Journal of Education for Pure Science- University of Thi-Qar
Vol.15, No.2 (2025)
DOI: https://doi.org/10.32792/jeps.v15i2.476
Website: jceps.utq.edu.iq Email: jceps@eps.utq.edu.iq

Exact Solution of Linear Fractional Telegraph Equation by Generalized Laplace Transform Method

Abubker Ahmed[1, 2,*]

[1]University of Science & Technology, College of Engineering, Sudan,
[2]AlMughtaribeen University, College of Engineering, Department of General Sciences, Sudan.

[*] Corresponding email: abobaker633@gmail.com

Received 5 / 7 /2024, Accepted 5 / 5 /2025, Published 1/ 6 /2025

Abstract: In this work, we investigated the generalized Laplace transform method (GLTM) for obtaining an exact solution to some linear partial differential equations of fractional order. It is shown that under specific conditions for our method, other related methods are deduced. Theorems about the existence and uniqueness of transform is established. Furthermore, we provide examples to demonstrate the method's applicability.

Keywords: Generalized Laplace transform; Integral transforms; Caputo fractional derivative; Partial differential equations.

1-Introduction

In 1880, Oliver Heaviside, developed the telegraph equation, a partial differential equation that appears in different fields of science and engineering [1]. This equation is used to represent electrical signal propagation down a telegraph line as well as reaction diffusion. Many scientists have researched the solution to the telegraph equation. For example, in [1], the authors used the double Laplace transform to obtain accurate solutions to linear and nonlinear space-time fractional telegraph equations. Elzaki et al. [2] solved a telegraph equation with the Elzaki-Laplace transform method. In their paper [3], the author's investigated the double Laplace-Sumudu transform strategy to solving partial differential equations. Furthermore, in their study [4], the authors have successfully used the natural transform and Adomian decomposition method to drive approximate solutions of the telegraph equation. In contrast, integral transformations are an effective technique for solving linear differential equations. It will enable us to convert a differential problem into an algebraic equation, and then, by solving this algebraic equation, we can quickly retrieve the unknown function by using the inverse transform [14]. H. Kim has recently presented a generalized integral transform [5]. Under certain conditions, this integral transform yields to other integral transforms, such as, the Laplace transform, the Aboodh transform [6], the Elzaki transform [7], and the Sumudu transform [8], and so on. Which was successfully applied to solve differential equations with fractional order [9]. Recently, the researchers in [10] investigated the generalized Laplace transform to determine the exact solution of the Burger's and the coupled Burger's equations. Ref. [11] studied a time-fractional Navier-Stokes equation in one and two dimensions utilizing the double Sumudu-generalized Laplace transform decomposition approach. The conventional form of the telegraph equation is given by [1]

$$\frac{\partial^2\psi(x,\tau)}{\partial x^2} = a_1\frac{\partial^2\psi(x,\tau)}{\partial \tau^2} + a_2\frac{\partial\psi(x,\tau)}{\partial \tau} + a_3\psi(x,\tau) + \mathscr{k}(x,\tau), \qquad x,\tau \geq 0. \tag{1.1}$$

where $\psi(x,\tau)$ represent the resistance and the constants a_1, a_2, and a_3 are related to the inductance, capacitance, and conductance of the cable, respectively, $\mathscr{k}(x,\tau)$ is the given function. It is obvious that the telegraph equation is a linear partial differential equation. Under specific scenarios, depending on the electrical parameters of the cable, two linear equations emerge: the heat diffusion equation and the wave equation. For more details, see [13].

The primary goal of this study is to investigate and develop the use of the Laplace-Generalized Laplace transform method (LGLTM) to obtain exact solutions to the linear homogeneous and nonhomogeneous telegraph equation using the Caputo fractional derivative. The fractional-order telegraph equation with the Caputo fractional derivative is given as

$$\frac{\partial^\varrho\psi(x,\tau)}{\partial x^\varrho} = a_1\frac{\partial^\sigma\psi(x,\tau)}{\partial \tau^\sigma} + a_2\frac{\partial^\varsigma\psi(x,\tau)}{\partial \tau^\varsigma} + a_3\psi(x,\tau) + \mathscr{k}(x,\tau), \qquad x,\tau \geq 0. \tag{1.2}$$

with initial and boundary conditions,

$$\begin{cases} ICs, & \psi(x,0) = \psi_1(x), \qquad \psi_t(x,0) = \psi_2(x), \\ BCs, & \psi(0,\tau) = \psi_3(\tau), \qquad \psi_x(0,\tau) = \psi_4(\tau). \end{cases} \tag{1.3}$$

where $1 < \varrho, \sigma \leq 2,\ \ 0 < \varsigma \leq 1$.

The paper is organized as follows: Section 2, introduces some basic principles. Section 3, discusses the existence and uniqueness of the suggested transform. In Section 4, the proposed method is used to solve the fractional telegraph equation. Section 5, contains some examples that demonstrate the applicability of the previous strategy. The final section has a conclusion.

1. Preliminaries

This section discuses some fundamental concepts and properties of the Laplace-generalized Laplace transform, which are useful in solving fractional-order partial differential equations.

Definition 2.1. [5] The generalized Laplace transform of the continuous function $\psi(\tau),\ \tau \geq 0$, is defined as:

$$\mathrm{G}_\alpha[\psi(\tau)] = \Psi_\alpha(\varphi) = \varphi^\alpha \int_0^\infty \psi(\tau)\, exp\left(-\frac{\tau}{\varphi}\right) d\tau\,, \qquad \varphi \in \mathbb{C}, \qquad \alpha \in \mathbb{Z}, \tag{2.1}$$

Specifically, for $\alpha = 0$ and $\vartheta = \frac{1}{\varphi}$, we have the Laplace transform as follows:

$$\mathcal{L}[\psi(\tau)] = \Psi(\varphi) = \int_0^\infty \psi(\tau)\, exp(-\vartheta\tau)\, d\tau. \tag{2.2}$$

Definition 2.2. The double integral of the continuous function $\psi(x,\tau)$, called as the Laplace-Generalized Laplace transform (LGLT) is defined as follows:

$$\mathcal{L}_x \mathrm{G}_\alpha[\psi(x,\tau)] = \Psi_\alpha(\vartheta,\varphi) = \varphi^\alpha \int_0^\infty \int_0^\infty \psi(x,\tau)\, exp\left(-\vartheta x - \frac{\tau}{\varphi}\right) d\tau dx. \quad (2.3)$$

The inverse Laplace-Generalized Laplace transform of the function $\Psi_\alpha(p,\varphi)$, is defined as:

$$\mathcal{L}_x^{-1} \mathrm{G}_\alpha^{-1}[\Psi_\alpha(\vartheta,\varphi)] = \psi(x,\tau) = \frac{-1}{(2\pi)^2} \int_{a-i\infty}^{a+i\infty} \int_{b-i\infty}^{b+i\infty} \varphi^\alpha\, \Psi_\alpha(\vartheta,\varphi)\, exp\left(\vartheta x + \frac{\tau}{\varphi}\right) d\vartheta d\varphi. \quad (2.4)$$

Corollary 1. Based on the previous definition of the Laplace-Generalized Laplace transform, we conclude the following definitions:

- If $\alpha = 0, \varphi = \frac{1}{\varphi}$, we obtain the double Laplace transform [1, 16].
$$\mathcal{L}_x \mathcal{L}_t[\psi(x,\tau)] = \Psi_\alpha(\vartheta,\varphi) = \int_0^\infty \int_0^\infty \psi(x,\tau)\, exp(-\vartheta x - \varphi\tau)\, d\tau dx.$$
- If $\alpha = 1$, we obtain the Laplace-Elzaki transform [2].
$$\mathcal{L}_x \mathrm{E}_t[\psi(x,\tau)] = \Psi_\alpha(\vartheta,\varphi) = \varphi \int_0^\infty \int_0^\infty \psi(x,\tau)\, exp\left(-\vartheta x - \frac{\tau}{\varphi}\right) d\tau dx.$$
- If $\alpha = -1$, we obtain the Laplace-Sumudu transform [3].
$$\mathcal{L}_x \mathrm{S}_t[\psi(x,\tau)] = \Psi_\alpha(\vartheta,\varphi) = \frac{1}{\varphi} \int_0^\infty \int_0^\infty \psi(x,\tau)\, exp\left(-\vartheta x - \frac{\tau}{\varphi}\right) d\tau dx.$$
- If $\alpha = -1, \varphi = \frac{1}{\varphi}$, we obtain the Laplace-Aboodh transform [15].
$$\mathcal{L}_x \mathrm{A}_t[\psi(x,\tau)] = \Psi_\alpha(\vartheta,\varphi) = \frac{1}{\varphi} \int_0^\infty \int_0^\infty \psi(x,\tau)\, exp(-\vartheta x - \varphi\tau)\, d\tau dx.$$
- If $\alpha = 2, \varphi = \frac{1}{\varphi}$, we obtain the Laplace-Mohand transform.
$$\mathcal{L}_x \mathrm{M}_t[\psi(x,\tau)] = \Psi_\alpha(\vartheta,\varphi) = \varphi^2 \int_0^\infty \int_0^\infty \psi(x,\tau)\, exp(-\vartheta x - \varphi\tau)\, d\tau dx.$$
- If $\alpha = -2$, we obtain the Laplace-Sawi transform.
$$\mathcal{L}_x \mathrm{S}_t[\psi(x,\tau)] = \Psi_\alpha(\vartheta,\varphi) = \frac{1}{\varphi^2} \int_0^\infty \int_0^\infty \psi(x,\tau)\, exp\left(-\vartheta x - \frac{\tau}{\varphi}\right) d\tau dx.$$
- If $\alpha = 0$, we obtain the Laplace-Kamal transform.
$$\mathcal{L}_x \mathrm{K}_t[\psi(x,\tau)] = \Psi_\alpha(\vartheta,\varphi) = \int_0^\infty \int_0^\infty \psi(x,\tau)\, exp\left(-\vartheta x - \frac{\tau}{\varphi}\right) d\tau dx.$$

Comment 1. Based on the preceding explanation, we may conclude that the definition of double integral known as Laplace-generalized Laplace transform (LGLT) is more generic than the aforementioned transforms.
We must note certain useful properties of LGLT employed in the paper, which are as follows: see [10, 16].
For that, let $\mathcal{L}_x \mathrm{G}_\alpha[\psi_1(x,\tau)] = \Psi_{1_\alpha}(\vartheta,\varphi)$ and $\mathcal{L}_x \mathrm{G}_\alpha[\psi_2(x,\tau)] = \Psi_{2_\alpha}(\vartheta,\varphi)$

- $\mathcal{L}_x \mathrm{G}_\alpha[\zeta\psi_1(x,\tau) + \xi\psi_2(x,\tau)] = \zeta\Psi_{1_\alpha}(\vartheta,\varphi) + \xi\Psi_{2_\alpha}(\vartheta,\varphi)$,
- $\mathcal{L}_x \mathrm{G}_\alpha[c] = c\frac{\varphi^{\alpha+1}}{\vartheta}$,
- $\mathcal{L}_x \mathrm{G}_\alpha[exp\,(\zeta x + \xi y)] = \frac{\varphi^{\alpha+1}}{(\vartheta-\zeta)(1-\xi\varphi)}$,

- $\mathcal{L}_x \mathrm{G}_\alpha[\ x^\zeta y^\xi] = \frac{\Gamma(\zeta+1)\Gamma(\xi+1)\varphi^{\alpha+\xi+1}}{\vartheta^{\zeta+1}}$,
- $\mathcal{L}_x \mathrm{G}_\alpha[\ \cos(\zeta x + \xi y)] = \frac{\vartheta\varphi^{\alpha+1} - \zeta\xi\varphi^{\alpha+2}}{(\vartheta^2+\zeta^2)(1+\xi^2\varphi^2)}$,
- $\mathcal{L}_x \mathrm{G}_\alpha[\ \sin(\zeta x + \xi y)] = \frac{\zeta\varphi^{\alpha+1} + \xi\vartheta\varphi^{\alpha+2}}{(\vartheta^2+\zeta^2)(1+\xi^2\varphi^2)}$.

where c, ζ, and ξ are constants, and Γ is the Gamma function.

Definition 2.3. [10] The Laplace-generalized Laplace transform for the partial derivatives of order $n \in \mathrm{N}$ is defined as:

$$\begin{cases} \mathcal{L}_x \mathrm{G}_\alpha\left[\dfrac{\partial^n \psi(x,\tau)}{\partial x^n}\right] = \vartheta^n \Psi_\alpha(\vartheta,\varphi) - \displaystyle\sum_{i=1}^{n} \vartheta^{n-i}\, \mathrm{G}_\alpha\left[\dfrac{\partial^{i-1}\psi(0,\tau)}{\partial x^{i-1}}\right], \\ \mathcal{L}_x \mathrm{G}_\alpha\left[\dfrac{\partial^n \psi(x,\tau)}{\partial t^n}\right] = \dfrac{\Psi_\alpha(p,s)}{\varphi^n} - \varphi^\alpha \displaystyle\sum_{i=1}^{n} \dfrac{1}{\varphi^{n-i}} \mathcal{L}_x\left[\dfrac{\partial^{i-1}\psi(x,0)}{\partial \tau^{i-1}}\right], \end{cases} \tag{2.5}$$

Definition 2.4. [1] The Caputo fractional derivative of the function $\psi(x,\tau)$ of order ϱ is defined as:

$$\frac{\partial^\varrho \psi(x,\tau)}{\partial x^\varrho} = \frac{1}{\Gamma(n-\varrho)} \int_0^x (x-\eta)^{n-\varrho-1} \frac{\partial^n \psi(\eta,\tau)}{\partial x^n} d\eta\,, \qquad n-1 < \varrho \leq n, \tag{2.6}$$

Definition 2.5. [1] The Laplace-generalized Laplace transform for the partial fractional Caputo derivatives operator is defined as:

$$\begin{cases} \mathcal{L}_x \mathrm{G}_\alpha\left[\dfrac{\partial^\varrho \psi(x,\tau)}{\partial x^\varrho}\right] = \vartheta^\varrho \Psi_\alpha(p,\varphi) - \displaystyle\sum_{i=1}^{n} \vartheta^{\varrho-i}\, \mathrm{G}_\alpha\left[\dfrac{\partial^{i-1}\psi(0,\tau)}{\partial x^{i-1}}\right], \\ \mathcal{L}_x \mathrm{G}_\alpha\left[\dfrac{\partial^\varrho \psi(x,t)}{\partial \tau^\varrho}\right] = \dfrac{\Psi_\alpha(p,s)}{\varphi^\varrho} - \varphi^\alpha \displaystyle\sum_{i=1}^{n} \dfrac{1}{\varphi^{\varrho-i}} \mathcal{L}_x\left[\dfrac{\partial^{i-1}\psi(x,0)}{\partial \tau^{i-1}}\right], \end{cases} \tag{2.7}$$

where $n-1 < \varrho \leq n,\ n \in \mathrm{N}$.

Definition 2.6. [12] The generalized Mittag-Leffler function with parameters ζ and ξ is defined as:

$$\mathrm{E}_{\zeta,\xi}(\tau) = \sum_{n=0}^{\infty} \frac{\tau^n}{\Gamma(\zeta n + \xi)}, \tag{2.8}$$

Specifically, for $\xi = 1$, the Mittag-Leffler function with parameter ζ is defined as:

$$\mathrm{E}_{\zeta}(\tau) = \sum_{n=0}^{\infty} \frac{\tau^n}{\Gamma(\zeta n + 1)}, \tag{2.9}$$

Therefore, the Laplace and generalized Laplace transforms for the function $\tau^{\xi-1}\mathrm{E}_{\zeta,\xi}(\lambda\tau^\zeta)$ take the form:

$$\mathcal{L}[\tau^{\xi-1}\mathrm{E}_{\zeta,\xi}(\lambda\tau^\zeta)] = \frac{\vartheta^{\zeta-\xi}}{\vartheta^\zeta - \lambda}, \qquad \mathrm{G}_\alpha[\tau^{\xi-1}\mathrm{E}_{\zeta,\xi}(\lambda\tau^\zeta)] = \frac{\varphi^{\alpha+\xi}}{1-\lambda\varphi^\zeta}. \tag{2.10}$$

2. The Existence and Uniqueness of Laplace-Generalized Laplace Transform

In this section, we will look at the existence and uniqueness of the Laplace-generalized Laplace transform.

Definition 3.1. The function $\psi(x,\tau),\ 0 \le\ x,\tau < \infty$, is called an exponential function of orders $\alpha,\beta > 0$, if we get a positive constants $P, X\ and\ T$, which make

$$|\psi(x,\tau)| \le Pe^{\alpha x+\beta\tau}, for\ all\ x > X, \qquad \tau > T, \tag{3.1}$$

when $x,\tau \to \infty$, we have

$$\psi(x,\tau) = Oe^{\alpha x+\beta\tau}. \tag{3.2}$$

In the same line, we get

$$\lim_{x\to\infty,\tau\to\infty} e^{-\vartheta x-\frac{\tau}{\varphi}}|\psi(x,\tau)| = P \lim_{x\to\infty,\tau\to\infty} e^{-(\vartheta-\alpha)x-(\frac{1}{\varphi}-\beta)\tau} = 0, \qquad \vartheta > \alpha, \qquad \varphi > \frac{1}{\beta}. \tag{3.3}$$

Theorem 1. The Laplace-generalized Laplace transform of the continuous function $\psi(x,\tau)$ in every finite interval $(0,X)$ and $(0,T)$ of exponential order $e^{\alpha x+\beta\tau}$ is exists for all ϑ and φ, and provided that $\vartheta > \alpha$ and $\frac{1}{\varphi} > \beta$.

Proof. Considering the definition 3.1 and definition 2, we have

$$|\Psi_\alpha(\vartheta,\varphi)| = \left|\varphi^\alpha \int_0^\infty \int_0^\infty \psi(x,\tau)\, exp\left(-\vartheta x - \frac{\tau}{\varphi}\right) d\tau dx\right| \le P\varphi^\alpha \int_0^\infty \int_0^\infty e^{-(\vartheta-\alpha)x-(\frac{1}{\varphi}-\beta)\tau} d\tau dx$$

$$= \frac{P\varphi^{\alpha+1}}{(\vartheta-\alpha)(1-\beta\varphi)}, \qquad \forall\, \vartheta > \alpha, \qquad \frac{1}{\varphi} > \beta\,. \tag{3.4}$$

Theorem 2. Let $\psi_1(x,\tau)$ and $\psi_2(x,\tau)$ are continuous functions on every finite interval $x,\tau \ge \infty$, and $\mathcal{L}_x \mathrm{G}_\alpha[\psi_1(x,\tau)] = \Psi_{1_\alpha}(\vartheta,\varphi)$ and $\mathcal{L}_x \mathrm{G}_\alpha[\psi_2(x,\tau)] = \Psi_{2_\alpha}(\vartheta,\varphi)$. If $\Psi_{1_\alpha}(\vartheta,\varphi) = \Psi_{2_\alpha}(\vartheta,\varphi)$, then $\psi_1(x,\tau) = \psi_2(x,\tau)$.

Proof. From Equation (2.5) we have

$$\mathcal{L}_x{}^{-1}\mathrm{G}_\alpha{}^{-1}[\Psi_\alpha(\vartheta,\varphi)] = \psi(x,\tau) = \frac{-1}{(2\pi)^2} \int_{a-i\infty}^{a+i\infty} \int_{b-i\infty}^{b+i\infty} \varphi^\alpha\, \Psi_\alpha(\vartheta,\varphi)\, exp\left(\vartheta x + \frac{\tau}{\varphi}\right) d\vartheta d\varphi, \tag{3.5}$$

we deduce that

$$\psi_1(x,\tau) = \frac{-1}{(2\pi)^2} \int_{a-i\infty}^{a+i\infty} \int_{b-i\infty}^{b+i\infty} \varphi^\alpha\, \Psi_{1_\alpha}(\vartheta,\varphi)\, exp\left(\vartheta x + \frac{\tau}{\varphi}\right) d\vartheta d\varphi$$

$$= \frac{-1}{(2\pi)^2} \int_{a-i\infty}^{a+i\infty} \int_{b-i\infty}^{b+i\infty} \varphi^\alpha\, \Psi_{2_\alpha}(\vartheta,\varphi)\, exp\left(\vartheta x + \frac{\tau}{\varphi}\right) d\vartheta d\varphi = \psi_2(x,\tau). \tag{3.6}$$

Thus, the demonstration of uniqueness is completed.

3. LGLTM Applied to the Fractional Telegraph Equation

In this section, we will examine the solution to the partial differential equation of the type (1.2) with initial and boundary conditions (1.3).

Firstly, by applying the Laplace-generalized Laplace transform to both sides of equation (1.2), we obtain

$$\vartheta^{\varrho}\Psi_{\alpha}(\vartheta,\varphi)-\vartheta^{\varrho-1}\Psi_{\alpha}(0,\varphi)-\vartheta^{\varrho-2}\frac{\partial\Psi_{\alpha}(0,\varphi)}{\partial x}$$
$$=a_1\left[\frac{\Psi_{\alpha}(\vartheta,\varphi)}{\varphi^{\sigma}}-\frac{\varphi^{\alpha+1}}{\varphi^{\sigma}}\Psi_{\alpha}(\vartheta,0)-\frac{\varphi^{\alpha+2}}{\varphi^{\sigma}}\frac{\partial\Psi_{\alpha}(\vartheta,0)}{\partial t}\right]+a_2\left[\frac{\Psi_{\alpha}(\vartheta,\varphi)}{\varphi^{\varsigma}}-\frac{\varphi^{\alpha+1}}{\varphi^{\varsigma}}\Psi_{\alpha}(\vartheta,0)\right]$$
$$+a_3\Psi_{\alpha}(\vartheta,\varphi)$$
$$+\mathcal{H}_{\alpha}(\vartheta,\varphi), \qquad (4.1)$$

where $\mathcal{H}_{\alpha}(\vartheta,\varphi)$ is the Laplace-generalized Laplace transform of the function $\hbar(x,\tau)$.

Secondly, using the Laplace transform for initial and boundary conditions (1.3), we get

$$\begin{cases}\Psi_{\alpha}(\vartheta,0)=\Psi_1(p), & \dfrac{\partial F_{\alpha}(\vartheta,0)}{\partial t}=\Psi_2(\vartheta),\\ \Psi_{\alpha}(0,\varphi)=\Psi_3(\varphi), & \dfrac{\partial F_{\alpha}(0,\varphi)}{\partial x}=\Psi_4(\varphi),\end{cases} \qquad (4.2)$$

Thirdly, by substituting equation (4.2) in equation (4.1) and simplifying, we get

$$\Psi_{\alpha}(\vartheta,\varphi)=\frac{1}{\left(\vartheta^{\varrho}-\frac{a_1}{\varphi^{\sigma}}-\frac{a_2}{\varphi^{\varsigma}}-a_3\right)}\left[\vartheta^{\varrho-1}\Psi_3(\varphi)+\vartheta^{\varrho-2}\Psi_4(\varphi)-a_1\frac{\varphi^{\alpha+1}}{\varphi^{\sigma}}\Psi_1(\vartheta)-a_1\frac{\varphi^{\alpha+2}}{\varphi^{\sigma}}\Psi_2(\vartheta)\right.$$
$$\left.-a_2\frac{\varphi^{\alpha+1}}{\varphi^{\varsigma}}\Psi_1(\vartheta)+\mathcal{H}_{\alpha}(\vartheta,\varphi)\right]. \qquad (4.3)$$

Finally, by taking the inverse Laplace-generalized Laplace transform (if it exists) to both sides of equation (4.3), we get the equivalent exact solution of equation (1.2) in the form

$$\psi(x,\tau)=\mathcal{L}_x^{-1}G_{\alpha}^{-1}\left[\frac{1}{\left(\vartheta^{\varrho}-\frac{a_1}{\varphi^{\sigma}}-\frac{a_2}{\varphi^{\varsigma}}-a_3\right)}\left[\vartheta^{\varrho-1}\Psi_3(\varphi)+\vartheta^{\varrho-2}\Psi_4(\varphi)-a_1\frac{\varphi^{\alpha+1}}{\varphi^{\sigma}}\Psi_1(\vartheta)-a_1\frac{\varphi^{\alpha+2}}{\varphi^{\sigma}}\Psi_2(\vartheta)\right.\right.$$
$$\left.\left.-a_2\frac{\varphi^{\alpha+1}}{\varphi^{\varsigma}}\Psi_1(\vartheta)+\mathcal{H}_{\alpha}(\vartheta,\varphi)\right]\right]. \qquad (4.4)$$

4. Illustrative Examples

To illustrate the usefulness of the earlier approach, we offer a few instances in this section.

Example 5.1. First, we take into consideration the linear homogeneous telegraph equation given bellow [1, 2].

$$\frac{\partial^{\varrho}\psi(x,\tau)}{\partial x^{\varrho}}=\frac{\partial^2\psi(x,\tau)}{\partial\tau^2}+\frac{\partial\psi(x,\tau)}{\partial\tau}+\psi(x,\tau), \qquad 1<\varrho\le 2, \qquad x,\tau\ge 0, \qquad (5.1)$$

with initial and boundary conditions,

$$\begin{cases}\psi(x,0)=-\psi_t(x,0)=\left[E_{\varrho}(x^{\varrho})+xE_{\varrho,2}(x^{\varrho})\right],\\ \qquad \psi(0,\tau)=\psi_x(0,\tau)=e^{-\tau}.\end{cases} \qquad (5.2)$$

The Laplace transform and generalized Laplace transform of the initial and boundary conditions (5.2) are given as

$$\begin{cases} \Psi_1(\vartheta) = \left(\frac{1}{\vartheta} + \frac{1}{\vartheta^2}\right)\frac{\vartheta^\varrho}{\vartheta^\varrho - 1}, \qquad \Psi_2(p) = -\left(\frac{1}{\vartheta} + \frac{1}{\vartheta^2}\right)\frac{\vartheta^\varrho}{\vartheta^\varrho - 1}, \\ \Psi_3(\varphi) = \Psi_4(\varphi) = \frac{\varphi^{\alpha+1}}{\varphi + 1}. \end{cases} \tag{5.3}$$

Considering equation (4.4), and using equation (5.3), we get

$$\psi(x,\tau) = \mathcal{L}_x{}^{-1}\mathrm{G}_\alpha{}^{-1}\left[\frac{1}{\left(\vartheta^\varrho - \frac{1}{\varphi^2} - \frac{1}{\varphi} - 1\right)}\left[\vartheta^{\varrho-1}\frac{\varphi^{\alpha+1}}{\varphi+1} + \vartheta^{\varrho-2}\frac{\varphi^{\alpha+1}}{\varphi+1} - \varphi^{\alpha-1}\left(\frac{1}{\vartheta} + \frac{1}{\vartheta^2}\right)\frac{\vartheta^\varrho}{\vartheta^\varrho - 1}\right]\right], \tag{5.4}$$

after some simplification, we have the solution to equation (5.1)

$$\psi(x,\tau) = \mathcal{L}_x{}^{-1}\mathrm{G}_\alpha{}^{-1}\left[\left[\frac{\varphi^{\alpha+1}}{\varphi+1}\left(\frac{\vartheta^{\varrho-1}}{\vartheta^\varrho - 1} + \frac{\vartheta^{\varrho-2}}{\vartheta^\varrho - 1}\right)\right]\right], \tag{5.5}$$

$$\psi(x,\tau) = e^{-\tau}\left[E_\varrho(x^\varrho) + xE_{\varrho,2}(x^\varrho)\right]. \tag{5.6}$$

Specifically, for $\varrho = 2$, we get $\psi(x,\tau) = e^{x-\tau}$, which the exact solution to equation (5.1) and the outcome is identical to the result obtained by [1].

Example 5.2. Finally, examine the linear non-homogeneous telegraph equation [2]:

$$\frac{\psi^\varrho f(x,\tau)}{\partial x^\varrho} = \frac{\partial^2\psi(x,\tau)}{\partial \tau^2} + \frac{\partial\psi(x,\tau)}{\partial\tau} - 2\psi(x,\tau) + e^{-2\tau}\left[E_\varrho(x^\varrho) + xE_{\varrho,2}(x^\varrho)\right], \qquad x,\tau \geq 0, \tag{5.7}$$

with initial and boundary conditions,

$$\begin{cases} \psi(x,0) = E_\varrho(x^\varrho) + xE_{\varrho,2}(x^\varrho), \qquad \psi_t(x,0) = -2\left[E_\varrho(x^\varrho) + xE_{\varrho,2}(x^\varrho)\right], \qquad 1 < \varrho \leq 2, \\ \psi(0,t) = \psi_x(0,\tau) = e^{-2\tau}. \end{cases} \tag{5.8}$$

The Laplace transform and generalized Laplace transform of the initial and boundary conditions (5.8) are given as

$$\begin{cases} \Psi_1(\vartheta) = \left(\frac{1}{\vartheta} + \frac{1}{\vartheta^2}\right)\frac{\vartheta^\varrho}{\vartheta^\varrho - 1}, \qquad \Psi_2(p) = -2\left(\frac{1}{\vartheta} + \frac{1}{\vartheta^2}\right)\frac{\vartheta^\varrho}{\vartheta^\varrho - 1}, \\ \Psi_3(\varphi) = \Psi_4(\varphi) = \frac{\varphi^{\alpha+1}}{1 + 2\varphi}. \end{cases} \tag{5.9}$$

Considering equation (4.4), and using equation (5.9), we get

$$\psi(x,\tau) = \mathcal{L}_x{}^{-1}\mathrm{G}_\alpha{}^{-1}\left[\frac{1}{\left(\vartheta^\varrho - \frac{1}{\varphi^2} - \frac{1}{\varphi} + 2\right)}\left[\vartheta^{\varrho-1}\frac{\varphi^{\alpha+1}}{1+2\varphi} + \vartheta^{\varrho-2}\frac{\varphi^{\alpha+1}}{1+2\varphi} - \varphi^{\alpha-1}\left(\frac{1}{\vartheta} + \frac{1}{\vartheta^2}\right)\frac{\vartheta^\varrho}{\vartheta^\varrho - 1}\right.\right.$$

$$\left.\left. + \varphi^\alpha\left(\frac{1}{\vartheta} + \frac{1}{\vartheta^2}\right)\frac{\vartheta^\varrho}{\vartheta^\varrho - 1} + \frac{\varphi^{\alpha+1}}{(1+2\varphi)}\left(\frac{1}{\vartheta} + \frac{1}{p^2}\right)\frac{\vartheta^\varrho}{\vartheta^\varrho - 1}\right]\right], \tag{5.10}$$

after some simplification, we have the solution to equation (5.7)

$$\psi(x,\tau) = \mathcal{L}_x{}^{-1}\mathrm{G}_\alpha{}^{-1}\left[\left[\frac{\varphi^{\alpha+1}}{1+2\varphi}\left(\frac{\vartheta^{\varrho-1}}{\vartheta^{\varrho}-1}+\frac{\vartheta^{\varrho-2}}{\vartheta^{\varrho}-1}\right)\right]\right], \tag{5.11}$$

$$\psi(x,\tau) = e^{-2\tau}\left[E_\varrho(x^\varrho) + xE_{\varrho,2}(x^\varrho)\right]. \tag{5.12}$$

Specifically, for $\varrho = 2$, we get $\psi(x,\tau) = e^{x-2\tau}$, which is the exact solution to equation (5.7). The outcome is identical to what was found by [2].

5. Conclusion

In this study, we have successfully utilized a double integral transform called the Laplace generalized Laplace transform to determine the exact solution of a linear telegraph problem with the Caputo fractional derivative. Moreover, the proposed method is more generic than the other methods. From the results, it is clear that the proposed method is a powerful tool and an efficient technique for obtaining the exact solution of a linear telegraph equation. In the future, we will utilize the coupling of the presented method with any of the iterative methods for solving non-linear fractional partial differential equations.

References

[1] Ranjit R D hunde1 and G. L. Waghmare, Double Laplace Transform Method for Solving Space and Time Fractional Telegraph Equations, *International Journal of Mathematics and Mathematical Sciences*, Volume **2016**, Article ID 1414595, 7 pages. http://dx.doi.org/10.1155/2016/1414595

[2] Tarig M. Elzaki and A.A. Ishag, Solution of Telegraph Equation by Elzaki-Laplace Transform, *African Journal of Engineering & Technology (AJET)*, **2022**, 2(8). https://doi.org/10.47959/AJET.2021.1.1.8

[3] Shams A. Ahmed, Tarig M. Elzaki, Mohamed Elbadri and Mohamed Z. Mohamed, Solution of partial differential equations by new double integral transform (Laplace - Sumudu transform), *Ain Shams Engineering Journal*, 12, **2021** 4045–4049, https://doi.org/10.1016/j.asej.2021.02.032

[4] Hassan Eltayeb, Yahya T. Abdalla, Imed Bachar and Mohamed H. Khabir, Fractional Telegraph Equation and Its Solution by Natural Transform Decomposition Method, Symmetry, **2019**, 11(334) https://doi:10.3390/sym11030334

[5] Kim, H. The intrinsic structure and properties of Laplace-typed integral transforms. *Math. Probl. Eng.* **2017**, 2017, 1762729

[6] Aboodh K. S. The new integral transform aboodh transform. *Global J Pure Appl Mathe*, **2013**, 9(1), 35–43.

[7] Elzaki T. M. The new integral transform Elzaki Transform. *Global J Pure Appl Mathe*, ***2011***, 7(1), 57–64.

[8] Watugala GK. Sumudu transform: a new integral transform to solve differential equations and control engineering problems. *Int J Math Educat Sci Technol*, **1993**, 24(1), 35–43.

[9] Nuruddeen, R.I.; Akbar, Y. and Kim, H. On the application of G_ integral transform to nonlinear dynamical models with non-integer order derivatives. *AIMS Math*. **2022**, 7, 17859–17878.

[10] Alhefthi, R.K.and Eltayeb, H. The Solution of Coupled Burger's Equation by G-Laplace Transform. *Symmetry*, **2023**, 15, 1764. https://doi.org/10.3390/sym15091764

[11] Eltayeb, H., Bachar, I., Mesloub, S. A Note on the Time-Fractional Navier–Stokes Equation and the Double Sumudu-Generalized Laplace Transform Decomposition Method. *Axioms*, **2024**, 13(44). https://doi.org/10.3390/axioms13010044

[12] Abubker Ahmed, Laplace Transform Method for Logistic Growth in a Population and Predator Models with Fractional Order, *Open J. Math. Sci.* **2023**, 7, 339-345; https://doi:10.30538/oms2023.0216

[13] A.M. Wazwaz, Partial Differential Equations: Methods and Applications, Balkema, Leiden, **2002**.

[14] Saeed Kazem, Exact Solution of Some Linear Fractional Differential Equations by Laplace Transform, *International Journal of Nonlinear Science*, **2013**, 16(1), 3-11.

[15] Mona Hunaiber and Ali Al-aati, the double laplace-aboodh transform and their properties with applications, *GSJ*, 10(3), March **2022**.

[16] Lokenath Debnath, the Double Laplace Transforms and Their Properties with Applications to Functional, Integral and Partial Differential Equations, *Int. J. Appl. Comput. Math*, **2016**, 2, 223–241. https://DOI10.1007/s40819-015-0057-3