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AB-Coupled Fixed Point Theorems Result in Partially Ordered S-Metric Spaces

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Abstract

The concepts presented in this paper pertain to the development and examination of AB-coupled fixed point results for mapping in partially ordered S-metric spaces that possess the strong mixed monotone property. The existence and uniqueness of AB-coupled fixed points are also demonstrated. We generalize the main theorems of Gnana Bhaskar and Lakshmikantham (2006) in [15] and Virendra Singh Chouhan and Richa Sharma (2015) [4].

Keywords: S-metric spaces, AB-coupled fixed points, Mixed monotone property, Partially ordered set Comparable partial ordered set.

1. Introduction

Fixed point theory dates back to the early 20th century, according to Banach (1922), who established fixed points for a contraction mapping of a complete metric space. This conclusion lays the groundwork for a wide range of extensions and generalizations. It is a widely used tool for solving current issues in a wide variety of mathematical subfields. Since then, researchers have developed various versions [6 ,7 ,8 ,11 ,13 ,14 ,16] of this conclusion, weakening its hypotheses as maintaining the convergence feature of consecutive iterates for a single fixed point of mappings. These generalizations follow the same pattern as the original result. Moreover, over the past few decades, various directions have led to the generalization of the concept of metric spaces. Some of the most important generalizations on rectangular metric spaces include the following: rectangular metric spaces, pseudo-metric spaces, fuzzy metric spaces, quasi-metric spaces, quasi-semi-metric spaces, probabilistic metric spaces, D-metric spaces, G-metric spaces, F-metric spaces, cone metric spaces, and so on. Fixed point theory incorporates the idea of partial orders to handle issues when the underlying structure is not always a linear order. Researchers such as Tarski (1955), who proved fixed point theorems for monotone functions on full lattices, gained prominence in the middle to late 20th century. The merging of metric spaces and partial orders led to significant advancements in the field. Notable among their works is that of Ran and Reurings (2004), who prov-ed that fixed points for monotone operators exist and are unique by extending the Banach Fixed Point Theorem to partially ordered metric spaces. Their results found applications in differential equations and dynamic systems . The concept of coupled fixed points emerged as a natural extension of single fixed points, aiming to solve systems of equations simultaneously. A coupled fixed point refers to a pair (a,b) such that (T(a,b) = (a,b)) for a given mapping T. This notion was formalized by Bhaskar and Lakshmikantham (2006), who provided sufficient conditions for the existence of coupled fixed points in partially ordered metric space Bhaskar and Lakshmikantham (2006) They introduced the concept of coupled fixed points and provided conditions under which such points exist for mappings in partially ordered metric space. Luong and Thuan (2011)They further generalized the coupled fixed point theorems for mixed monotone operators, expanding the applicability of these result. The theory has been extensively used to prove the existence of solutions to various types of differential equations and boundary value problems, demonstrating its practical significance. The development of coupled fixed points in partially ordered spaces represents a rich and evolving field of mathematical research. It builds upon classical fixed point theory, integrating the additional complexity of partial orders and addressing more sophisticated problems in mathematical analysis and applied mathematics. The ongoing research continues to find new applications and generalizations, solidifying the importance of this theory in contemporary mathematics.

2.Materials and Methods

This part will delve into a variety of definitions and theorems and outcomes previously explored by other authors and we will benefit the reader

because it will help them understand the primary findings of this research. It is a development of previous papers [4] and [15].

Definition 2.1[12]:Assum that the set $X \neq \emptyset$ and $S: X^3 \to [0, \infty)$ to be a function that meet all of the criteria

$\forall \mu, v, \omega, x \in X$.

1. $S(\mu, v, \omega) \geq 0$;

2. $S(\mu, v, \omega) = 0$ if and only if $\mu = v = \omega$;

3. $S(\mu, v, \omega) \leq S(\mu, \mu, x) + S(v, v, x) + S\,\omega, \omega, x)$

Then (X, S) is said to be an S-metric space.

Definition 2.2 [12]: Consider that (N, S) be an S-metric space and asequence $\{\mu_n\}$of N is defined as follows :

1. The sequence $\{\mu_n\}$in N converges to μ if $S(\mu_n, \mu_n, \mu) \to 0$ as $n \to \infty$. In this section, we write $\lim_{n\to\infty} \mu_n = \mu$.

2. $\{\mu_n\}$ is named a Cauchy sequence if $\forall \epsilon > 0$, there is $n_0 \in \mathbb{N}$ in which

$S(\mu_n, \mu_n, \mu_m) < \epsilon, \ \forall\, n, m \geq n_0$;

3. If every Cauchy sequence is convergent in S-metric space, then it is termed the complete space.

Lemma 2.3[12]: Consider the (N, S) be an S-metric space .If $\{\mu_n\}$,$\{\omega_n\}$are sequences in which $\lim_{n\to\infty} \mu_n = \mu$ and $\lim_{n\to\infty} \omega_n = \omega$ then $\lim_{n\to\infty} S(\mu_n, \mu_n, w_n) = S(\mu, \mu, \omega)$.

Definition 2.4[10]: A Partialy ordered set is a set P and abinary relation $\leq$ denoted by $(N, \leq)$,such that $\forall \mu, v, \omega \in P$.

1. $\mu \leq \mu$ (reflexivty) ;

2. $\mu \leq v$ and $v \leq \omega$ implies $\mu \leq \omega$ (transitivity);

3. $\mu \leq v$ and $v \leq \mu$ implies $\mu = v$ (anti-symmetry).

Definition 2.5[4]:Consider that $N \neq \emptyset$ we can say $(N, d, \leq)$ a partially ordered metric space, under certain conditions

1. $(N, \leq)$ is a partially ordered set ;

2. $(N, \leq)$ is a metric space.

Definition 2.6[4]:Consider that $(N, \leq)$ a paritial ordered set .Then $\mu, \omega \in N$ are called comparable if $\mu \leq \omega$ or $\omega \leq \mu$ holds.

Definition 2.7[4]: An element $(\mu, \omega) \in N \times N$ is called a coupled fixed point of the mapping $A: N \times N \to N$ if $A(\mu, \omega) = \mu$, $A(\omega, \mu) = \omega$.

Definition 2. 8 [9] Consider that $(N, \leq)$ be a partially ordered set. A self-mapping $f: N \to N$ is named to be strictly increasing if $f(\mu) < f(\omega), \forall \mu, \omega \in N$,

with $\mu < \omega$, also is called be strictly decreasing if

$f(\mu) > f(\omega), \forall \mu, \omega \in N.$ with $\mu < \omega$.

Definition 2.9[5]. Consider that $N \neq \emptyset$ and $M \neq \emptyset$, $A: N \times M \to N$ and $B: M \times N \to M$ be two mapping

An element $(\mu, \omega) \in N \times M$ iscalled an AB-coupled fixed point if $A(\mu, \omega) = \mu$, $B(\omega, \mu) = \omega$.

Definition 2.10[5] Consider that $(N, \leq)$, $(M, \leq)$ be twe partially ordered sets and $A: N \times M \to N$ and $B: M \times N \to M$ be two mappings .

So, A and B have mixed monotone property i.e., if

$\forall(\mu,\omega) \in N \times M$, $\mu_1, \mu_2 \in N$, $\mu_1 \leq \mu_2$ implies $A(\mu_1,\omega) \leq A(\mu_2,\omega)$ and $B(\omega,\mu_2) \leq B(\omega,\mu_1)$ and $\omega_1, \omega_2 \in M$, $\omega_1 \leq \omega_2$ implies $A(\mu,\omega_2) \leq A(\mu,\omega_1)$and $B(\omega_1,\mu) \leq B(\omega_2,\mu)$.

Definition 2.11[9] The triple $(N, d, \leq)$ is referred to as a partially ordered complete metric space if (N, d) is a complete metric space.

Theorem 2.12 [4] Consider that the set $(N, \leq)$ be a partially ordered with a complete metric space(N, d). Let $F: N \times N \to N$

be a mapping with the mixed monotone property on N and there is $\beta_0, \gamma_0 \in N$ in which $\beta_0 \leq F(\beta_0,\gamma_0)$, $\gamma_0 \geq F(\gamma_0,\beta_0)$ .Let $\psi: [0,\infty) \to [0,\infty)$

is a continuous and increasing function in which it is non-negative in (0 ,∞), $\psi(0) = 0$, $\lim\limits_{n\to\infty} \psi(t) = \infty$. In which

$$d(F(\beta,\gamma), F(\mu,\omega)) \leq d(\beta,\mu) + \psi(d(\gamma,\omega)), \ \forall \beta,\gamma,\mu,\omega \in N, \quad \beta \geq \mu, \quad \gamma \leq \omega.$$

Consider that either

1. F is continuous or

2. N which has fulfilling the next requirement;

a) If an increasing sequence$\{\beta_n\}$in N converges to $\beta \in N$ then $\beta_n \leq \beta$, $\forall n$.

b) If a decreasing sequence $\{\gamma_n\}$ in N converges to $\gamma \in N$ then $\gamma_n \leq \gamma, \forall n$.

Consequently, F presents a coupled fixed point $(\mu *, \omega *) \in N \times N$.

Theorem 2.13[4] Consider that supposition of Theorem 2.12 hold. And there is $z \in N$ which is comparable to μ, ω. $\forall \mu, \omega \in N$, then F has a unique coupled fixed point .

3.Results

In this section, we demonstrate the existence of a few AB-coupled fixed point results for self-mapping inside the framework of an ordered S-metric space.

We devote the subsequent sections to the growth and enhancement of the theory discussed in [4] and [15].

Theorem 3.1 Consider that $(N, S. \leq)$. $(M, S, \leq)$be twe partially ordered complete S-metric spaces. $A: N \times M \to N$ and $B: M \times N \to M$ be two mappings have mixed monotone property and there is in which $\beta_0 \leq A(\beta_0, \gamma_0)$ and $B(\gamma_0, \beta_0) \leq \gamma_0$ suppose there is $\psi: [0, \infty) \to [0, \infty)$ is a continuous and non-decreasung function in which it is non-negative in $(0, \infty)$, $\psi(0) = 0$, $\lim_{t\to\infty} \psi(t) = 0$. from this Lemma ($\psi \in \Psi$, . forall $t > 0$ and $\Psi: [0, \infty) \to [0, \infty)$, we have $\lim_{n\to\infty} \psi^n(t)=0$)in which $\forall\ \beta, \mu \in N, \forall\ \gamma, \omega \in M$.

$$S\big(A(\beta.\gamma), A(\beta.\gamma), A(\mu.\omega)\big) \leq S(\beta.\beta.\mu) + \psi\big(S(\gamma.\gamma.\omega)\big), \qquad (\beta, \gamma) \leq (\mu.\omega) \tag{3.1}$$

$$S(B(\gamma.\beta), B(\gamma.\beta), B(\omega, \mu)) \leq S(\gamma.\gamma.\omega) + \psi\big(S(\beta.\beta.\mu)\big), \qquad (\gamma.\beta) \leq (\omega, \mu) \tag{3.2}$$

Suppose either

1) A and B is continuous or

2) N and M have the following properties

a) if $\{\beta_n\}$ is an increasing sequence in N converges to $\beta \in N$ then $\beta_n \leq \beta, \forall n$;

b) if $\{\gamma_n\}$ is a decreasing sequence in M converges to $\gamma \in M$then $\gamma \leq \gamma_n, \forall n$.

If there is $\beta_0 \in N$ and $\gamma_0 \in M$, in which $(\beta_0, \gamma_0) \leq (A(\beta_0, \gamma_0), B(\gamma_0, \beta_0))$ then AB-coupled fixed point (μ_0, ω_0).

Proof: Given $\beta_0 \in N$ and $\gamma_0 \in M$ in which $(\beta_0, \gamma_0) \leq (A(\beta_0, \gamma_0), B(\beta_0, \gamma_0))$ if $(\beta_0, \gamma_0) = (A(\beta_0, \gamma_0), B(\gamma_0, \beta_0))$

then (β_0, γ_0) if is an AB-coupled fixed point,by definition of Partial order on $N \times M$.

we have $\beta_0 \leq A(\beta_0, \gamma_0)$ and $B(\gamma_0, \beta_0) \leq \gamma_0$
(3.3)

Let $(\beta_0, \gamma_0) \in N \times M$, $\quad \beta_1 = A(\beta_0, \gamma_0), \quad \gamma_1 = B(\gamma_0, \beta_0)$,

From (3.3) we have $\beta_0 \leq \beta_1$ and $\gamma_1 \leq \gamma_0$

The mixed monotone property of A and B a follows us to derive the following:

$A(\beta_0, \gamma_0) \leq A(\beta_1, \gamma_0), \quad A(\beta_0, \gamma_0) \leq A(\beta_1, \gamma_1)$ and $B(\gamma_1, \beta_1) \leq B(\gamma_0, \beta_1), \quad B(\gamma_1, \beta_1) \leq B(\gamma_0, \beta_0)$

Let $\beta_2 = A(\beta_1, \gamma_1), \quad \gamma_2 = B(\gamma_1, \beta_1)$

As, a result of A and B is mixed monotone property, we get

$\gamma_2 \leq \gamma_1$, $\beta_1 \leq \beta_2$

This approach will be continued by utilizing the mixed monotone property of A and B, as well as by employing the definition of Partially ordered on $N \times M$ variables, we get

$\beta_{n+1} = A(\beta_n, \beta_n)$, and $\gamma_{n+1} = B(\gamma_n, \gamma_n)$
(3.4)

by equations (3.1) and (3.2) we have

$S(\beta_n, \beta_n, \beta_{n+1}) = S\big(A(\beta_{n-1}, \gamma_{n-1}), A(\beta_{n-1}, \gamma_{n-1}), A(\beta_n, \gamma_n)\big) \leq S(\beta_{n-1}, \beta_{n-1}, \beta_n) + \psi\big(S(\gamma_{n-1}, \gamma_{n-1}, \gamma_n)\big)$.

$S(\gamma_n, \gamma_n, \gamma_{n+1}) = S\big(B(\gamma_{n-1}, \beta_{n-1}), B(\gamma_{n-1}, \beta_{n-1}), B(\gamma_n, \beta_n)\big) \leq S(\gamma_{n-1}, \gamma_{n-1}, \gamma_n) + \psi(S(\beta_{n-1}, \beta_{n-1}, \beta_n))$.

By adding we have $S_n \leq S_{n-1} + \psi(S_{n-1})$. Let S_n=$(\beta_{n_1}, \beta_{n_1}, \beta_{n_1-1}) + S(\gamma_{n_1}, \gamma_{n_1}, \gamma_{n_1-1})$. There is $n_1 \in \mathbb{N}$,

in which $S\big(\beta_{n_1}, \beta_{n_1}, \beta_{n_1-1}\big) = 0$, $S\big(\gamma_{n_1}, \gamma_{n_1}, \gamma_{n_1-1}\big) = 0$ then

$\beta_{n_1-1} = \beta_{n_1} = A\big(\beta_{n_1-1}, \beta_{n_1-1}, \gamma_{n_1-1}\big)$, $\gamma_{n_1-1} = \gamma_{n_1} = B(\gamma_{n_1-1}, \gamma_{n_1-1}, \beta_{n_1-1})$

And $\beta_{n_1-1}, \gamma_{n_1-1}$ is fixed point of A, B the proof is complete.

In other case $S(\beta_{n+1}, \beta_{n+1}, \beta_n) \neq 0$, $S(\gamma_{n+1}, \gamma_{n+1}, \gamma_n) \neq 0$, $\forall\, n \in \mathbb{N}$

By using supposition of ψ we have $S_n \leq S_{n-1} + \psi(S_{n-1}) \leq S_{n-1}$

S_n is positive sequence and has alimit S^* taking limit when $n \to \infty$ we get $S^* \leq S^* + \psi(S^*)$ and subsequently $\psi(S^*) = 0$ by our assmption on ψ.

If $S^* = 0$ i.e., $\lim_{n\to\infty}(S_n) = 0$, $\lim_{n\to\infty} S(\beta_{n+1}, \beta_{n+1}, \beta_n) + S(\gamma_{n+1}, \gamma_{n+1}, \gamma_n) = 0$,

Then $\lim_{n\to\infty} S(\beta_{n+1}, \beta_{n+1}, \beta_n) = \lim_{n\to\infty} S(\gamma_{n+1}, \gamma_{n+1}, \gamma_n) = 0$ (3.5)

Now we need showing$\{\beta_n\}$is a Cauchy sequence in N and $\{\gamma_n\}$ is a Cauchy sequence in M there is two subsequence of integers with $n_k, m_k \in \mathbb{N}$, $n_k > m_k \geq k$

$r_k = S(\beta_{m_k}, \beta_{m_k}, \beta_{n_k}) + S(\gamma_{m_k}, \gamma_{m_k}, \gamma_{n_k}) \geq \epsilon$ (3.6)

in which (3.6) Further,corresponding m_k , we choose n_k , to be the smallest for which (3.6) holds hance

$S(\beta_{m_k}, \beta_{m_k}, \beta_{n_k}) + S(\gamma_{m_k}, \gamma_{m_k}, \gamma_{n_k}) < \epsilon$
(3.7)

Using (3.6) and (3.7) and triangle intequality we get

$\epsilon \leq r_k$

$\leq S(\beta_{m_k}, \beta_{m_k}, \beta_{n_k}) + S(\gamma_{m_k}, \gamma_{m_k}, \gamma_{n_k})$

$\leq S(\beta_{m_k}, \beta_{m_k}, \beta_{n_k-1}) + S(\beta_{m_k}, \beta_{m_k}, \beta_{n_k-1}) + S(\beta_{n_k-1}, \beta_{n_k-1}, \beta_{n_k}) + S(\gamma_{m_k}, \gamma_{m_k}, \gamma_{n_k-1}) + S(\gamma_{m_k}, \gamma_{m_k}, \gamma_{n_k-1}) + S(\gamma_{n_k-1}, \gamma_{n_k-1}, \gamma_{n_k})$

$\leq 2\epsilon + S_{n_k-1}$

(3.8

Using the formula (3.5) and a fllowing $k \to \infty$, we obtain

$$\lim_{n,m\to\infty} r_k = \epsilon > 0$$

(3.9)

Presently, we acquire

$$S(\beta_{n_k+1}, \beta_{n_k+1}, \beta_{m_k+1}) \leq S(\beta_{n_k}, \beta_{n_k}, \beta_{m_k}) + \psi(S(\gamma_{n_k}, \gamma_{n_k}, \gamma_{m_k})$$

$$S(\gamma_{n_k+1}, \gamma_{n_k+1}, \gamma_{m_k+1}) \leq S(\gamma_{n_k}, \gamma_{n_k}, \gamma_{m_k}) + \psi(S(\beta_{n_k}, \beta_{n_k}, \beta_{m_k})$$

Then we get $r_{k+1} \leq r_k + \psi(r_k)$

(3.10)

Letting $k \to \infty$ of both sides of eguation (3.10) and from equation (3.9) it follows that

$$\epsilon = \lim_{k\to\infty} r_{k+1} \leq \lim_{n\to\infty} r_k + \lim_{k\to\infty} \psi(r_k) < \epsilon$$

Which is contraction.

Therefore $\{\beta_n\}$ is Cauchy sequence in N ,also $\{\gamma_n\}$is Cauchy sequence in M .Since N and M are complete S-metric spaces there is $\mu_0 \in \mathbb{N}, \omega_0 \in \mathbb{N}$,in which

$$\lim_{n\to\infty} \beta_n = \mu_0, \qquad \lim_{n\to\infty} \gamma_n = \omega_0.$$

(3.11)

Assume that A and B are continuous function therefore by equation (3.11)and (3.4)we get

$$\mu_0 = \lim_{n\to\infty} \beta_{n+1} = \lim_{n\to\infty} A(\beta_n, \gamma_n) = A(\mu_0, \omega_0),$$

$$\omega_0 = \lim_{n\to\infty} \gamma_{n+1} = \lim_{n\to\infty} B(\gamma_n, \beta_n) = B(\omega_0, \gamma_0).$$

That is $A(\mu_0 , \omega_0) = \mu_0$, $B(\omega_0 , \mu_0) = \omega_0$

Thus (μ_0 , ω_0) is an AB-coupled fixed point Consider that the conditions 2(a) and 2(b) of the theorem holds.

We get $\{\beta_n\}$ is non-decreasing sequence in N and $\{\gamma_n\}$ is non-increasing in M and by equation (3.11)we have

$$\lim_{n\to\infty} \beta_n = \mu_0 , \lim_{n\to\infty} \gamma_n=\omega_0.$$

From definition of S-metric space and equations (3.1) and (3.4).

$$S(A(\mu_0 , \omega_0), A(\mu_0 , \omega_0), \mu_0) \leq S(A(\mu_0 , \omega_0), A(\mu_0 , \omega_0), \beta_{n+1}) + S(\beta_{n+1} , \beta_{n+1} , \mu_0)$$

$$= S\big(A(\mu_0 , \omega_0), A(\mu_0 , \omega_0), A(\beta_n , \gamma_n)\big) + S(\beta_{n+1} , \beta_{n+1} , \mu_0)$$

$$\leq S(\mu_0 , \mu_0 , \beta_n) + \psi(S(\omega_0 , \omega_0 , \gamma_n) + S(\beta_n , \beta_n , \mu_0)$$

Letting $n \to \infty$, we have $S(A(\mu_0 , \omega_0), A(\mu_0 , \omega_0), \mu_0) \leq 0 + \psi(0) = 0$

This implies that $A(\mu_0 , \omega_0) = \mu_0$

And from definition of S-metric space and equations (3.2) and (3.4)

$$S(B(\omega_0 , \mu_0), B(\omega_0, \mu_0), \omega_0) \leq S(B(\omega_0 , \mu_0), B(\omega_0 , \mu_0), \beta_{n+1}) + S(\beta_{n+1} , \beta_{n+1} , \omega_0)$$

$$=S\big(B(\omega_0 , \mu_0), B(\omega_0 , \mu_0), B(\gamma_n , \beta_n)\big) + S(\gamma_{n+1} , \gamma_{n+1} , \omega_0)$$

$$\leq S(\omega_0 , \omega_0, \gamma_n) + \psi\big(S(\mu_0 , \mu_0 , \beta_n)\big) + S(\gamma_n , \gamma_n , \omega_0)$$

Letting $n \to \infty$ we have

$S(B(\omega_0, \mu_0), B(\omega_0, \mu_0), \omega_0) \leq 0 + \psi(0) = 0$

Then $B(\omega_0, \mu_0) = \omega_0$

This complete the theorem.

Theorem 3.2. Let the hypotheses stated in Theorem(3.1) and suppose that
$\forall(\beta, \gamma), (\mu, \omega) \in N \times M$ and there is (t, s)

Such that $(A(t, s), B(s, t))$ is comparable to $(A(\beta, \gamma), B(\gamma, \beta))$ and $(A(\mu, \omega), B(\omega, \mu))$ then there exist unique an AB-coupled fixed point in $N \times M$.

Proof:- As, we know from Theorem (3.1) Suppose that $(\beta, \gamma), (\mu, \omega) \in N \times M$ are coupled Fixed Points Then $\beta = A(\beta, \gamma)$, $y = B(\gamma, \beta)$, $\mu = A(\mu, \omega)$, $\omega = B(\omega, \mu)$.

We have show that $\beta = \mu, \gamma = \omega$ by our supposition there is $(t, s) \in N \times M$ Such that $(A(t, s), B(s, t))$ is comparable to $(A(\beta, \gamma), B(\gamma, \beta))$,$(A(\mu, \omega), B(\omega, \mu))$

Case 1

We construct$\{t_n\}$ an increasing sequence in N and $\{s_n\}$a decreasing sequence in M defined

$t_{n+1} = A(t_n, s_n)$,$s_{n+1} = B(s_n, t_n)$, $\forall n$; are comparable, we have

$S(\beta, \beta, \mu) = S(A(\beta, \gamma), A(\beta, \gamma), A(\mu, \omega)) \leq S(\beta, \beta, \mu) + \psi(S(\gamma, \gamma, \omega))$

Similarly

$S(\gamma, \gamma, \omega) \leq S(B(\gamma, \beta), B(\gamma, \beta), B(\omega, \mu) \leq S(\gamma, \gamma, \omega) + \psi(S(\beta, \beta, \mu))$

Then $S(\beta,\beta,\mu)+S(\gamma,\gamma,\omega)=0$

So, $\beta=\mu,\gamma=\omega$ This proof is complete

Case 2

Suppose that(β,γ) and (μ,ω) are not comparable there exist $(t,s)\in N\times M$ which comparable with both (β,γ) and (μ,ω) consider

$$S(\beta,\beta,\mu) = S(A^n(\beta,\gamma),A^n(\beta,\gamma),A^n(\mu,\omega)),\quad S(\gamma,\gamma,\omega) = S(B^n(\gamma,\beta),B^n(\gamma,\beta),B^n(\omega,\mu))$$

Monotonicity $(A^n(t,s),B^n(s,t))$

$$S\big((\beta,\beta,\gamma),(\mu,\mu,\omega)\big)$$

$$= S((A^n(\beta,\gamma),A^n(\beta,\gamma),B^n(\gamma,\beta)),(A^n(\mu,\omega),A^n(\mu,\omega),B^n(\omega,\mu))$$

$$= S(A^n(\beta,\gamma),A^n(\beta,\gamma),B^n(\gamma,\beta)),(A^n(t,s),A^n(t,s),B^n(s,t)) + S(A^n(t,s),A^n(t,s),B^n(s,t)),(A^n(\mu,\omega),A^n(\mu,\omega),B^n(\omega,\mu))$$

$$\leq (S(\beta,\beta,t)+\psi(S(\gamma,\gamma,s)))+(S(\gamma,\gamma,s)+\psi(S(\beta,\beta,t)))+(S(t,t,\mu)+\psi(S(s,s,\omega)))+(S(s,s,\omega)+\psi(S(t,t,\mu)))$$

$=0$

So ,$\beta=\mu,\gamma=\omega$ The proof is complete.

Example 3.3. Let $N=[0,\infty)$and $M=[-\infty,0)$ by the usual order in $\mathbb{R}$ Consider the S-metric on N and M ,as $S(\beta,\beta,\gamma)=|\beta-\gamma|$,$\forall\beta\in N,\gamma\in M$.

Consider the mapping $A: N\times M\to N$, $B: M\times N\to M$.

defined by $A(\beta\,,\gamma)=\frac{\beta-7\gamma}{12}$,$B(\gamma\,,\beta)=\frac{\gamma-7\beta}{12}$,$\forall\beta\,,\mu\in N$, $\forall\gamma\,,\omega\in M$.

Let us take $\psi:[0\,,\infty)\to[0\,,\infty)$, in which = $\psi(t)=\frac{7t}{12}$,

We have $A(\beta,\gamma)\ \leq\ A(\mu.\gamma)$, $B(\gamma.\beta)\ \geq\ B(\gamma,\mu)$ and $A(\beta,\gamma)\leq\ A(\beta,\omega), B(\gamma,\beta)\geq$ $B(\omega,\beta)$.

Therefor A and B are continuous and have mixed monotone property. Also there are $(\beta_0\,,\gamma_0)\in N\times M$,

Then (0,0) is the unique AB-coupled fixed point $\beta_0=0\,,\gamma_0=0$,

$$S\big(A(\beta,\gamma),A(\beta,\gamma),A(\mu,\omega)\big)=\left|\frac{\beta-7\gamma}{12}-\frac{\mu-7\omega}{12}\right|$$

$$=\frac{1}{12}|(\beta-7\gamma)-(\mu-7\omega)|$$

$$=\frac{1}{12}\ |(\beta-\mu)-7(\gamma-\omega)|$$

$$\leq\frac{1}{12}|\beta-\mu|-\frac{7}{12}|\gamma-\omega|$$

$$\leq S(\beta\,,\beta\,,\mu)-\psi(S(\gamma\,,\gamma\,,\omega))$$

and

$$S\big(B(\gamma,\beta),B(\gamma,\beta),B(\omega,\mu)\big)=\left|\frac{\gamma-7\beta}{12}-\frac{\omega-7\mu}{12}\right|$$

$$=\frac{1}{12}|(\gamma-7\beta)-(\omega-7\mu)|$$

$$=\frac{1}{12}\ |(\gamma-\omega)-7(\beta-\mu)|$$

$$\leq \frac{1}{12}\ |(\gamma - \omega| - \frac{7}{12}\ |(\beta - \mu)|$$

$$\leq S(\gamma\,,\gamma\,,\omega) - \psi(S(\beta\,,\beta\,,\mu))$$

There for A and B are satisfied all conditions of theorem (3.1).Then, (0,0) is the unique AB-coupled fixed point.

4.Conclusions

In this paper, we have study and explored AB-coupled fixed point theorems results in partially ordered of S-metric spaces In addition, we have proven uniqueness and existence and give an illustrative example.

When compared to the writings of earlier experts Our findings represent developments and improvements for the field (see[1 ,2 ,3]).

5.References.

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