Journal of Education for Pure Science- University of Thi-Qar
Vol.14, No. 3 (2024)
Website: jceps.utq.edu.iq Email:jceps@eps.utq.edu.iq

DOI: https://doi.org/10.32792/jeps.v14i3.454

The Approximate Solutions of 2D- Fractional Burger's Equations

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Received 24/05/2024, Accepted 28 /07/2024, Published 01 /09/2024

Abstract

This paper investigates the use of the fractional variational iteration method (FVIM) to obtain approximate analytical solutions to two dimensional Burger's Equations with the Atangana-Baleanu fractional operator (ABFO). This study provides insight on the fractional variational iteration method's accuracy and reliability while approximating fractional differential equation solutions.

Keywords: Burger's Equations; Atangana-Baleanu fractional operator; fractional variational iteration method.

1. Introduction

Fractional calculus is one of the branches of mathematics that studies the properties of integrals and derivatives of non-integer orders which are called fractional integrals and derivatives, fractional calculus is three centuries older than standard calculus, it is not widely used in science and engineering (fluid flow, electrical network, singnal processing and optics, etc) . This subject is unique since fractional derivatives and integrals are not limited to a certain location or amount. This takes into account both historical and non-local dispersed impacts. In other words, perhaps this subject better represents the truth of nature! Making this subject more accessible to the scientific and engineering community enhances our understanding of basic nature. It's possible that nature knows fractional calculus, making communicating with it more efficient. [1-4].

There are many analytical and numerical methods to solve FPDEs [5-23]. The J.M. Burger's equation, also known as Burger's equation, isignificant and commonly used non-linear PDE. The first person to introduced it was Bateman and later corrected by Burger's. This equation is employed to simulate numerous physical phenomena, for example (acoustics, diffraction, heat conduction water waves, shock waves and turbulence issues, among others) .This research focuses on the approximate solutions of the two-dimensional Burger's equations[25-28]. The fractional variational iteration method (FVIM) is utilized for solving Burger's equations. Approximate results obtained using the FVIM approach are then compared with the exact results of Burgers's equation, the suggested strategy's convergence is illustrated through graphs of approximate solutions.

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2- Preliminaries

Definition 1. The ABFD of order α is given as follows [24]:

$$ {}^{AB}_{\alpha}D^{\alpha}_{t}\, g(t) = \frac{M(\alpha)}{1-\alpha}\int_{\alpha}^{t} E_{\alpha}\left(\frac{-\alpha(t-x)^{\alpha}}{\alpha-1}\right) g'(x)dx \tag{2.1} $$

where $0<\alpha<1$ and $M(0)=M(1)=1$.

Definition 2. The ABFI of order α defined as follows [24]:

$$ {}^{AB}_{a}I^{\alpha}_{t}\, g(t) = \frac{1-\alpha}{M(\alpha)} g(t) + \frac{\alpha}{M(\alpha)} \frac{1}{\Gamma(\alpha)} \int_{a}^{t} (t-x)^{\alpha-1} g(x)\, dx. \qquad (2.2)$$

The properties of ABFI is defined as follows**:**

1. ${}^{AB}_{a}I^{\alpha}_{t}\; {}^{AB}_{a}D^{\alpha}_{t}\; g(t) = g(t) - g(0).$
2. ${}^{AB}_{a}I^{\alpha}_{t}\; t^{k} = \frac{t^{k}}{M(\alpha)} \left(1-\alpha + \frac{\alpha\,\Gamma(k+1)\; t^{\alpha}}{\Gamma(\alpha+k+1)}\right).$

3- Analysis of FVIM

Consider the following: partial differential equation with fractions

$$ {}^{AB}D^{\omega}_{t} g(x,t) + R\, g(x,t) + N g(x,t) = h(x,t), \qquad 0< \omega \leq 1 \qquad (3.1)$$

with the initial condition

$$ g(x,0) = F(x)$$

where ${}^{AB}D^{\omega}_{t}$ is ABFD , R is the linear differential operator ,N denotes the nonlinear term,and h(x,t) denotes the source term .

The correctional functional for (3.1) is approximately expressed as follows :

$$ g_{n+1}(x,t) = g_n(x,t) + {}^{AB}_{a}I^{\omega}_{t} \left[\lambda(\mu)\left(AB_{D}{}^{\omega}_{\mu} g_n(x,\mu) + R\tilde{g}(x,\mu) + N\breve{g}(x,t) - h(x,\mu)\right)\right], \qquad (3.2)$$

where $\lambda(\mu)$ is general Lagrange's multiplier. $\widetilde{g}$ is considered as restricted variations. The relevant adjustment in place and making it functioning and noticing $\delta\tilde{g} = 0$, we obtain

$$ \delta g_{n+1}(x,t) = \delta g_n(x,t) + {}^{AB}I^{\omega}_{t}\left[\delta\lambda(\mu)\left({}^{AB}D^{\omega}_{\mu} g_n(x,\mu)\right)\right].$$

Or

$$ \delta g_{n+1}(x,t) = \delta g_n(x,t) + \lambda(\mu)\delta g_n(x,t) - {}^{AB}I^{\omega}_{t}\left[\delta\lambda(\mu)\left({}^{AB}D^{\omega}_{\mu} g_n(x,\mu)\right)\right].$$

This produces the stationary conditions

$$ \lambda'(\mu) = 0$$

$$ 1 + \lambda(\mu) = 0$$

There for, we identified $\lambda = -1$,

$$ g_{n+1}(x,t) = g_n(x,t) + {}^{AB}_{a}I^{\omega}_{t} \left[\lambda(\mu)\left({}^{AB}D^{\omega}_{t} g_n(x,\mu) + R\, g_n(x,\mu) + N g_n(x,t) - h(x,\mu)\right)\right]. \qquad (3.3)$$

Finally,we have

$$g(\mathrm{x},\mathrm{t}) = \lim_{n\to\infty} g_n$$

4- Applications of FVIM

Example 1. Consider the 2D-fractional Burger equation

$$ {}^{AB}D_t^{\omega} + gg_x + gg_y = g_{xx} + g_{yy}, \quad (4.1)$$

with $g(x,y,0) = x + y$

From (3.3) and) (4.1),

$$g_{n+1}(x,y,t) = g_n(x,y,t) - \int_0^t \left(\frac{\partial^{\omega} g_n(x,y,\tau)}{\partial\tau^{\alpha}} + g_n(g_n)_x + g_n(g_n)_y - (g_n)_{xx} - (g_n)_{yy}\right) d(\tau)^{\omega}.$$

Then,

$$g_0(x,y,t) = x + y$$

$$g_1(x,y,t) = (x+y) - \int_0^t \left(\frac{\partial^{\omega} g_0(x,y,\tau)}{\partial\tau^{\alpha}} + g_0(g_0)_x + g_0(g_0)_y - (g_0)_{xx} - (g_0)_{yy}\right)$$

$$= (x+y)(1 - 2\left[1 - \omega + \frac{\omega t^{\omega}}{\Gamma(\omega+1)}\right])$$

$$g_2(x,y,t) = (x+y)\left(1 - 2\left[1 - \omega + \frac{1}{\Gamma(\omega+1)}\right]\right)$$

$$- \int_0^t \left(\frac{\partial^{\omega} g_n(x,y,\tau)}{\partial\tau^{\alpha}} + g_1(g_1)_x + g_1(g_1)_y - (g_1)_{xx} - (g_1)_{yy}\right)$$

$$= (x+y) - 2(x+y)\left(1 - \alpha + \frac{\alpha t^{\alpha}}{\Gamma(\alpha+1)}\right) - 8(x+y)\left[(1-\alpha)^2 + 2(1-\alpha)\frac{\alpha t^{\alpha}}{\Gamma(\alpha+1)} + \frac{\alpha^2 t^{2\alpha}}{\Gamma(2\alpha+1)}\right] - 8(x+y)[(1-\alpha)^2 + 2(1-\alpha)\frac{\alpha t^{\alpha}}{\Gamma(\alpha+1)} + \frac{\alpha^2 t^{2\alpha}}{\Gamma(2\alpha+1)}](1 - \alpha + \frac{\alpha t^{\alpha}}{\Gamma(\alpha+1)})$$

$$\vdots$$

$$g_n(x,y,t) = (x+y)\left[1 - 2\left(1-\alpha+\frac{\alpha t^{\alpha}}{\Gamma(\alpha+1)}\right) + 8\left[(1-\alpha)^2 + 2(1-\alpha)\frac{\alpha t^{\alpha}}{\Gamma(\alpha+1)} + \frac{\alpha^2 t^{2\alpha}}{\Gamma(2\alpha+1)}\right] - 8\left[(1-\alpha)^2 + 2(1-\alpha)\frac{\alpha t^{\alpha}}{\Gamma(\alpha+1)} + \frac{\alpha^2 t^{2\alpha}}{\Gamma(2\alpha+1)}\right]\left(1-\alpha+\frac{\alpha t^{\alpha}}{\Gamma(\alpha+1)}\right) + \cdots\right].$$

When $\alpha = 1$,

$$g(x,y,t) = (x+y)[\,1 - 2t + 4(t)^2 + \cdots\,] = \frac{x+y}{1+2t}$$

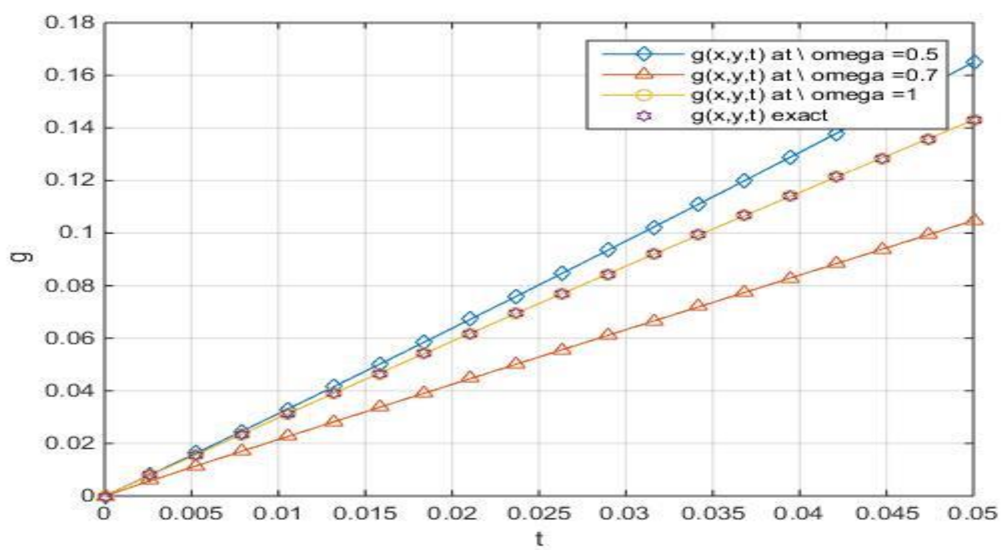

Figure 1. Plot of the exact and approximate solutions $g(x,y,t)$ for different values of ω with fixed values x=1

Example 2. Consider the 2D-fractional coupled Burger's equations

$$^{AB}D_t^{\omega} g(x,y,t) + g g_x + w g_y = g_{xx} + g_{yy} \qquad 0 < \omega \le 1$$

$$^{AB}D_t^{\beta} w(x,y,t) + g w_x + w w_y = w_{xx} + w_{yy}\,, \qquad 0 < \beta \le 1 \qquad (4.2)$$

with

$$g(x,y,0) = x + y$$

$$w(x,y,0) = x - y$$

In view of (3.3) and (4.2),

$$g_{n+1}(x,y,t) = g_n(x,y,t) - \int_0^t \left(\frac{\partial^\omega g_n(x,y,\tau)}{\partial \tau^\alpha} + g_n (g_n)_x + w_n (g_n)_y - (g_n)_{xx} - (g_n)_{yy} \right) d(\tau)^\omega$$

$$w_{n+1}(x,y,t) = w_n(x,y,t) - \int_0^t \left(\frac{\partial^\beta z_n(x,y,\tau)}{\partial \tau^\beta} + g_n (w_n)_x + w_n (w_n)_y - (w_n)_{xx} - (w_n)_{yy} \right) d(\tau)^\beta.$$

Then,

$$g_0(x,y,t) = x + y$$

$$w_0(x,y,t) = x - y$$

$$g_1(x,y,t) = (x+y) - \int_0^t \left(\frac{\partial^\omega g_0(x,y,\tau)}{\partial \tau^\alpha} + g_0 (g_0)_x + w_0 (g_0)_y - (g_0)_{xx} - (g_0)_{yy} \right) d(\tau)^\omega$$

$$= (x+y) - 2x(1 - \omega + \frac{\omega t^\omega}{\Gamma(\omega+1)})$$

$$w_1(x,y,t) = (x-y) - \int_0^t \left(\frac{\partial^\beta w_0(x,y,\tau)}{\partial \tau^\beta} + g_0 (w_0)_x + w_0 (w_0)_y - (w_0)_{xx} - (w_0)_{yy} \right) d(\tau)^\beta$$

$$= (x-y) - 2y(1 - \beta + \frac{\beta t^\beta}{\Gamma(\beta+1)})$$

$$g_2(x,y,t) = (x+y) - 2x\left(1 - \omega + \frac{\omega t^\omega}{\Gamma(\omega+1)}\right)$$

$$- \int_0^t \left(\frac{\partial^\omega g_1(x,y,\tau)}{\partial \tau^\alpha} + g_1 (g_1)_x + w_1 (g_1)_y - (g_1)_{xx} - (g_1)_{yy} \right) d(\tau)^\omega$$

$$= (x+y) - 2x\left(1 - \omega + \frac{\omega t^\omega}{\Gamma(\omega+1)}\right) - 2x\left(1 - \omega + \frac{\omega t^\omega}{\Gamma(\omega+1)}\right) + 4x\Big[(1-\omega)^2 + 2(1-\omega)\left(1 - \omega + \frac{\omega t^\omega}{\Gamma(\omega+1)}\right) + \frac{\omega^2 t^{2\omega}}{\Gamma(2\omega+1)}\Big] + 2y[(1-\omega)^2 + 2(1-\omega)\left(1 - \omega + \frac{\omega t^\omega}{\Gamma(\omega+1)}\right) + \frac{\omega^2 t^{2\omega}}{\Gamma(2\omega+1)}] + 2y[(1 - \omega - \beta + \beta\omega) + (1-\beta)\frac{t^\omega}{\Gamma(\omega+1)} + (1-\omega)\frac{t^\beta}{\Gamma(\beta+1)} + \frac{\omega\beta t^{\omega+\beta}}{\Gamma(\omega+1)\Gamma(\beta+1)}]$$

$$w_2(x,y,t) = (x-y) - 2y\left(1 - \beta + \frac{\beta t^\beta}{\Gamma(\beta+1)}\right)$$

$$-\int_0^t\left(\frac{\partial^\beta w_1(x,y,\tau)}{\partial\tau^\beta}+g_1(w_1)_x+w_1(w_1)_y-(w_1)_{xx}-(w_1)_{yy}\right)d(\tau)^\beta$$

$$=(x-y)-2y(1-\beta+\frac{\beta t^\beta}{\Gamma(\beta+1)})-2y\left(1-\beta+\frac{\beta t^\beta}{\Gamma(\beta+1)}\right)+4y\left[(1-\beta)^2+\right.$$
$$\left.2(1-\beta)\frac{\beta t^\beta}{\Gamma(\beta+1)}+\frac{\beta^2t^{2\beta}}{\Gamma(2\beta+1)}\right]+2x[(1-\beta)^2+2(1-\beta)\frac{\beta t^\beta}{\Gamma(\beta+1)}+\frac{\beta^2t^{2\beta}}{\Gamma(2\beta+1)}]+$$
$$2x[(1-\omega-\beta+\beta\omega)+(1-\beta)\frac{t^\omega}{\Gamma(\omega+1)}+(1-\omega)\frac{t^\beta}{\Gamma(\beta+1)}+\frac{\omega\beta t^{\omega+\beta}}{\Gamma(\omega+1)\Gamma(\beta+1)}]$$

$$\vdots$$

$$g(x,y,t)=\lim_{n\to\infty}g_n(x,y,t)=x+y-2x\left(1-\omega+\frac{\omega t^\omega}{\Gamma(\omega+1)}\right)+4x\left[(1-\omega)^2+2(1-\right.$$
$$\left.\omega)\frac{\omega t^\omega}{\Gamma(\omega+1)}+\frac{\omega^2t^{2\omega}}{\Gamma(2\omega+1)}\right]+2y[(1-\omega)^2+2(1-\omega)\left(1-\omega+\frac{\omega t^\omega}{\Gamma(\omega+1)}\right)+\frac{\omega^2t^{2\omega}}{\Gamma(2\omega+1)}]+$$
$$2y[(1-\omega-\beta+\beta\omega)+(1-\beta)\frac{t^\omega}{\Gamma(\omega+1)}+(1-\omega)\frac{t^\beta}{\Gamma(\beta+1)}+\frac{\omega\beta t^{\omega+\beta}}{\Gamma(\omega+1)\Gamma(\beta+1)}]+\cdots$$

$$w(x,y,t)=\lim_{n\to\infty}w_n(x,y,t)=(x-y)-2y\left(1-\beta+\frac{\beta t^\beta}{\Gamma(\beta+1)}\right)-4y\left[(1-\beta)^2+\right.$$
$$\left.2(1-\beta)\frac{\beta t^\beta}{\Gamma(\beta+1)}+\frac{\beta^2t^{2\beta}}{\Gamma(2\beta+1)}\right]+2x[(1-\beta)^2+2(1-\beta)\frac{\beta t^\beta}{\Gamma(\beta+1)}+\frac{\beta^2t^{2\beta}}{\Gamma(2\beta+1)}]+2x[(1-\omega-\beta+$$
$$\beta\omega)+(1-\beta)\frac{t^\omega}{\Gamma(\omega+1)}+(1-\omega)\frac{t^\beta}{\Gamma(\beta+1)}+\frac{\omega\beta t^{\omega+\beta}}{\Gamma(\omega+1)\Gamma(\beta+1)}]+\cdots$$

When $\omega=\beta=1$,

$$g(x,y,t)=x+y-2xt+2xt^2+2yt^2+\cdots$$
$$=\frac{x-2xt+y}{1-2t^2}$$

$$w(x,y,t)=x-y-2yt+2xt^2-2yt^2+\cdots$$
$$=\frac{x-2yt-y}{1-2t^2}$$

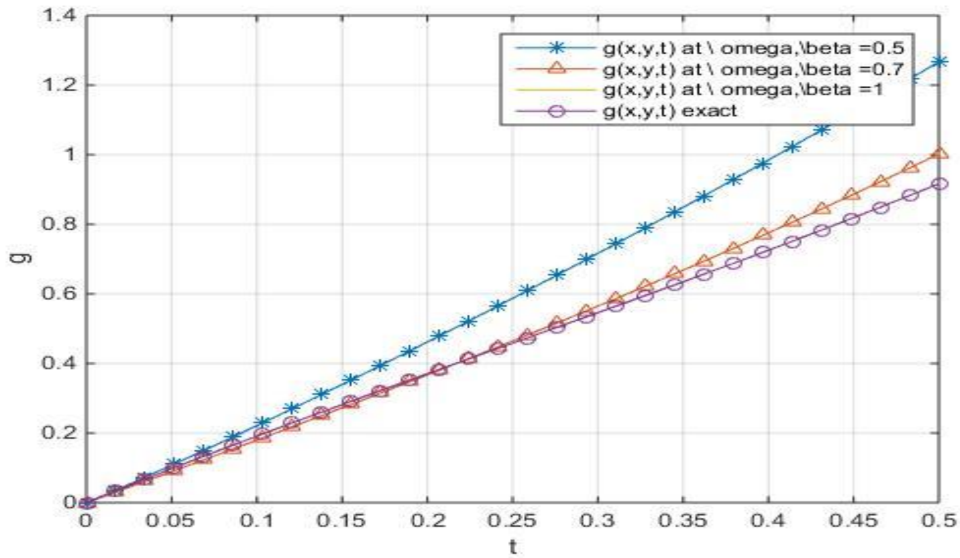

Figure 2. Plot of the exact and approximate solutions $g(x, y, t)$for different values of β, ω with fixed values x=1

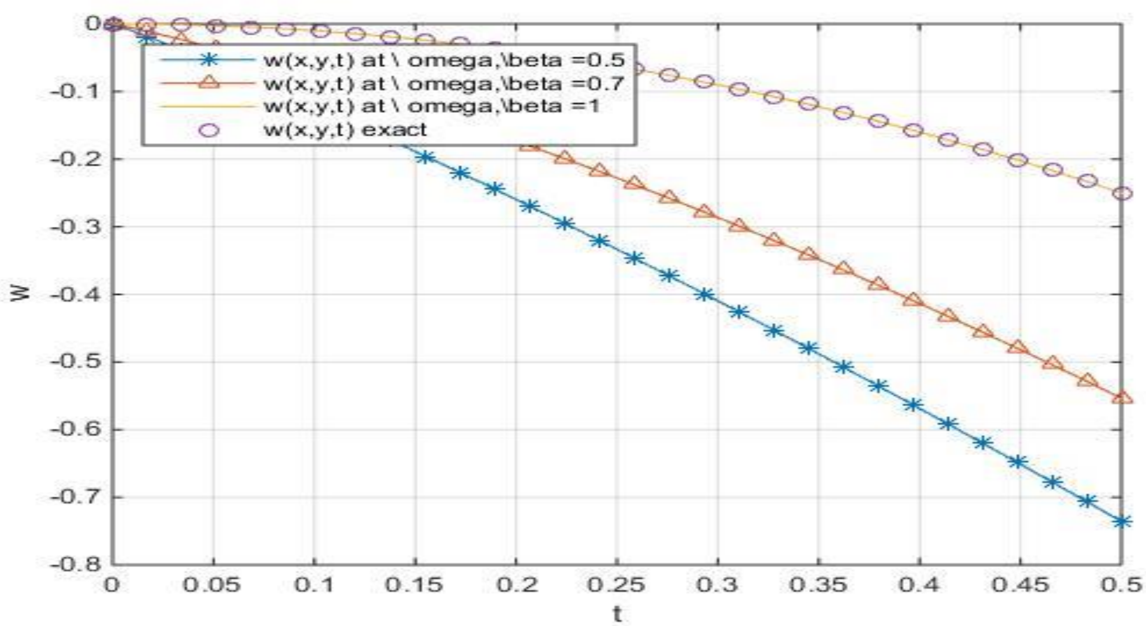

Figure 3. Plot of the exact and approximate solutions $w(x, y, t)$for different values of β, ω with fixed values x=1

5. Conclusions

In the idea of the Atangana-Baleanu fractional operator, the variational iteration method (VIM) was shown to be extremely successful in solving 2D-FPDEs. The solution is provided in a series form by the suggested algorithm, if there is an exact solution, it converges quickly. It is obvious from the findings that the VIM produces solutions that are extremely precise with only a few iterates. As a result of the efficiency and flexibility demonstrated in the provided examples, VIM may be used to additional FPDEs of higher order, according to the findings of this study.

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