} $$

where ${}^{AB}D_t^{\gamma}$ Atangana-Baleanu operator, h, a, b, r are real numbers. The initial condition is $\varphi(x, y, 0) = \lambda$.

By taking the fractional integral of Atangana-Baleanu ${}^{AB}I_t^{\gamma}$, we get

$$\varphi(x,y,t) = \varphi(x,y,0) + (1-\gamma)\left((\varphi^2)_{xx} + (\varphi^2)_{yy} + h\varphi^a(1 - r\varphi^b)\right) + \frac{\gamma}{\Gamma(\gamma)}\int_0^t (t-\xi)^{\gamma-1}\left((\varphi^2)_{xx} + (\varphi^2)_{yy} + h\varphi^a(1 - r\varphi^b)\right) d\xi \quad (3.2)$$

Putting initial condition, we get

$$\varphi(x,y,t) = \lambda + (1-\gamma)\left((\varphi^2)_{xx} + (\varphi^2)_{yy} + h\varphi^a(1 - r\varphi^b)\right) + \frac{\gamma}{\Gamma(\gamma)}\int_0^t (t-\xi)^{\gamma-1}\left((\varphi^2)_{xx} + (\varphi^2)_{yy} + h\varphi^a(1 - r\varphi^b)\right) d\xi \quad (3.3)$$

By applying homotopy permutation method,

$$\varphi(x,y,t) = \sum_{n=0}^{\infty} \wp^n \varphi_n, \quad \mathcal{N}[\varphi(x,y,t)] = \sum_{n=0}^{\infty} \wp^n \mathcal{H}_n(\varphi) \quad (3.4)$$

Where

$$\mathcal{H}_n(\varphi) = \frac{1}{n!}\frac{\partial^n}{\partial \wp^n}[\mathcal{N}(\sum_{i=0}^{n} \wp^i \varphi_i(x,t))]_{\wp=0} \quad n = 0,1,2, \ldots. \quad (3.5)$$

Substituting (3.4) into (3.3) gives us the result that,

$$\sum_{n=0}^{\infty} \wp^n \varphi_n(x,y,t) = \lambda + (1-\gamma)\begin{pmatrix} \left(\sum_{i=0}^{\infty} p^n A_n\right)_{xx} + \left(\sum_{i=0}^{\infty} p^n A_n\right)_{yy} \\ +h\left(\sum_{i=0}^{\infty} p^n \varphi_i\right)^a \left(1 - r\left(\sum_{i=0}^{\infty} p^n \varphi_i\right)^b\right) \end{pmatrix} + \frac{\gamma}{\Gamma(\gamma)}\int_0^t (t-\xi)^{\gamma-1}\begin{pmatrix} \left(\sum_{i=0}^{\infty} p^n A_n\right)_{xx} + \left(\sum_{i=0}^{\infty} p^n A_n\right)_{yy} \\ +h\left(\sum_{i=0}^{\infty} p^n \varphi_i\right)^a \left(1 - r\left(\sum_{i=0}^{\infty} p^n \varphi_i\right)^b\right) \end{pmatrix} d\xi \quad (3.6)$$

By comparing both sides of (3.6), we get

$p^0: \varphi_0 = \lambda,$

$$p^1: \varphi_1 = (1-\gamma)\begin{pmatrix} (A_0)_{xx} + (A_0)_{yy} \\ +h\varphi_0^a\left(1 - r\varphi_0^b\right) \end{pmatrix}$$

$$+\frac{\gamma}{\Gamma(\gamma)}\int_0^t (t-\xi)^{\gamma-1}\begin{pmatrix} (A_0)_{xx} + (A_0)_{yy} \\ +h\varphi_0^a\left(1 - r\varphi_0^b\right) \end{pmatrix} d\xi$$

$$p^{n+1}: \varphi_{n+1} = (1-\gamma)\begin{pmatrix} (A_n)_{xx} + (A_n)_{yy} \\ +h\varphi_n^a(1 - r\varphi_n^b) \end{pmatrix}$$

$$+\frac{\gamma}{\Gamma(\gamma)}\int_0^t (t-\xi)^{\gamma-1}\begin{pmatrix} (A_n)_{xx} + (A_n)_{yy} \\ +h\varphi_n^a(1 - r\varphi_n^b) \end{pmatrix} d\xi \tag{3.7}$$

Using the parameter $\wp$, we expand the solution in the following form

$$\varphi(x,t) = \sum_{n=0}^{\infty} \wp^n \varphi_n(x,t). \tag{3.8}$$

Setting p = 1 results in the solution of (3.8):

$$\varphi(x,t) = \lim_{\wp\to 1}\sum_{n=0}^{\infty} \wp^n \varphi_n(x,t) = \sum_{n=0}^{\infty} \varphi_n(x,t) \tag{3.9}$$

4. Applications

This section will provide approximation solutions to the three previously mentioned equations with the fractional derivative Atangana-Baleanu operator.

Example 4.1. Consider the BPM at $a = b = h = 1$.

$${}^{AB}D_t^{\gamma}\varphi(x,y,t) = \frac{\partial^2\varphi^2}{\partial x^2} + \frac{\partial^2\varphi^2}{\partial y^2} + \varphi - r\varphi, \tag{4.1}$$

where $0 < \gamma \leq 1$ and subject to the initial condition

$$\varphi(x,y,0) = e^{\frac{\sqrt{2r}(x+y)}{4}}. \tag{4.2}$$

By taking the fractional integral of Atangana-Baleanu ${}^{AB}I_t^{\gamma}$, we get

$$\varphi(x,y,t) = \varphi(x,y,0) + (1-\gamma)\left((\varphi^2)_{xx} + (\varphi^2)_{yy} + \varphi - r\varphi\right)$$
$$+\frac{\gamma}{\Gamma(\gamma)}\int_0^t (t-\xi)^{\gamma-1}\left((\varphi^2)_{xx} + (\varphi^2)_{yy} + \varphi - r\varphi\right)d\xi \tag{4.3}$$

Putting initial condition, we get

$$\varphi(x,y,t) = e^{\frac{\sqrt{2r}(x+y)}{4}} + (1-\gamma)\big((\varphi^2)_{xx} + (\varphi^2)_{yy} + \varphi - r\varphi\big)$$
$$+\frac{\gamma}{\Gamma(\gamma)}\int_0^t (t-\xi)^{\gamma-1}\big((\varphi^2)_{xx} + (\varphi^2)_{yy} + \varphi - r\varphi\big)d\xi \qquad (4.4)$$

Now, Applying HPM on (4.4), we get

$$\sum_{n=0}^{\infty} p^n\varphi_n(x,y,t) = e^{\frac{\sqrt{2r}(x+y)}{4}} + (1-\gamma)\begin{pmatrix}\left(\sum_{i=0}^{\infty} p^n A_n\right)_{xx} + \left(\sum_{i=0}^{\infty} p^n A_n\right)_{yy} \\ +\sum_{i=0}^{\infty} p^n\varphi_n - r\sum_{i=0}^{\infty} p^n\varphi_n\end{pmatrix}$$
$$+\frac{\gamma}{\Gamma(\gamma)}\int_0^t (\boldsymbol{t}-\xi)^{\gamma-1}\begin{pmatrix}\left(\sum_{i=0}^{\infty} p^n A_n\right)_{xx} + \left(\sum_{i=0}^{\infty} p^n A_n\right)_{yy} \\ +\sum_{i=0}^{\infty} p^n\varphi_n - r\sum_{i=0}^{\infty} p^n\varphi_n\end{pmatrix}d\xi \qquad (4.5)$$

By comparing both sides of the above equation, we get

$p^0 : \varphi_0 = e^{\frac{\sqrt{2r}(x+y)}{4}}$,

$$p^1 : \varphi_1 = (1-\gamma)\begin{pmatrix}(A_0)_{xx} + (A_0)_{yy} \\ +\varphi_0 - r\varphi_0\end{pmatrix}$$
$$+\frac{\gamma}{\Gamma(\gamma)}\int_0^t (\boldsymbol{t}-\xi)^{\gamma-1}\begin{pmatrix}(A_0)_{xx} + (A_0)_{yy} \\ +\varphi_0 - r\varphi_0\end{pmatrix}d\xi$$

$$p^{n+1} : \varphi_{n+1} = (1-\gamma)\begin{pmatrix}(A_n)_{xx} + (A_n)_{yy} \\ +\varphi_n - r\varphi_n\end{pmatrix}$$
$$+\frac{\gamma}{\Gamma(\gamma)}\int_0^t (t-\xi)^{\gamma-1}\begin{pmatrix}(A_n)_{xx} + (A_n)_{yy} \\ +\varphi_n - r\varphi_n\end{pmatrix}d\xi \qquad (4.6)$$

Then, we have:

$$\varphi_0 = e^{\frac{\sqrt{2}\sqrt{r}(x+y)}{4}}$$
$$A_0 = \left(e^{\frac{\sqrt{2}\sqrt{r}(x+y)}{4}}\right)^2$$
$$\varphi_1 = e^{\frac{\sqrt{2}\sqrt{r}(x+y)}{4}}\left(1-\gamma+\frac{\gamma t^\gamma}{\Gamma(1+\gamma)}\right)$$
$$A_1 = 2\left(e^{\frac{\sqrt{2}\sqrt{r}(x+y)}{4}}\right)^2\left(1-\gamma+\frac{\gamma t^\gamma}{\Gamma(1+\gamma)}\right)$$

$$\varphi_2 = e^{\frac{\sqrt{2}\sqrt{r}(x+y)}{4}}\left(1 - 2\gamma + \gamma^2 + \frac{\gamma^2 t^{2\gamma}}{\Gamma(1+2\gamma)} - \frac{2t^\gamma \gamma(-1+\gamma)}{\Gamma(1+\gamma)}\right)$$

$$A_2 = 2\left(e^{\frac{\sqrt{2}\sqrt{r}(x+y)}{4}}\right)^2\left(1 - 2\gamma + \gamma^2 + \frac{\gamma^2 t^{2\gamma}}{\Gamma(1+2\gamma)} - \frac{2t^\gamma \gamma(-1+\gamma)}{\Gamma(1+\gamma)}\right) + \left(e^{\frac{\sqrt{2}\sqrt{r}(x+y)}{4}}\right)^2\left(1 - \gamma + \frac{\gamma t^\gamma}{\Gamma(1+\gamma)}\right)^2$$

$$\varphi_3 = e^{\frac{\sqrt{2}\sqrt{r}(x+y)}{4}}\left(\begin{array}{c} 1 + 3\gamma^2 - 3\gamma - \gamma^3 + \frac{\gamma^3 t^{3\gamma}}{\Gamma(1+3\gamma)} \\ + \frac{3t^\gamma \gamma(-1+\gamma)^2}{\Gamma(1+\gamma)} - \frac{3t^{2\gamma}\gamma^2(-1+\gamma)}{\Gamma(1+2\gamma)} \end{array}\right).$$

⋮

The approximate solution is given by

$$\varphi = \varphi_0 + \varphi_1 + \varphi_2 + \cdots$$

The approximate solution at $\gamma = 1$ is

$$\varphi = e^{\frac{1}{2}\sqrt{\frac{r}{2}}(x+y)}\left(1 + t + \frac{t^2}{2!} + \cdots\right).$$

The exact solution is

$$\varphi = e^{\frac{1}{2}\sqrt{\frac{r}{2}}(x+y)} e^t.$$

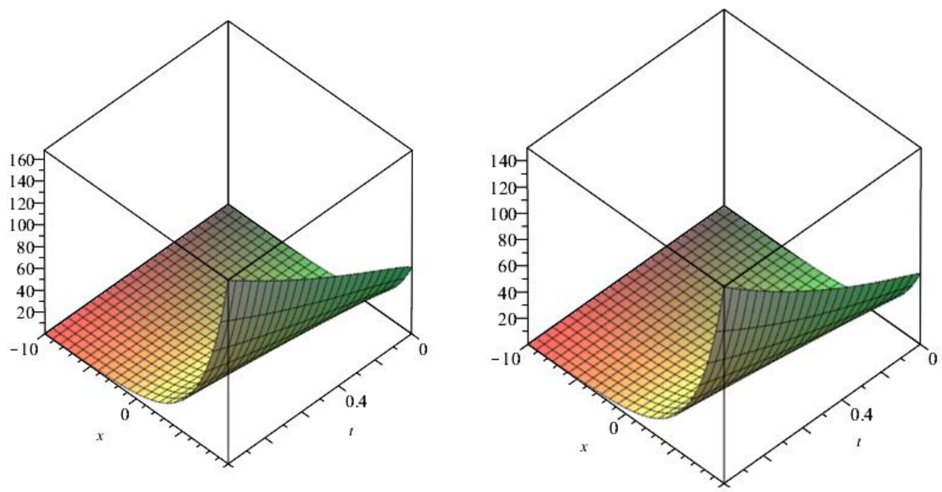

Figure 4.1. ***The surface graph of the approximate solution $\varphi(x, y, t)$ of (4.1) when r=1 and $\gamma = 0.8, 0.9$.***

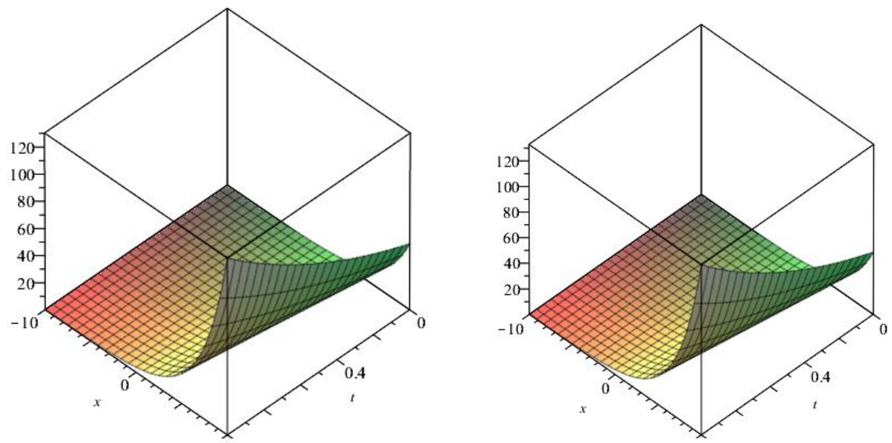

Figure 4.2. ***The surface graph of the approximate solution at r=1 and $\gamma = 1$ and exact solution $\varphi(x, y, t)$ of (4.1)***

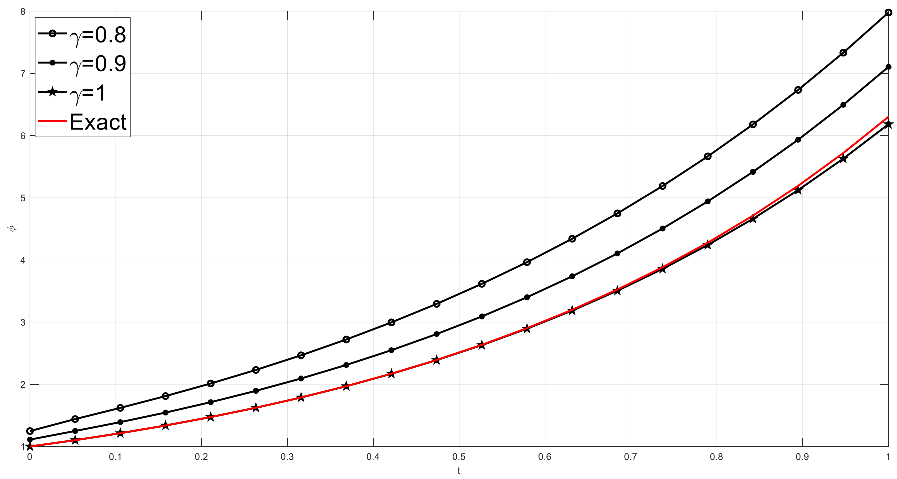

Figure 4.3. ***The 2D graph of the approximate and exact solutions at r=1 and difference values of γ of $\varphi(x, y, t)$ of (4.1).***

Table 1 This table includes the values of the differential equation Eq (4.1) at different γ values.

x, y, t	$\varphi_{\gamma=0.5}$	$\varphi_{\gamma=0.6}$	$\varphi_{\gamma=0.7}$	$\varphi_{\gamma=0.8}$	$\varphi_{\gamma=0.9}$	$\varphi_{\gamma=1}$	φ_E	$\lvert\varphi_{\gamma=1} - \varphi_E\rvert$
0.0526	2.3675	1.9936	1.6859	1.4408	1.2496	1.1017	1.1017	0.0000
0.1053	2.6739	2.2628	1.9102	1.6212	1.3920	1.2138	1.2138	0.0000
0.1579	2.9647	2.5272	2.1384	1.8107	1.5456	1.3373	1.3373	0.0000
0.2105	3.2558	2.7976	2.3772	2.0133	1.7127	1.4733	1.4734	0.0001
0.2632	3.5535	3.0790	2.6300	2.2312	1.8949	1.6230	1.6233	0.0003
0.3158	3.8615	3.3740	2.8991	2.4665	2.0939	1.7878	1.7884	0.0006
0.3684	4.1822	3.6849	3.1864	2.7207	2.3112	1.9692	1.9704	0.0011
0.4211	4.5176	4.0136	3.4936	2.9958	2.5486	2.1688	2.1708	0.0020
0.4737	4.8694	4.3617	3.8226	3.2935	2.8080	2.3883	2.3917	0.0034
0.5263	5.2390	4.7309	4.1750	3.6156	3.0911	2.6295	2.6350	0.0056
0.5789	5.6280	5.1227	4.5526	3.9640	3.4001	2.8945	2.9031	0.0086
0.6316	6.0377	5.5387	4.9573	4.3410	3.7372	3.1856	3.1985	0.0129
0.6842	6.4696	5.9807	5.3910	4.7485	4.1048	3.5051	3.5239	0.0187
0.7368	6.9252	6.4504	5.8557	5.1891	4.5053	3.8557	3.8824	0.0266
0.7895	7.4058	6.9496	6.3537	5.6651	4.9415	4.2402	4.2774	0.0371

0.8421	7.9131	7.4801	6.8871	6.1792	5.4164	4.6617	4.7125	0.0508
0.8947	8.4487	8.0440	7.4583	6.7342	5.9331	5.1235	5.1920	0.0685
0.9474	9.0141	8.6432	8.0700	7.3331	6.4951	5.6291	5.7202	0.0911
1.0000	9.6112	9.2800	8.7247	7.9792	7.1058	6.1825	6.3022	0.1197

Example 4.2. Consider the BPM at $a = b = 1$ and $r = 0$

$$ {}^{AB}D_t^{\gamma}\varphi(x,y,t) = \frac{\partial^2\varphi^2}{\partial x^2} + \frac{\partial^2\varphi^2}{\partial y^2} + \mathrm{h}\varphi, \tag{4.8}$$

where $0 < \gamma \leq 1$ and subject to the initial condition

$$\varphi(x,y,0) = \sqrt{xy}\,. \tag{4.9}$$

By taking the fractional integral of Atangana-Baleanu ${}^{AB}I_t^{\gamma}$, we get

$$\begin{aligned}\varphi(x,y,t) &= \varphi(x,y,0) + (1-\gamma)\big((\varphi^2)_{xx} + (\varphi^2)_{yy} + \mathrm{h}\varphi\big)\\ &+ \frac{\gamma}{\Gamma(\gamma)}\int_0^t (t-\xi)^{\gamma-1}\big((\varphi^2)_{xx} + (\varphi^2)_{yy} + \mathrm{h}\varphi\big)d\xi\end{aligned} \tag{4.10}$$

Putting initial condition, we get

$$\begin{aligned}\varphi(x,y,t) &= \sqrt{xy} + (1-\gamma)\big((\varphi^2)_{xx} + (\varphi^2)_{yy} + \mathrm{h}\varphi\big)\\ &+ \frac{\gamma}{\Gamma(\gamma)}\int_0^t (t-\xi)^{\gamma-1}\big((\varphi^2)_{xx} + (\varphi^2)_{yy} + \mathrm{h}\varphi\big)d\xi\end{aligned} \tag{4.11}$$

Now, Applying HPM on (4.10), we get

$$\begin{aligned}\sum_{n=0}^{\infty} p^n\varphi_n(x,y,t) &= \sqrt{xy} + (1-\gamma)\begin{pmatrix}\left(\sum_{i=0}^{\infty} p^n A_n\right)_{xx} + \left(\sum_{i=0}^{\infty} p^n A_n\right)_{yy}\\ +\mathrm{h}\sum_{i=0}^{\infty} p^n\varphi_n\end{pmatrix}\\ &+ \frac{\gamma}{\Gamma(\gamma)}\int_0^t (t-\xi)^{\gamma-1}\begin{pmatrix}\left(\sum_{i=0}^{\infty} p^n A_n\right)_{xx} + \left(\sum_{i=0}^{\infty} p^n A_n\right)_{yy}\\ +\mathrm{h}\sum_{i=0}^{\infty} p^n\varphi_n\end{pmatrix}d\xi\end{aligned} \tag{4.12}$$

By comparing both sides of the above equation, we get

$p^0: \varphi_0 = \sqrt{xy}\,,$

$p^1: \varphi_1 = (1-\gamma)\begin{pmatrix}(A_0)_{xx} + (A_0)_{yy}\\ +\mathrm{h}\varphi_0\end{pmatrix}$

$$+\frac{\gamma}{\Gamma(\gamma)}\int_0^{t}(t-\xi)^{\gamma-1}\begin{pmatrix}(A_0)_{xx}+(A_0)_{yy}\\ +h\varphi_0\end{pmatrix}d\xi$$

$$p^{n+1}:\varphi_{n+1}=(1-\gamma)\begin{pmatrix}(A_n)_{xx}+(A_n)_{yy}\\ +h\varphi_n\end{pmatrix}$$

$$+\frac{\gamma}{\Gamma(\gamma)}\int_0^{t}(t-\xi)^{\gamma-1}\begin{pmatrix}(A_n)_{xx}+(A_n)_{yy}\\ +h\varphi_n\end{pmatrix}d\xi \tag{4.13}$$

Then, we have:

$$\varphi_0=\sqrt{xy},$$

$$A_0=xy,$$

$$\varphi_1=h\sqrt{xy}\left(1-\gamma+\frac{\gamma t^{\gamma}}{\Gamma(1+\gamma)}\right),$$

$$A_1=2xyh\left(1-\gamma+\frac{\gamma t^{\gamma}}{\Gamma(1+\gamma)}\right),$$

$$\varphi_2=h^2\sqrt{xy}\left(1-2\gamma+\gamma^2+\frac{\gamma^2t^{2\gamma}}{\Gamma(1+2\gamma)}-\frac{2t^{\gamma}\gamma(-1+\gamma)}{\Gamma(1+\gamma)}\right),$$

$$A_2=2xyh^2\begin{pmatrix}1-2\gamma+\gamma^2+\dfrac{\gamma^2t^{2\gamma}}{\Gamma(1+2\gamma)}\\ -\dfrac{2t^{\gamma}\gamma(-1+\gamma)}{\Gamma(1+\gamma)}\end{pmatrix}+h^2xy\left(1-\gamma+\frac{\gamma t^{\gamma}}{\Gamma(1+\gamma)}\right)^2$$

$$\varphi_3=h^3\sqrt{xy}\begin{pmatrix}1+3\gamma^2-3\gamma-\gamma^3+\dfrac{\gamma^3t^{3\gamma}}{\Gamma(1+3\gamma)}\\ +\dfrac{3t^{\gamma}\gamma(-1+\gamma)^2}{\Gamma(1+\gamma)}-\dfrac{3t^{2\gamma}\gamma^2(-1+\gamma)}{\Gamma(1+2\gamma)}\end{pmatrix}.$$

$$\vdots$$

The approximate solution is given by

$$\varphi=\varphi_0+\varphi_1+\varphi_2+\cdots \tag{4.14}$$

The approximate solution at $\alpha=1$ is

$$\varphi=\sqrt{xy}+\frac{h\sqrt{xy}t}{\Gamma(2)}+\frac{h^2\sqrt{xy}t^2}{\Gamma(3)}+\frac{h^3\sqrt{xy}t^3}{\Gamma(4)} \tag{4.15}$$

The exact solution is given by

$$\varphi=\sqrt{xy}\,e^{ht}$$

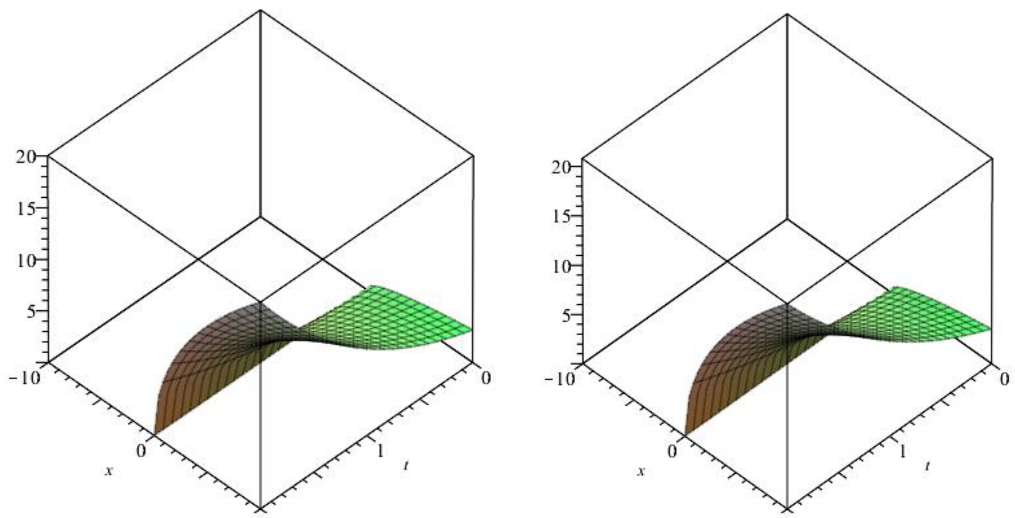

Figure 4.5. ***The surface graph of the approximate solution $\varphi(x, y, t)$ of (4.8) when $h = 1$ and $\gamma = 0.8, 0.9$.***

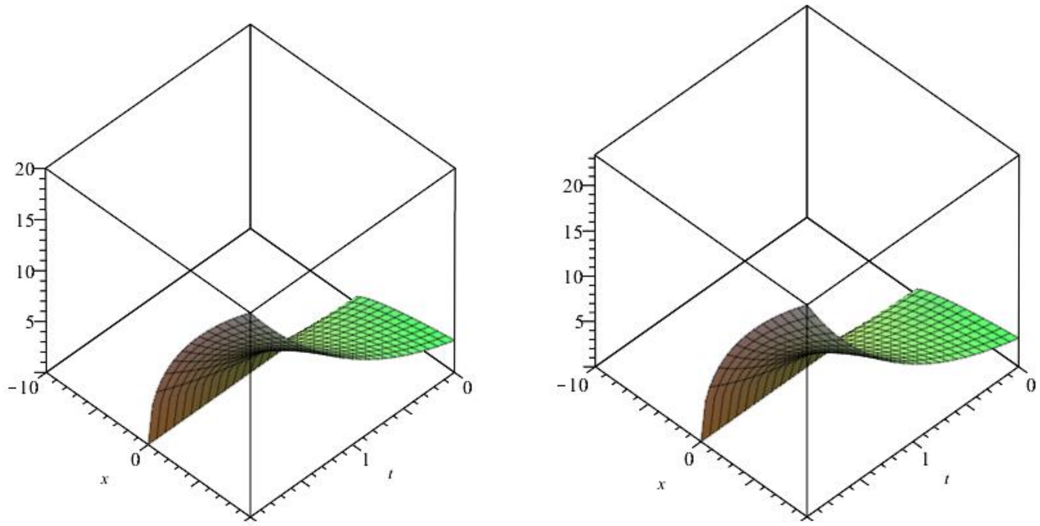

Figure 4.6. ***The surface graph of the approximate solution at h=1 and $\gamma = 1$ and the exact solution $\varphi(x, y, t)$ of (4.8).***

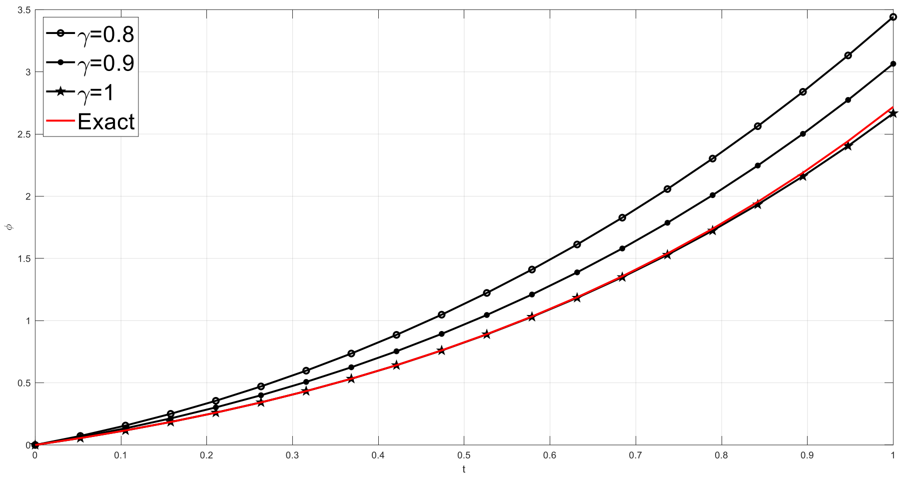

Figure 4.7. ***The 2D graph of the approximate and exact solutions at h=1 and difference values of γ of $\varphi(x, y, t)$ of (4.8).***

Table 2 This table includes the values of the differential equation Eq (4.8) at different γ values.

x, y, t	$\varphi_{\gamma=0.5}$	$\varphi_{\gamma=0.6}$	$\varphi_{\gamma=0.7}$	$\varphi_{\gamma=0.8}$	$\varphi_{\gamma=0.9}$	$\varphi_{\gamma=1}$	φ_E	$\lvert\varphi_{\gamma=1} - \varphi_E\rvert$
0.0526	0.1192	0.1004	0.0849	0.0725	0.0629	0.0555	0.0555	0.0000
0.1053	0.2576	0.2180	0.1840	0.1562	0.1341	0.1169	0.1169	0.0000
0.1579	0.4099	0.3494	0.2957	0.2504	0.2137	0.1849	0.1849	0.0000
0.2105	0.5742	0.4934	0.4193	0.3551	0.3021	0.2598	0.2599	0.0000
0.2632	0.7495	0.6494	0.5547	0.4706	0.3997	0.3423	0.3424	0.0001
0.3158	0.9350	0.8170	0.7020	0.5972	0.5070	0.4329	0.4331	0.0001
0.3684	1.1303	0.9959	0.8612	0.7353	0.6246	0.5322	0.5325	0.0003
0.4211	1.3350	1.1861	1.0324	0.8853	0.7531	0.6409	0.6415	0.0006
0.4737	1.5487	1.3873	1.2158	1.0475	0.8931	0.7596	0.7607	0.0011
0.5263	1.7713	1.5995	1.4115	1.2224	1.0451	0.8890	0.8909	0.0019
0.5789	2.0025	1.8227	1.6198	1.4104	1.2098	1.0299	1.0329	0.0031
0.6316	2.2421	2.0568	1.8409	1.6120	1.3878	1.1830	1.1877	0.0048
0.6842	2.4900	2.3018	2.0749	1.8276	1.5798	1.3490	1.3562	0.0072
0.7368	2.7461	2.5578	2.3220	2.0577	1.7865	1.5289	1.5395	0.0106
0.7895	3.0102	2.8248	2.5826	2.3027	2.0086	1.7235	1.7386	0.0151
0.8421	3.2823	3.1027	2.8567	2.5631	2.2467	1.9336	1.9547	0.0211
0.8947	3.5623	3.3916	3.1447	2.8394	2.5016	2.1602	2.1891	0.0289
0.9474	3.8501	3.6916	3.4468	3.1321	2.7741	2.4043	2.4432	0.0389
1.0000	4.1456	4.0027	3.7632	3.4416	3.0649	2.6667	2.7183	0.0516

Example 4.3. Consider the BPM at $a = b = h = 1$ and $r = 0$.

$$^{AB}D_t^{\gamma}\varphi(x, y, t) = \frac{\partial^2\varphi^2}{\partial x^2} + \frac{\partial^2\varphi^2}{\partial y^2} + \varphi, \tag{4.16}$$

where $0 < \gamma \leq 1$ and subject to the initial condition

$$\varphi(x, y, 0) = \sqrt{\sin(x)\sinh(y)}. \tag{4.17}$$

By taking the fractional integral of Atangana-Baleanu $^{AB}I_t^{\gamma}$, we get

$$\varphi(x,y,t) = \varphi(x,y,0) + (1-\gamma)\big((\varphi^2)_{xx} + (\varphi^2)_{yy} + \varphi\big) + \frac{\gamma}{\Gamma(\gamma)}\int_0^t (t-\xi)^{\gamma-1}\big((\varphi^2)_{xx} + (\varphi^2)_{yy} + \varphi\big)d\xi \quad (4.18)$$

Putting initial condition, we get

$$\varphi(x,y,t) = \sqrt{\sin(x)\sinh(y)} + (1-\gamma)\big((\varphi^2)_{xx} + (\varphi^2)_{yy} + \varphi\big) + \frac{\gamma}{\Gamma(\gamma)}\int_0^t (t-\xi)^{\gamma-1}\big((\varphi^2)_{xx} + (\varphi^2)_{yy} + \varphi\big)d\xi \quad (4.19)$$

Now, Applying HPM on (4.18), we get

$$\sum_{n=0}^{\infty} p^n\varphi_n(x,y,t) = \sqrt{\sin(x)\sinh(y)} + (1-\gamma)\left(\left(\sum_{i=0}^{\infty} p^n A_n\right)_{xx} + \left(\sum_{i=0}^{\infty} p^n A_n\right)_{yy} + \sum_{i=0}^{\infty} p^n\varphi_n\right) + \frac{\gamma}{\Gamma(\gamma)}\int_0^t (t-\xi)^{\gamma-1}\left(\left(\sum_{i=0}^{\infty} p^n A_n\right)_{xx} + \left(\sum_{i=0}^{\infty} p^n A_n\right)_{yy} + \sum_{i=0}^{\infty} p^n\varphi_n\right)d\xi \quad (4.20)$$

By comparing both sides of the above equation, we get

$p^0: \varphi_0 = \sqrt{\sin(x)\sinh(y)}$,

$$p^1: \varphi_1 = (1-\gamma)\big((A_0)_{xx} + (A_0)_{yy} + \varphi_0\big) + \frac{\gamma}{\Gamma(\gamma)}\int_0^t (t-\xi)^{\gamma-1}\big((A_0)_{xx} + (A_0)_{yy} + \varphi_0\big)d\xi$$

$$p^{n+1}: \varphi_{n+1} = (1-\gamma)\big((A_n)_{xx} + (A_n)_{yy} + \varphi_n\big) + \frac{\gamma}{\Gamma(\gamma)}\int_0^t (t-\xi)^{\gamma-1}\big((A_n)_{xx} + (A_n)_{yy} + \varphi_n\big)d\xi \quad (4.21)$$

Then, we have:

$$\varphi_0 = \sqrt{\sin(x)\sinh(y)}$$

$$A_0 = \sqrt{\sin(x)\sinh(y)}$$

$$\varphi_1 = \sqrt{\sin(x)\sinh(y)}\left(1-\gamma+\frac{\gamma t^{\gamma}}{\Gamma(1+\gamma)}\right)$$

$$A_1 = 2\sin(x)\sinh(y)\left(1-\gamma+\frac{\gamma t^{\gamma}}{\Gamma(1+\gamma)}\right)$$

$$\varphi_2 = \sqrt{\sin(x)\sinh(y)}\left(1-2\gamma+\gamma^2+\frac{\gamma^2 t^{2\gamma}}{\Gamma(1+2\gamma)}-\frac{2t^{\gamma}\gamma(-1+\gamma)}{\Gamma(1+\gamma)}\right)$$

$$A_2 = \sin(x)\sinh(y)\left(1-2\gamma+\gamma^2+\frac{\gamma^2 t^{2\gamma}}{\Gamma(1+2\gamma)}-\frac{2t^{\gamma}\gamma(-1+\gamma)}{\Gamma(1+\gamma)}\right) + \sin(x)\sinh(y)\left(1-\gamma+\frac{\gamma t^{\gamma}}{\Gamma(1+\gamma)}\right)^2$$

$$\varphi_3 = \sqrt{\sin(x)\sinh(y)}\left(\begin{array}{c} 1+3\gamma^2-3\gamma-\gamma^3+\frac{\gamma^3 t^{3\gamma}}{\Gamma(1+3\gamma)} \\ +\frac{3t^{\gamma}\gamma(-1+\gamma)^2}{\Gamma(1+\gamma)}-\frac{3t^{2\gamma}\gamma^2(-1+\gamma)}{\Gamma(1+2\gamma)} \end{array}\right)$$

The approximate solution is given by,

$$\varphi = \varphi_0+\varphi_1+\varphi_2+\cdots \tag{4.22}$$

The approximate solution at $\gamma = 1$ is

$$\varphi = \sqrt{\sin(x)\sinh(y)}+\frac{\sqrt{\sin(x)\sinh(y)}\,t}{\Gamma(2)}+\frac{\sqrt{\sin(x)\sinh(y)}\,t^2}{\Gamma(3)} + \frac{\sqrt{\sin(x)\sinh(y)}\,t^3}{\Gamma(4)}+\cdots \tag{4.23}$$

The exact solution is

$$\varphi = \sqrt{\sin(x)\sinh(y)}\,e^t$$

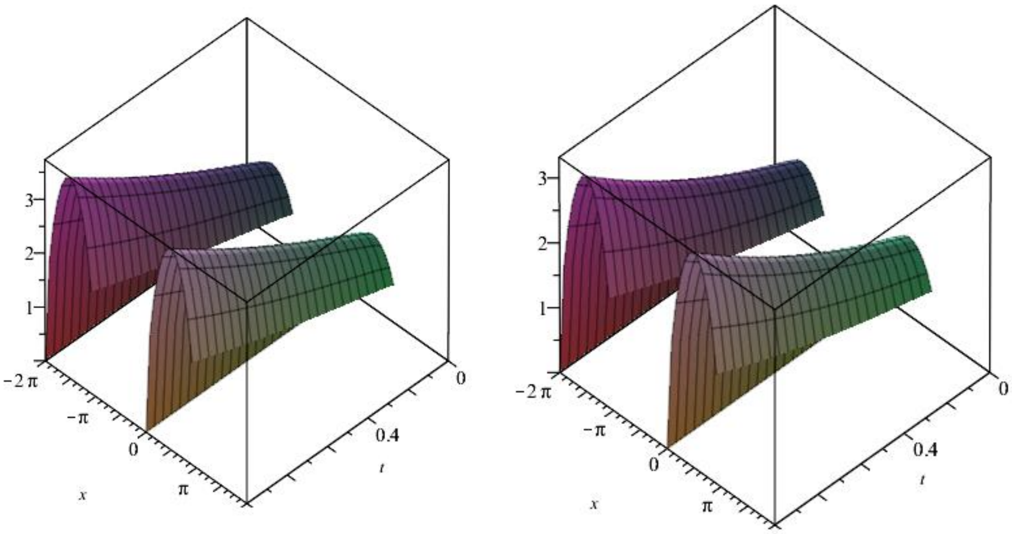

Figure 4.8 The surface graph of the approximate solution $\varphi(x,y,t)$ of Eq. (4.16) when $\gamma = 0.8, 0.9$.

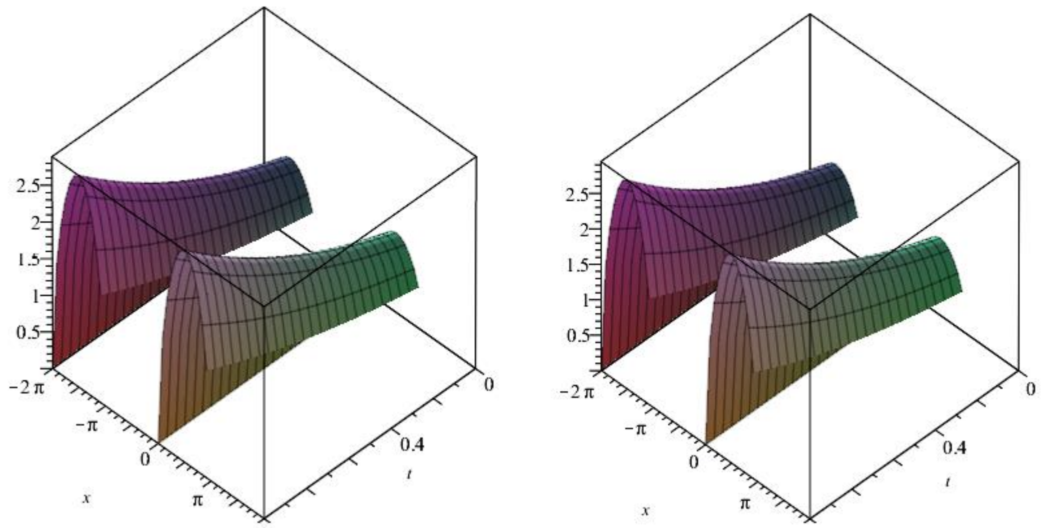

Figure 4.9. ***The surface graph of the approximate solution at $\gamma = 1$ and exact solution $\varphi(x, y, t)$ of (4.16).***

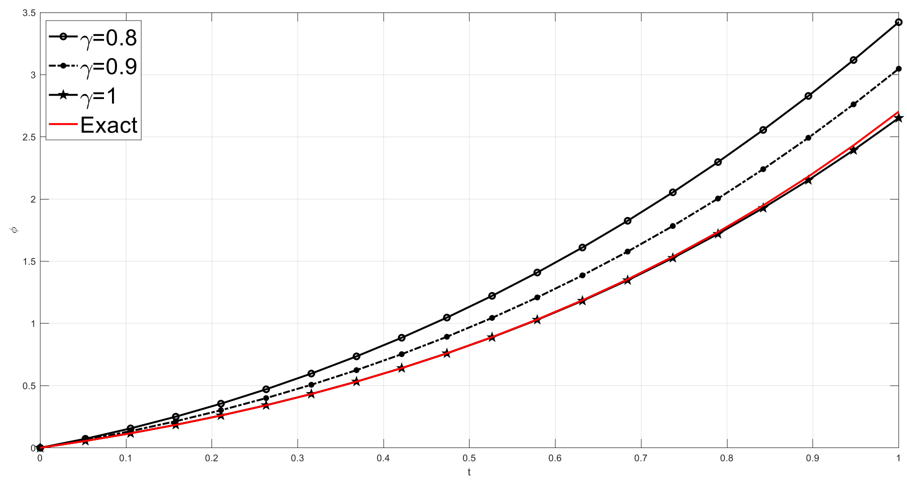

Figure 4.10. ***The 2D graph of the approximate and exact solutions at difference values of γ of $\varphi(x, y, t)$ of Eq. (4.16).***

Table 3 This table includes the values of the differential equation Eq(4.16) at different γ values.

x, y, t	$\varphi_{\gamma=0.5}$	$\varphi_{\gamma=0.6}$	$\varphi_{\gamma=0.7}$	$\varphi_{\gamma=0.8}$	$\varphi_{\gamma=0.9}$	$\varphi_{\gamma=1}$	φ_E	$\lvert\varphi_{\gamma=1} - \varphi_E\rvert$
0.0526	*0.1192*	*0.1004*	*0.0849*	*0.0725*	*0.0629*	*0.0555*	*0.0555*	*0.0000*
0.1053	*0.2576*	*0.2180*	*0.1840*	*0.1562*	*0.1341*	*0.1169*	*0.1169*	*0.0000*
0.1579	*0.4099*	*0.3494*	*0.2957*	*0.2504*	*0.2137*	*0.1849*	*0.1849*	*0.0000*
0.2105	*0.5742*	*0.4934*	*0.4193*	*0.3551*	*0.3021*	*0.2598*	*0.2599*	*0.0000*
0.2632	*0.7495*	*0.6494*	*0.5547*	*0.4706*	*0.3997*	*0.3423*	*0.3424*	*0.0001*
0.3158	*0.9350*	*0.8169*	*0.7020*	*0.5972*	*0.5070*	*0.4329*	*0.4330*	*0.0001*
0.3684	*1.1302*	*0.9958*	*0.8611*	*0.7353*	*0.6246*	*0.5322*	*0.5325*	*0.0003*
0.4211	*1.3348*	*1.1858*	*1.0322*	*0.8851*	*0.7530*	*0.6408*	*0.6414*	*0.0006*

0.4737	*1.5483*	*1.3869*	*1.2155*	*1.0472*	*0.8928*	*0.7594*	*0.7605*	*0.0011*
0.5263	*1.7705*	*1.5988*	*1.4109*	*1.2219*	*1.0446*	*0.8886*	*0.8905*	*0.0019*
0.5789	*2.0012*	*1.8215*	*1.6188*	*1.4095*	*1.2090*	*1.0292*	*1.0323*	*0.0031*
0.6316	*2.2401*	*2.0550*	*1.8392*	*1.6106*	*1.3866*	*1.1819*	*1.1867*	*0.0048*
0.6842	*2.4870*	*2.2990*	*2.0723*	*1.8254*	*1.5779*	*1.3474*	*1.3546*	*0.0072*
0.7368	*2.7416*	*2.5536*	*2.3182*	*2.0543*	*1.7836*	*1.5264*	*1.5370*	*0.0105*
0.7895	*3.0037*	*2.8187*	*2.5770*	*2.2977*	*2.0042*	*1.7198*	*1.7349*	*0.0151*
0.8421	*3.2731*	*3.0940*	*2.8487*	*2.5559*	*2.2404*	*1.9282*	*1.9493*	*0.0210*
0.8947	*3.5496*	*3.3796*	*3.1335*	*2.8293*	*2.4927*	*2.1525*	*2.1813*	*0.0288*
0.9474	*3.8328*	*3.6751*	*3.4314*	*3.1180*	*2.7617*	*2.3935*	*2.4322*	*0.0387*
1.0000	*4.1225*	*3.9804*	*3.7422*	*3.4224*	*3.0478*	*2.6518*	*2.7032*	*0.0513*

Conclusion

This thesis discussed analyzes the application of HPM to develop approximate analytical solutions for FOBPM. The investigation gives information on the accuracy and confidence of the HPM in solving fractional differential equations by carefully comparing these approximate answers to precise solutions, backed by 2D and 3D graphs made using the Maple platform.

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