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Non-Comaximal Graphs of Commutative Semirings

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Abstract:

Let S be a commutative semiring with unity. The non-comaximal graph of a semiring S, denoted by $G(S)$ is an undirected graph where S is the set of vertices in $G(S)$ and $a, b \in S$ remain adjacent if and only if $aS + bS \neq S$. We look at the connectedness and the diameter of this graph. The concepts of independent set, clique and the girth of $G(S)$ are discussed.

Keywords: *Non-comaximal graph, Semiring, Maximal ideal, Diameter, Girth.*

1-Introduction Section

Semirings remain helpful tools aimed at resolving issues in a variety of in aimed atmation sciences besides applied mathematics fields, including automata, coding, graph, and optimization theories, besides computer program analysis. This is because the structure of semirings offers an algebraic method aimed at analyzing and modeling the important variables in these fields.

In the past several years, the study of algebraic structures with graph possessions has gained a lot of attention and produced many intriguing findings in addition to intriguing questions. Assigning a graph to a ring is the subject of multiple works, aimed at instance see, [1, 4, 7, 10, 11, 16, 17, 18, 19]. In addition, there remain several paperson assigning a graph to semirings, aimed at instance [2, 3, 5, 8, 9, 13, 14].

The zero divisor graph of a commutative ring is a milestone of this trend which was introduced in 1988 by Beck [11]. There remain more than five hundred papers on zero divisor graphs. Some other graphs remain introduced, which make a bridge between algebraic structure and graph. Sharma and Bhatwadekar [17] introduced the comaximal graph of a commutative ring with unity. Miamian in [19] was replaced the set of vertices by the set of all proper ideals in a ring K. In [10] non-comaximal graphs remain defined where the set of proper ideals of K is the set of vertices and N is adjacent to M if and only if $N + M \neq K$. Generalizing the definition of [10], In [16] consider a graph $G(K)$ where K is the set of vertices and $d, f \in K$ remain adjacent if and only if $dK + fK \neq K$. In this paper, we generalize the non-comaximal graph of a commutative ring in [16] to the non-comaximal graph of a commutative semiring under semiring theoretic settings.

Throughout this paper S will be a commutative semiring with identity, $U\ (= U(S))$ be the set of all units of S, and the intersection of all maximal ideals of S is called the Jacobson radical of S and is denoted by $J(S)$. A semiring is a set S equipped with binary operations $+$ and $\cdot$ where $(S, +)$ is a commutative mooned with identity element 0, besides $(S, \cdot)$ is a mooned with identity element 1. Too, operations $+$ and $\cdot$ connected by distributive and 0 annihilates S. A semiring is commutative if $xy = yx$ for all $x, y \in S$. Throughout Section two of this paper presume S is commutative semiring with unity. The simplest example of commutative semirings is $\{0, 1\}$ the Boolean semiring, in which $1 + 1 = 0 \neq 1$. Besides, the set of nonnegative integers (or reals) with the standard operations in addition and multiplication, is commutative semiring. A non-empty subset R in S is named an ideal in S if the next two conditions hold: (i) $I + L \in R$ aimed at $I, L \in R$ (ii) $SI \in R$ aimed at $v \in S$ besides $I \in R$. An ideal R in S is named k-ideal (subtractive ideal) if u, $u + V \in R$, then $V \in R$. $\{0\}$ is k-ideal in S by page 66 of [15]. S is named a subtractive semiring if every ideal in S is subtractive ideal. A semiring S is named semidomain whenever $d, f \in S$ with $df = 0$ involves that either $d = 0$ or $f = 0$ [11]. A semifield is a semiring where a group under multiplication is aimed atmed by non-zero members [8] besides ([15], p. 52). A commutative semiring S is said to be a local semiring if it has a unique maximal subtractive ideal and it is semilocal semiring if it has finitely many maximal ideals. The Jacobson radical of a semiring S is the intersection of all maximal ideals is denoted by $G(S)$. Moreover, I is a unit of S if and only if I lies outside of each maximal k-ideal of S [6]. We denote the characteristic, the set of all maximal ideals of a semiring S by $Max(S)$. For undefined terminology and concept of semiring theory, we refer to Golan [15].

Let G be a simple undirected graph with vertex set $V(G)$ besides edge set $E(G)$. A path from I toward L is series of adjacent vertices $I — I_1 — I_2 — \cdots — I_n — L$. A graph G is connected if a path connects each of G's two unique vertices; if not, it is disconnected. Aimed at $I, L \in P(G)$ with $I \neq L$, $d(I, L)$ indicates the length of shortest path from I into L, if such a path does not exist, one uses the convention $d(I, L) = \infty$. The diameter to G is defined as $diam(G) = sup\{d(I, L) | I$ and L are vertices of $G\}$. Aimed at any $I \in P(G)$, $deg(I)$ symbolizes the number of edges incident with I, named the degree to I. A cycle is a path that starts and ends at the same vertex, has no edges that remain repeated, and has different vertices at every point except the starting and finishing vertices. The girth of G, symbolized by gr(G), is the length of shortest cycle in G ($gr(G) = \infty$ if G contains no cycles). When a graph's vertex set can be divided into two subsets, I and L, such that each edge hasone end in I plusone end in L, the graph is named bipartite. Any bipartite graph with two partitions (I and L) in which any vertex in I is linked toward any vertex in L is said to be complete. A bipartite graph with part sizes 1other than a aimed at a given positive integer a is called a star graph. By a clique in a graph G, we mean a complete subgraph of G and the number of vertices in the maximal clique of G, is named the clique number of G and is indicated by $\omega(G)$. Aimed at a graph G, let χ(G), denotes the vertex chromatic number of G, i.e., the minimum number of colors which can be assigned to the vertices of G such that every two adjacent vertices must different colors. A graph G is perfect, if aimed at every induced subgraph N of G, $\chi(N) = \omega(N)$. A graph is named weakly perfect, if its vertex chromatic number equals its clique number. $\bar{G}$ is the complement graph of the graph G. Apart from the notion of graph theory, we resort to Bondy & Marty [12] aimed at any ambiguouss terminology.

In section 2, in addition to studying non-comaximal graphs over commutative semirings, we generalize conclusions from [16]. We examined the girth, diameter, and connectedness of $G(S)$. We show that an element $a \in S$ is an isolated vertex of $G(S)$ if and only if a is a unit. Also, we proved that the graph $G(S)$ is weakly perfect.

2- Non-comaximal graphs of semilocal semirings

Similar to [16] in this section we introduce the concept of non-comaximal graph $G(S)$ of a commutative semiring S.

Here we consider any semiring is a semilocal and u is set . We find the girth and the diameter of $G(S)$. We begin with the next definition.

Definition 2-1 Let S be a commutative semiring with identity. The non-comaximal graph of S, denoted by $G(S)$

where S is the set of vertices in $G(S)$ and $a, b \in S$ are adjacent if and only if $aS + bS \neq S$.

Remark 2-2 In this section we consider S is a semilocal semiring thereaimed ate S contains finitely many ideals. As Sd is also an ideal it is contained in a maximal ideal in S. This implies that d is adjacent to e if dS, fS remain contained in the same maximal ideal M in S or $d, f \in M$.

Example 2-3 An inspection will shows that a set $SP_4 = \{0, 1, 2, b\}$ equipped with operations $+$ and $\cdot$ defined as:

+	0	1	2	b
0	0	1	2	b
1	1	2	1	2
2	2	1	2	1
B	b	2	1	0

·	0	1	2	b
0	0	0	0	0
1	0	1	2	b
2	0	2	2	0
b	0	b	0	b

is a semiring (which is not a ring) with unity. Here, $m_1 = \{0,2\}$ and $m_2 = \{0, b\}$ remain two maximal subtractive ideals of SP_4 ([12]). Then we must $G(SP_4)$ is not connected as in the next figure 1.

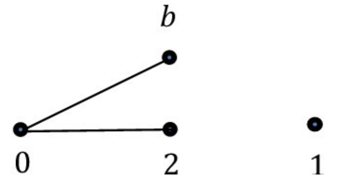

Figure 1: G(SP_4)

Example 2-4 Consider the set $S = \{0, 1\}$. On S we define the operations as follows: 0 + 0 = 1 + 1 = 0, 1 + 0 = 0 + 1 = 1 and 0.0 = 0.1 = 1.0 = 1.1 = 0. Then $(S, +, \cdot)$ forms a commutative semiring without unity thus $G(S)$ is connected graph.

Proposition 2-5 An element $x \in S$ is an isolated vertex of S if and only if x is a unit.

Proof: If x is a unit, then $Sx = S$ and a is not adjacent to any other element of S. if x is not a unit then Sx is an ideal in S contained in some maximal ideal M in S. Now $x\mathrm{S} + \mathrm{yS} \neq \mathrm{S}$. aimed at every $\mathrm{y} \in M$ and a cannot be isolated. Then $x \in \mathrm{S}$ is isolated if and only if x is a unit and the set of all units in S, $U(S)$ is an independent set in $G(S)$. □

Proposition 2-6 In a semilocal semiring S,

i. if $d \in \mathrm{G(S)} \neq 0$, then $\deg(d) \neq 0$.

ii. $\mathrm{Max}\{|M_i|\}$ is the clique number of $G(S)$.

Proof: (i) Mean S is a semilocal semiring it has finitely many maximal ideals $M_1, M_2, \dots, M_k$ (say). if $a \in \cap M_i = \mathrm{G(S)}$, then a is adjacent to every other element of every M_i. Hence $\deg(a) \neq 0$.

(ii) If $G_i(S)$ is the subgraph of $G(S)$ generated by the elements of the maximal ideal M_i, then it is complete and any element in $G_i(S)$ is not adjacent to any element which is not in M_i. Thus we must atleast

K complete subgraphs of $G(S)$. The $G_i(S)$ generated by the largest set of vertices is the clique of $G(S)$ and its order i.e Max$\{|M_i|\}$ is the clique number $\omega(G(S))$ of $G(S)$. □

Remark 2-7 If $n = p_1 p_2$, then $\mathbb{Z}_n$ has two ideals $\prec p_1 \succ$ and $\prec p_2 \succ$ where $\prec p_1 \succ \cap \prec p_2 \succ = 0$ i.e., $G(S) = 0$. Now all the elements remain either unitsor they contained in $\prec p_1 \succ$ or $\prec p_2 \succ$. if $p_1^{\propto_1}.p_2^{\propto_2} \dots p_k^{\propto_k}$, then there remain K maximal ideals in $\mathbb{Z}_n$ and $G(S) = \cap \prec p_i \succ \neq 0$. Then every element of $G(S)$ is adjacent to every element of $\prec p_i \succ$ aimed at every i.

Proposition 2-8 Non-comaximal graph $G(S)$ is not connected and $G_1(S)$ is connected íf has more than two maximal ideals and $G(S) \neq 0$. Here $G_1(S)$ is the subgraph of G generated by non-units of S.

***Proof*:** Let S be a semilocal semiring with more than two maximal ideals. Aimed at $d, \mathrm{f} \in S\backslash \mathrm{U}$ and $\mathrm{q} \in J(S)$. íf d and f remain contained in the same maximal ideal $M_i$of S , then $dS + \mathrm{q}S \neq S$ and they remain adjacent, otherwise íf $d \in M_i$ and $f \in M_j$, then $dS + \mathrm{q}S \neq S(d, \mathrm{q} \in M_i)$ and $\mathrm{f}S + \mathrm{q}S \neq S$ as $f, c \in M_j$. Thus we must a path $d - \mathrm{q} - \mathrm{f}$. We conclude that the subgraph $G_1(S)$ generated by the elements of $S\backslash \mathrm{U}(S)$ is connected but the subgraph $G_2(S)$ generated by $\mathrm{G}(S)$ is a null graph. We may say $G(S)$ is the union of a connected graph and the complement of a complete graph. If $|\mathrm{U}(S)| = \mathrm{n}$, then $G_2(S)$ is $\overline{K_n}$ and $G(S) = G_1(S) \cup \overline{K_n}$. Since $G(S)$ is not connected its subgraph $G_1(S)$ is connected. □

Proposition 2-9 The chromatic number $\chi(G(S)) = \max|\,M_i|$, thus $G(S)$ is weakly perfect.

***Proof*:** To color a $G(S)$ graph we need $\max|M_i| = n_j$ colors. 'If $\mathrm{J(S)} \neq 0$ such that $|\mathrm{J(S)}| = \mathrm{t}$, thenout of these n_j colors t colors remain assigned to the elementsof $\mathrm{J(S)}$. Now M_j generates a complete subgraph $G_j(\mathrm{S})$ such that the elementsof $M_j \backslash \mathrm{J(S)}$ require $n_j - \mathrm{t}$ colors. The elements of $M_i \backslash \mathrm{J(S)}$ are not adjacent to the elements of $M_l \backslash \mathrm{J(S)}$ aimed at any l. This implies that $n_j - \mathrm{t}$ colors remain sufficient to color the elementsof $M_i \backslash \mathrm{J(S)}$ aimed at all j. We must seen that $\omega(G(S)) = \max\{|M_i|\}$. Thereaimed ate $\chi(G(S)) = \omega(G(S))$ and $G(S)$ is weakly perfect. □

Proposition 2-10 For the semiring $\mathbb{Z}_n$, $n = p_1 p_2$, the diameter of $G(\mathbb{Z}_n)$ is infinite.

***Proof*:** If $n = p_1 p_2$, formerly there remain two maximal ideals M_1,M_2 such that $M_1 \cap M_2 = 0$ i.e, $\mathrm{J(S)} = 0$. Now $d, f \in \mathbb{Z}_n$ remain adjacent íf they belong to same M_i and $d(d, f) = 1$. 'If they belong to dífferent M_i's, then there is no path connecting then as in this cases $dS + \mathrm{f}S = \mathrm{S}$. Thereaimed ate $\mathrm{d(d, f)} = \infty$. This implies that $diam(G(\mathbb{Z}_n)) = \infty$. □

Now, we determine the diameter and girth of $G_1(S)$.

Proposition 2-11 The $diam(G_1(S)) = 1{,}2$ or ∞ and girth of $G_1(S)$, $\mathrm{gr}(G_1(S)) \leq 4$.

***Proof*:** Let $I, L \in S\backslash U(S)$ and let S be a semilocal semiring such that $\{M_i\}, i = 1,2, \dots, n$ remain maximal idealsof S. Now I, L remain contained in a maximal ideals M_i, M_j in S. If I and L remain contained in same maximal ideal M_i, formerly $d(I, L) = 1$ as they remain adjacent ootherwise we find an element $u \in G(S)$ to get a path $I - u - L$ and $d(I, L) = 2$. In both cases $d(I, L) \leq 2$, there aimed ate $diam(G(S)) = 2$. If $J(\mathrm{S}) = 0$ and I, L remain in different maximal ideals, then $d(I, L) = \infty$.

If $J(S) \neq 0$ which has at least two elements $d, f \in J(S)$. Aimed at any vertices I, L of $G_1(S)$ such that I, L remain in different maximal ideals, the cycle $I - d - L - L - I$ is the shortest cycle of length 4. Hence girth of $G_1(S) \leq 4$. □

Proposition 2-12 Let S be a semiring. Then $G\ (S\setminus U\ )$ is complete if and only if S is isomorphic to $\mathbb{Z}_{p^k}$ or it has a unique maximal ideal.

***Proof*:** Suppose S is isomorphic to $\mathbb{Z}_{p^k}$ or it has a unique maximal ideal. Aimed at a commutative semiring S every ideal is contained in a maximal ideal. Now every $d \in S$ is either a unit or dS is contained in the unique maximal ideal. Clearly aimed at any $d, f \in G(S\setminus U)$, $dS + fS \neq S$ and d is adjacent to r.
Conversely, if $G(S\setminus U)$ is complete, formerly the sum of $dS + fS \neq S$ aimed at any $d, f \in S\setminus U$ implying that all the ideals dS remain contained in a unique maximal ideal. As a result S is isomorphic to $\mathbb{Z}_{p^k}$ or it has a unique maximal ideal. □

Proposition 2-13 For the semiring $\mathbb{Z}_{p^k}, k \geq 5$ the non-comaximal graph $G(\mathbb{Z}_{p^k})$ is not planar.

***Proof*:** It is sufficient to show that $G(\mathbb{Z}_{p^k})$ has a complete subgraph K_5 or a bipartite graph $K_{3,3}$ as subgraphs in $G(\mathbb{Z}_{p^k})$. Now all the elementsof $G(\mathbb{Z}_{p^k})$ which remain not units, remain contained in the maximal ideal $\prec p \succ$. Thereaimed ate aimed at any two elements d, f in $S\setminus U$, $dS + fS \neq S$ hence they remain adjacent. meanwhile $\prec p \succ$ has more than 5 elements we may consider any 5 elements d_1, d_2, d_3, d_4, and d_5 such that they aimed atm a clique. Thus $G(S\setminus U)$ has K_5 as a subgraph and $G(S\setminus U)$ is not planar. □

3- Conclusion

In this work, we defined and study the non-comaximal graph of a commutative semiring S, $G(S)$, an undirected graph. Here, we consider S is a semi-local semiring. We study the connectivity, the chromatic number and the clique number of $G(S)$. We examined the girth, and diameter, of $G(S)$. We observed that an element $x \in S$ is an isolated vertex of S if and only if x is a unit.

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