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New analytical and numerical solutions for the fractional differential heat-like equation

Nadia Jabbar Enad

Education Directorate of Thi-Qar, Ministry of Education, Nasiriyah 64001, Iraq

* Corresponding email: amwyd5699@gmail.com

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Abstract

This study presented new analytical and numerical solutions for the two-dimensional heat-like differential equation with four different fractional derivatives using the Laplace-Adomian decomposition method. This paper introduced the algorithm of the method used with each of the four fractional derivatives and then presents the analytical solutions of the heat equation as well as the numerical solutions and two-dimensional graphs with the help of Maplesoft and MATLAB.

Keywords: heat-like equation; Laplace transform; Adomian decomposition method; Caputo derivative; Atangana-Baleanu derivative; Caputo-Fabrizio derivative; Hussein-Jassim derivative.

1-Introduction

Fractional calculus offers a profound approach to model and analyze complex systems that exhibit non-local behaviors and memory effects, transcending the limitations of traditional integer-order calculus. The fractional Caputo derivative, fractional Caputo-Fabrizio derivative, fractional Atangana-Baleanu derivative and fractional Hussein-Jassim derivative are among the advanced tools in this field [1]-[4], each tailored to capture specific characteristics such as memory and non-locality in dynamical systems. These derivatives find application in various domains, including physics, engineering, and biology. The Adomian decomposition method (ADM) complements fractional calculus by providing an efficient numerical technique to solve fractional differential equations, such as the fractional heat-like differential equation. This equation describes heat-like conduction in materials with anomalous diffusion characteristics [20]-[24]. The two-dimensional heat-like differential equation (HLDE) is given as follows

$$\frac{\partial}{\partial \tau}\varphi(x,y,\tau) = \lambda\, \nabla^2\varphi(x,y,\tau), \tag{1.1}$$

where λ is a positive constant defining the thermal diffusivity and ∇^2 is the Laplacian operator defined as

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}. \tag{1.2}$$

In this study, we will use the Adomian decomposition method to obtain approximate analytical solutions to the linear heat-like equation with some fractional derivatives such as (the Caputo operator, the Caputo Fabrizio operator, the Atangana-Baleanu operator, and the Hussein Jassim operator) [5],[6],[7]. We will discuss this paper as follows: In the second section we present some basic concepts related to our study, and in the third section we will discuss the algorithm of the Adomian decomposition method for the heat-like equation. The fourth section will address the analytical solutions for the heat-like equation with the fractional operator using the Adomian decomposition technique. In the fifth section, we will discuss the numerical solutions and graphs for the equation above, and in the final section, we will discuss the results, recommendations, and future studies.

- Preliminaries

This section includes some definitions and theories relevant to our study, so that the following sections are clear to the reader.

Definition 1. [8] Let ω is a complex number such that $Re(\omega) > 0$, then the gamma function (GF) is denoted by $\Gamma(\omega)$, and is defined as follows

$$\Gamma(\omega) = \int_0^\infty t^{\omega-1} \exp(-t)\, dt. \tag{2.1}$$

There are some properties of the gamma function that should be mentioned,

$$\Gamma(\omega+1) = \omega\Gamma(\omega), \text{for all } \omega. \tag{2.2}$$

$$\Gamma(\omega+1)=\omega!,\ \ \text{If } \omega \text{ is a non negetive integer number.} \tag{2.3}$$

Definition 2. [9] The Mittag-Leffler function (MLF) of one parameter is defined by the following power series

$$E_{\omega}(z)=\sum_{\eta=0}^{\infty}\frac{z^{\eta}}{\Gamma(\eta\omega+1)},\ Re(\omega)>0, \tag{2.4}$$

and of two parameters is defined by

$$E_{\omega,\gamma}(z)=\sum_{\eta=0}^{\infty}\frac{z^{\eta}}{\Gamma(\eta\omega+\gamma)},\ Re(\omega),Re(\gamma)>0, \tag{2.5}$$

There are some properties of the Mittag-Leffler function that should be mentioned

$$E_{\omega,1}(z)=E_{\omega}(z), \tag{2.6}$$

$$E_{1,1}(z)=\exp{(z)}. \tag{2.7}$$

Definition 3. [10] Let ψ is a function such that $\psi\in C^{m}[a,b]$, $b>a$ and ω is greater than or equal zero, then The Riemann-Liouville integral of fractional order (RLFI) of $\eta-1<\omega\leq\eta$, η is a non-negative integer number is given by

$$_{a}J_{\tau}^{\omega}\psi(\tau)=\frac{1}{\Gamma(\omega)}\int_{a}^{\tau}(\tau-z)^{\omega-1}\,\psi(z)dz, \tag{2.8}$$

and the Caputo derivative of fractional order (CFO) is given by [11]

$$_{a}^{C}D_{\tau}^{\omega}\psi(\tau)=J_{\tau}^{\eta-\omega}\left(D_{\tau}^{\eta}\psi(\tau)\right)=\frac{1}{\Gamma(\eta-\omega)}\int_{a}^{\tau}(\tau-z)^{\eta-\omega-1}\,\psi^{(\eta)}(z)dz. \tag{2.9}$$

Definition 4. [12] Let ϕ is a function such that $\phi\in H^{1}[a,b]$,$b>a$ and $0<\gamma\leq1$, then the Caputo-Fabrizio fractional operator (CFFO) is defined as

$$_{a}^{CF}D_{\tau}^{\gamma}\phi(\tau)=\frac{M(\gamma)}{1-\gamma}\int_{a}^{\tau}\exp\left(-\frac{\gamma(\tau-z)}{1-\gamma}\right)\phi'(z)dz, \tag{2.10}$$

and the Atangana-Baleanu fractional operator (ABFO) is given by [13]

$$ {}^{AB}_{a}D^{\gamma}_{\tau}\phi(\tau) = \frac{\mathrm{M}(\gamma)}{1-\gamma}\int_{a}^{\tau} \mathrm{E}_{\gamma}\left(-\frac{\gamma(\tau-\mathrm{z})^{\gamma}}{1-\gamma}\right)\phi'(z)\mathrm{dz}\,, \tag{2.11} $$

and the Hussein-Jassim fractional operator (HJO) of order $\eta - 1 < \omega \leq \eta$ [14]

$$ {}^{HJ}_{a}D^{\omega}_{\tau}\phi(\tau) = H_{\omega}\int_{a}^{\tau} \exp(-H_{\omega}(\tau - z))\,\phi^{(\eta)}(\mathrm{z})\mathrm{dz}, \tag{2.12} $$

where $\mathrm{M}(\gamma)$ is a function such that $\mathrm{M}(0) = \mathrm{M}(1) = 1$, and H_{ω} is a function of ω such that $\lim_{\omega\to\eta} H_{\omega} = \infty$.

Definition 5. [15] Let φ is an integrable function, then the Laplace transform (LT) of φ is given by

$$ F(\varsigma) = L(\varphi(\tau)) = \int_{0}^{\infty} \varphi(\tau)\,\mathrm{e}^{-\varsigma\tau}d\tau. \tag{2.13} $$

Theorem 1. [16],[17] Let $L(.)$ is the Laplace transform and λ is a constant, then

$$ L(\lambda) = \frac{\lambda}{\varsigma}, \tag{2.14} $$

$$ L(\tau^{\omega}) = \frac{\Gamma(\omega+1)}{\varsigma^{\omega+1}}, \tag{2.15} $$

$$ L(\mathrm{e}^{\lambda\tau}) = \frac{1}{\varsigma-\lambda}, \tag{2.16} $$

$$ L\left(\tau^{\gamma-1}E_{\omega,\gamma}(\lambda\tau^{\omega})\right) = \frac{\varsigma^{\omega-\gamma}}{\varsigma^{\omega}-\lambda}. \tag{2.17} $$

Theorem 2. [18],19] Let $L(.)$ is the Laplace transform, $0 < \gamma \leq 1$ and $\eta - 1 < \omega \leq \eta$, then

$$ L\left({}_{a}J^{\omega}_{\tau}\psi(\tau)\right) = \frac{L(\psi(\tau))}{\varsigma^{\omega}}, \tag{2.18} $$

$$ L\left({}^{C}_{a}D^{\omega}_{\tau}\psi(\tau)\right) = \varsigma^{\omega}L(\psi(\tau)) - \sum_{k=0}^{\eta-1}\varsigma^{\omega-k-1}\,\psi^{(k)}(0), \tag{2.19} $$

$$ L\left({}^{CF}_{a}D^{\gamma}_{\tau}\phi(\tau)\right) = \frac{M(\gamma)}{\gamma+\varsigma-\gamma\varsigma}\left(\varsigma L(\psi(\tau)) - \psi(0)\right), \tag{2.20} $$

$$L\left({}_{a}^{AB}D_{\tau}^{\gamma}\phi(\tau)\right)=\frac{M(\gamma)}{\gamma+\varsigma^{\gamma}-\gamma\varsigma^{\gamma}}\left(\varsigma^{\gamma}L\big(\psi(\tau)\big)-\psi(0)\right), \quad (2.21)$$

$$L\left({}_{a}^{HJ}D_{\tau}^{\omega}\phi(\tau)\right)=\frac{H_{\omega}}{\varsigma+H_{\omega}}\left(\varsigma^{\eta}L\big(\psi(\tau)\big)-\sum_{k=0}^{\eta-1}\varsigma^{\eta-k-1}\psi^{(k)}(0)\right). \quad (2.22)$$

3- Analysis of Method

This section deals with the algorithm of the Laplace Adomian decomposition method for the heat-like equation with the fractional differential operators that we mentioned in the previous section.

1. Analysis Of the Method with CFO

We will first start with the algorithm of the method with FCO, consider the following two-dimensional fractional heat-like equation with Caputo operator and $\gamma \in (0,1]$

$${}_{a}^{C}D_{\tau}^{\gamma}\psi(x,y,\tau)=\lambda\left(\frac{\partial^2}{\partial x^2}\psi(x,y,\tau)+\frac{\partial^2}{\partial y^2}\psi(x,y,\tau)\right), \quad (3.1)$$

with initial condition $\psi(x,y,0)=g(x,y)$. By taking Laplace transform to both sides of Eq.(3.1) and replacing the initial condition, we obtain

$$L\big(\psi(x,y,\tau)\big)=\frac{1}{\varsigma}\psi(x,y,0)+\frac{\lambda}{\varsigma^{\gamma}}L\left(\frac{\partial^2}{\partial x^2}\psi(x,y,\tau)+\frac{\partial^2}{\partial y^2}\psi(x,y,\tau)\right), \quad (3.2)$$

by using inverse Laplace transform, we get

$$\psi(x,y,\tau)=g(x,y)+\lambda L^{-1}\left(\frac{1}{\varsigma^{\gamma}}L\left(\frac{\partial^2}{\partial x^2}\psi(x,y,\tau)+\frac{\partial^2}{\partial y^2}\psi(x,y,\tau)\right)\right), \quad (3.3)$$

Now, we apply Adomian decomposition method. Let's rewrite $\psi(x,y,\tau)$ as follows

$$\psi(x,y,\tau)=\sum_{n=0}^{\infty}\psi_n(x,y,\tau). \quad (3.4)$$

By substituting Eq.(3.4) in Eq.(3.3), we get

$$\sum_{n=0}^{\infty}\psi_n(x,y,\tau)=g(x,y)+\lambda L^{-1}\left(\frac{1}{\varsigma^{\gamma}}L\left(\frac{\partial^2}{\partial x^2}\sum_{n=0}^{\infty}\psi_n(x,y,\tau)+\frac{\partial^2}{\partial y^2}\sum_{n=0}^{\infty}\psi_n(x,y,\tau)\right)\right), \quad (3.5)$$

putting $n = n + 1$ in left side, we obtain

$$\psi_0 + \sum_{n=0}^{\infty} \psi_{n+1} = g(x, y) + \lambda L^{-1}\left(\frac{1}{\varsigma^{\gamma}} L\left(\frac{\partial^2}{\partial x^2}\sum_{n=0}^{\infty} \psi_n + \frac{\partial^2}{\partial y^2}\sum_{n=0}^{\infty} \psi_n\right)\right), \tag{3.6}$$

by comparing both sides of Eq.(3.6), we have

$$\psi_0 = g(x, y), \tag{3.7}$$

$$\psi_1 = \lambda L^{-1}\left(\frac{1}{\varsigma^{\gamma}} L(\nabla^2 \psi_0)\right), \tag{3.8}$$

$$\psi_2 = \lambda L^{-1}\left(\frac{1}{\varsigma^{\gamma}} L(\nabla^2 \psi_1)\right), \tag{3.9}$$

$$\vdots$$

$$\psi_{n+1} = \lambda L^{-1}\left(\frac{1}{\varsigma^{\gamma}} L(\nabla^2 \psi_n)\right). \tag{3.10}$$

Therefore, the approximate solution to the heat-like equation with FCD is given by the following formula

$$\psi(x, y, \tau) = \psi_0 + \psi_1 + \psi_2 + \cdots = \sum_{n=0}^{\infty} \psi_n(x, y, \tau). \tag{3.11}$$

2. Analysis Of the Method with CFFO

Let's assume the following two-dimensional fractional heat-like equation with Caputo-Fabrizio operator and $\gamma \in (0,1]$

$${}^{CF}_{a}D^{\gamma}_{\tau}\psi(x, y, \tau) = \lambda\left(\frac{\partial^2}{\partial x^2}\psi(x, y, \tau) + \frac{\partial^2}{\partial y^2}\psi(x, y, \tau)\right), \tag{3.12}$$

with initial condition $\psi(x, y, 0) = g(x, y)$. The algorithm of the Laplace ADM for the heat equation with the fractional derivative Caputo-Fabrizio, in steps similar to the previous steps and using Eq.(2.20), we obtain

$$\psi_0 = g(x, y), \tag{3.13}$$

$$\psi_1 = \frac{\lambda}{M(\gamma)} L^{-1}\big((1 - \gamma + \gamma\varsigma^{-1}) L(\nabla^2 \psi_0)\big), \tag{3.14}$$

$$\psi_2 = \frac{\lambda}{M(\gamma)} L^{-1}\big((1-\gamma+\gamma\varsigma^{-1})L(\nabla^2\psi_1)\big), \tag{3.15}$$

$$\vdots$$

$$\psi_{n+1} = \frac{\lambda}{M(\gamma)} L^{-1}\big((1-\gamma+\gamma\varsigma^{-1})L(\nabla^2\psi_n)\big). \tag{3.16}$$

Thus, the approximate solution is given by

$$\psi(x,y,\tau) = \psi_0 + \psi_1 + \psi_2 + \cdots = \sum_{n=0}^{\infty} \psi_n(x,y,\tau). \tag{3.17}$$

3. Analysis Of the Method with ABFO

Let's assume the following two-dimensional fractional HLDE with ABFO and $\gamma \in (0,1]$

$${}^{AB}_{a}D^{\gamma}_{\tau}\psi(x,y,\tau) = \lambda\left(\frac{\partial^2}{\partial x^2}\psi(x,y,\tau) + \frac{\partial^2}{\partial y^2}\psi(x,y,\tau)\right), \tag{3.18}$$

with initial condition $\psi(x,y,0) = g(x,y)$. Now, in the same way and using Eq.(2.21), the approximate solution by using the Laplace ADM for the heat-like equation can be obtained with the fractional derivative Atangana-Baleanu

$$\psi_0 = g(x,y), \tag{3.19}$$

$$\psi_1 = \frac{\lambda}{M(\gamma)} L^{-1}\big((1-\gamma+\gamma\varsigma^{-\gamma})L(\nabla^2\psi_0)\big), \tag{3.20}$$

$$\psi_2 = \frac{\lambda}{M(\gamma)} L^{-1}\big((1-\gamma+\gamma\varsigma^{-\gamma})L(\nabla^2\psi_1)\big), \tag{3.21}$$

$$\vdots$$

$$\psi_{n+1} = \frac{\lambda}{M(\gamma)} L^{-1}\big((1-\gamma+\gamma\varsigma^{-\gamma})L(\nabla^2\psi_n)\big). \tag{3.22}$$

Thus, the approximate solution is given by

$$\psi(x,y,\tau) = \psi_0 + \psi_1 + \psi_2 + \cdots = \sum_{n=0}^{\infty} \psi_n(x,y,\tau). \tag{3.23}$$

4. Analysis Of the Method with HJFO

Let's assume the following two-dimensional fractional HLDE with HJFO and $\gamma \in (0,1]$

$${}^{CF}_{a}D^{\gamma}_{\tau}\psi(x,y,\tau) = \lambda\left(\frac{\partial^2}{\partial x^2}\psi(x,y,\tau) + \frac{\partial^2}{\partial y^2}\psi(x,y,\tau)\right), \tag{3.24}$$

with initial condition $\psi(x,y,0)=g(x,y)$. Finally, in the same way and using Eq.(2.22), the algorithm of the Laplace ADM for the approximate solution of the heat-like equation with the fractional derivative Hussein-Jassim, is given below:

$$\psi_0 = g(x,y), \tag{3.25}$$

$$\psi_1 = \frac{\lambda}{H_\gamma} L^{-1}\left(\left(1+H_\gamma \varsigma^{-1}\right) L\left(\nabla^2 \psi_0\right)\right), \tag{3.26}$$

$$\psi_2 = \frac{\lambda}{H_\gamma} L^{-1}\left(\left(1+H_\gamma \varsigma^{-1}\right) L\left(\nabla^2 \psi_1\right)\right), \tag{3.27}$$

$$\vdots$$

$$\psi_{n+1} = \frac{\lambda}{H_\gamma} L^{-1}\left(\left(1+H_\gamma \varsigma^{-1}\right) L\left(\nabla^2 \psi_n\right)\right). \tag{3.28}$$

Thus, the approximate solution is given by

$$\psi(x,y,\tau) = \psi_0 + \psi_1 + \psi_2 + \cdots = \sum_{n=0}^{\infty} \psi_n(x,y,\tau). \tag{3.29}$$

Analytical and Numerical Solutions

This section deals with the approximate and exact analytical solution of the two-dimensional fractional differential heat-like equation with the fractional derivatives mentioned previously and initial condition $\psi(x,y,0)=g(x,y)=\sin(x)\sin(y)$ using the Laplace Adomian decomposition method discussed in the previous section. Using the algorithm of Eq.(3.1) in the previous section, the approximate solution to the heat-like equation with the fractional derivative Caputo is in the following form

$$\begin{aligned}\psi_C = \sin(x)\sin(y) &- 2\frac{\sin(x)\sin(y)\,\tau^{\gamma}}{\Gamma(1+\gamma)} + 4\frac{\sin(x)\sin(y)\,\tau^{2\gamma}}{\Gamma(1+2\gamma)} - 8\frac{\sin(x)\sin(y)\,\tau^{3\gamma}}{\Gamma(1+3\gamma)} \\ &+ 16\frac{\sin(x)\sin(y)\,\tau^{4\gamma}}{\Gamma(1+4\gamma)} - 32\frac{\sin(x)\sin(y)\,\tau^{5\gamma}}{\Gamma(1+5\gamma)} + 64\frac{\sin(x)\sin(y)\,\tau^{6\gamma}}{\Gamma(1+6\gamma)} \\ &- 128\frac{\sin(x)\sin(y)\,\tau^{7\gamma}}{\Gamma(1+7\gamma)} + \cdots.\end{aligned}$$

Applying the algorithm of Eq.(3.11), the approximate solution to the heat-like equation with the Atangana-Baleanu fractional operator is as follows.

$$
\begin{aligned}
\psi_{AB} &= 16z\gamma^4 \sin(x)\sin(y) - 56\gamma^3 \sin(x)\sin(y) \\
&+ 8\frac{\sin(x)\sin(y)\sqrt{2}\pi^{\frac{3}{2}}\sqrt{4}\gamma^3 (t^\gamma)^4}{(4^\gamma)^4\Gamma(\gamma)\Gamma\left(\gamma+\frac{1}{4}\right)\Gamma\left(\frac{1}{2}+\gamma\right)\Gamma\left(\gamma+\frac{3}{4}\right)} 16\gamma^4 si\,n(x)\,si\,n(y) - 56\gamma^3 si\,n(x)\,si\,n(y) \\
&+ 8\frac{\sin(x)\sin(y)\sqrt{2}\pi^{\frac{3}{2}}\sqrt{4}\gamma^3 (t^\gamma)^4}{(4^\gamma)^4\Gamma(\gamma)\Gamma\left(\gamma+\frac{1}{4}\right)\Gamma\left(\frac{1}{2}+\gamma\right)\Gamma\left(\gamma+\frac{3}{4}\right)} - 64\frac{\sin(x)\sin(y)\,t^\gamma\gamma^3}{\Gamma(\gamma)} \\
&+ 96\frac{\sin(x)\sin(y)\sqrt{\pi}(t^\gamma)^2\gamma^3}{\Gamma(\gamma)\Gamma\left(\frac{1}{2}+\gamma\right)(2^\gamma)^2} - \frac{128\sin(x)\sin(y)\,\pi\sqrt{3}\gamma^3(t\tau^\gamma)^3}{3\Gamma(\gamma)(3^3)^3\Gamma\left(\frac{1}{3}+\gamma\right)\Gamma\left(\gamma+\frac{2}{3}\right)} + 76\gamma^2 si\,n(x)\,si\,n(y) \\
&+ 168\frac{\sin(x)\sin(y)\,t^\gamma\gamma^2}{\Gamma(\gamma)} - 168\frac{\sin(x)\sin(y)\sqrt{\pi}(t^\gamma)^2\gamma^2}{\Gamma(\gamma)\Gamma\left(\frac{1}{2}+\gamma\right)(2^\gamma)^2} \\
&+ \frac{112\sin(x)\sin(y)\,\pi\sqrt{3}\gamma^2(t^\gamma)^3}{3\Gamma(\gamma)(3^2)^3\Gamma\left(\frac{1}{3}+\gamma\right)\Gamma\left(\gamma+\frac{2}{3}\right)} - 46\gamma si\,n(x)\,si\,n(y) - 152\frac{\sin(x)\sin(y)\,t^\gamma\gamma}{\Gamma(\gamma)} \\
&+ 76\frac{\sin(x)\sin(y)\sqrt{\pi}\gamma(t^\gamma)^2}{\Gamma(\gamma)\Gamma\left(\frac{1}{2}+\gamma\right)(2^\gamma)^2} + 11si\,n(x)\,si\,n(y) + 46\frac{\sin(x)\sin(y)\,t^\gamma}{\Gamma(\gamma)} + \cdots.
\end{aligned}
$$

Using the algorithm of Eq.(3.18), the approximate solution to the heat-like equation with the Caputo-Fabrizio fractional operator is shown below.

$$\begin{aligned}\psi_{CF} = &-85\sin(x)\sin(y) - \frac{8\sin(x)\sin(y)\tau^7\gamma^7}{315} + \frac{56\sin(x)\sin(y)\tau^6\gamma^7}{45} - \frac{52\sin(x)\sin(y)\tau^6\gamma^6}{45} \\
&- \frac{112\sin(x)\sin(y)\tau^5\gamma^7}{5} + \frac{208\sin(x)\sin(y)\tau^5\gamma^6}{5} + \frac{560\sin(x)\sin(y)\tau^4\gamma^7}{3} \\
&- \frac{292\sin(x)\sin(y)\tau^5\gamma^5}{15} - 520\sin(x)\sin(y)\tau^4\gamma^6 - \frac{2240\sin(x)\sin(y)\tau^3\gamma^7}{3} \\
&+ \frac{1460\sin(x)\sin(y)\tau^4\gamma^5}{3} + \frac{8320\sin(x)\sin(y)\tau^3\gamma^6}{3} + 1344\sin(x)\sin(y)\tau^2\gamma^7 \\
&- \frac{458\sin(x)\sin(y)\tau^4\gamma^4}{3} - \frac{11680\sin(x)\sin(y)\tau^3\gamma^5}{3} - 6240\sin(x)\sin(y)\tau^2\gamma^6 \\
&- 896\sin(x)\sin(y)\tau\gamma^7 + \frac{7328\sin(x)\sin(y)\tau^3\gamma^4}{3} + 11680\sin(x)\sin(y)\tau^2\gamma^5 \\
&+ 4992\sin(x)\sin(y)\tau\gamma^6 - \frac{1732\sin(x)\sin(y)\tau^3\gamma^3}{3} - 10992\sin(x)\sin(y)\tau^2\gamma^4 \\
&- 11680\sin(x)\sin(y)\tau\gamma^5 + 5196\sin(x)\sin(y)\tau^2\gamma^3 + 14656\sin(x)\sin(y)\tau\gamma^4 \\
&- 986\sin(x)\sin(y)\tau^2\gamma^2 - 10392\sin(x)\sin(y)\tau\gamma^3 + 3944\sin(x)\sin(y)\tau\gamma^2 \\
&- 626\sin(x)\sin(y)\tau\gamma + 128\sin(x)\sin(y)\gamma^7 + 2336\sin(x)\sin(y)\gamma^5 \\
&- 3664\sin(x)\sin(y)\gamma^4 + \cdots.\end{aligned}$$

Applying the algorithm of Eq.(3.24), the approximate solution to the heat-like equation with the Hussein-Jassim fractional operator is as follows

$$\psi_{HJ} = -85\sin(x)\sin(y) + \frac{56\sin(x)\sin(y)\tau^6\gamma}{45} - \frac{112\sin(x)\sin(y)\tau^5\gamma^2}{5} + \frac{560\sin(x)\sin(y)\tau^4\gamma^3}{3} - \frac{2240\sin(x)\sin(y)\tau^3\gamma^4}{3} + 1344\sin(x)\sin(y)\tau^2\gamma^5 - 896\sin(x)\sin(y)\tau\gamma^6 + \frac{208\sin(x)\sin(y)\tau^5\gamma}{5} - 520\sin(x)\sin(y)\tau^4\gamma^2 + \frac{8320\sin(x)\sin(y)\tau^3\gamma^3}{3} - 6240\sin(x)\sin(y)\tau^2\gamma^4 + 4992\sin(x)\sin(y)\tau\gamma^5 + \frac{1460\sin(x)\sin(y)\tau^4\gamma}{3} - \frac{11680\sin(x)\sin(y)\tau^3\gamma^2}{3} + 11680\sin(x)\sin(y)\tau^2\gamma^3 - 11680\sin(x)\sin(y)\tau\gamma^4 + \frac{7328\sin(x)\sin(y)\tau^3\gamma}{3} - 10992\sin(x)\sin(y)\tau^2\gamma^2 + 14656\sin(x)\sin(y)\tau\gamma^3 + 5196\sin(x)\sin(y)\tau^2\gamma - 10392\sin(x)\sin(y)\tau\gamma^2 + 3944\sin(x)\sin(y)\tau\gamma - \frac{8\sin(x)\sin(y)\tau^7}{315} + 128\sin(x)\sin(y)\gamma^7 - \frac{52\sin(x)\sin(y)\tau^6}{45} - 832\sin(x)\sin(y)\gamma^6 - \frac{292\sin(x)\sin(y)\tau^5}{15} + 2336\sin(x)\sin(y)\gamma^5 - \frac{458\sin(x)\sin(y)\tau^4}{3} - 3664\sin(x)\sin(y)\gamma^4 - \frac{1732\sin(x)\sin(y)\tau^3}{3} + 3464\sin(x)\sin(y)\gamma^3 + \cdots.$$

Thus, the approximate solution of Eq.(3.1), Eq.(3.12), Eq.(3.18) and Eq.(3.24) at $\gamma \to 1$ is given by

$$\psi_{\gamma\to 1} = \sin(x)\sin(y) - \frac{8\sin(x)\sin(y)t^7}{315} + \frac{4\sin(x)\sin(y)t^6}{45} - \frac{4\sin(x)\sin(y)t^5}{15} + \frac{2}{3}\sin(x)\sin(y)\,t^4 - \frac{4}{3\sin(x)\sin(y)\,t^3} + 2\sin(x)\sin(y)\,t^2 - 2\sin(x)\sin(y)\,t + \cdots.$$

Table 1. The values of Eq.(3.1) with deferent values of x, y, τ, γ.

x, y, τ	$\psi_{\gamma\to0.8}$	$\psi_{\gamma\to0.9}$	$\psi_{\gamma\to1}$	ψ_{Exact}	$\lvert\psi_{Exact}-\psi_{\gamma\to1}\rvert$
0.001254638763000	0.000001558083065	0.000001566129455	0.000001570172653	0.000001570172653	0.000000000000000
0.101129174886700	0.007322021361579	0.007852961468998	0.008325918635910	0.008325919577721	0.000000000941812
0.201003711010400	0.022830427993020	0.024721601331540	0.026666019442540	0.026666240348711	0.000000220906171
0.300878247134100	0.041367264886107	0.044555364715519	0.048111505601532	0.048116834785285	0.000005329183754
0.400752783257800	0.059624723131874	0.063762003984836	0.068228963187319	0.068279174877708	0.000050211690389
0.500627319381500	0.074164058324520	0.079889940902193	0.084363211690669	0.084644671357633	0.000281459666964
0.600501855505200	0.079882267498073	0.090584431701093	0.094938828262316	0.096071511108056	0.001132682845741
0.700376391628900	0.068615068962795	0.092578889804851	0.098738351758099	0.102356182685923	0.003617830927824
0.800250927752600	0.027996470415254	0.080717291530307	0.094157400401064	0.103894386346209	0.009736985945145
0.900125463876300	-0.059186432302692	0.047111980081588	0.078473340108954	0.101422314144214	0.022948974035259
1.000000000000000	-0.214886281308912	-0.019395843250291	0.047204894551572	0.095827316614370	0.048622422062799

Table 2. The values of Eq.(3.12) with deferent values of x, y, τ, γ.

x, y, τ	$\psi_{\gamma\to0.8}$	$\psi_{\gamma\to0.9}$	$\psi_{\gamma\to1}$	ψ_{Exact}	$\lvert\psi_{Exact}-\psi_{\gamma\to1}\rvert$
0.001254638763000	0.000001112646805	0.000001306673061	0.000001570172653	0.000001570172653	0.000000000000000
0.101129174886700	0.005697407785631	0.006965327142862	0.008325918635909	0.008325919577721	0.000000000941812
0.201003711010400	0.017149284736820	0.022914866863428	0.026666019442542	0.026666240348711	0.000000220906169
0.300878247134100	0.024716690119127	0.042291894225796	0.048111505601535	0.048116834785285	0.000005329183750
0.400752783257800	0.014489333417997	0.059813727714014	0.068228963187317	0.068279174877708	0.000050211690391
0.500627319381500	-0.033093097421554	0.069192408313578	0.084363211690677	0.084644671357633	0.000281459666956
0.600501855505200	-0.143759274217139	0.062011538085400	0.094938828262347	0.096071511108056	0.001132682845709
0.700376391628900	-0.349000749685544	0.026801722621862	0.098738351758108	0.102356182685923	0.003617830927815
0.800250927752600	-0.684638325319206	-0.051533129511149	0.094157400401066	0.103894386346209	0.009736985945143
0.900125463876300	-1.188325355027324	-0.191762202486800	0.078473340108971	0.101422314144214	0.022948974035243
1.000000000000000	-1.896056682310331	-0.415687024166189	0.047204894551570	0.095827316614370	0.048622422062801

Table 3. The values of Eq.(3.18) with deferent values of x, y, τ, γ.

x, y, τ	$\psi_{\gamma\to0.8}$	$\psi_{\gamma\to0.9}$	$\psi_{\gamma\to1}$	ψ_{Exact}	$\lvert\psi_{Exact}-\psi_{\gamma\to1}\rvert$
0.001254638763000	0.000001118020399	0.000001309208743	0.000001570172653	0.000001570172653	0.000000000000000
0.101129174886700	0.006348555617105	0.007290124745694	0.008325918635909	0.008325919577721	0.000000000941812
0.201003711010400	0.021286766565994	0.024448040985776	0.026666019442541	0.026666240348711	0.000000220906170

0.300878247134100	0.038511290150165	0.045863554840736	0.048111505601533	0.048116834785285	0.000005329183753
0.400752783257800	0.050252075694614	0.066620214174826	0.068228963187316	0.068279174877708	0.000050211690392
0.500627319381500	0.046570034539933	0.081861476225392	0.084363211690679	0.084644671357633	0.000281459666954
0.600501855505200	0.014760012409180	0.086056864971710	0.094938828262322	0.096071511108056	0.001132682845734
0.700376391628900	-0.060991408669679	0.072259945127236	0.098738351758058	0.102356182685923	0.003617830927864
0.800250927752600	-0.199544153001277	0.031477801486282	0.094157400401116	0.103894386346209	0.009736985945093
0.900125463876300	-0.422207836658650	-0.047708796053662	0.078473340108900	0.101422314144214	0.022948974035314
1.000000000000000	-0.751517026827301	-0.179131245648520	0.047204894551541	0.095827316614370	0.048622422062829

Table 4. The values of Eq.(3.24) with deferent values of x, y, τ, γ.

x, y, τ	$\psi_{\gamma\to 0.8}$	$\psi_{\gamma\to 0.9}$	$\psi_{\gamma\to 1}$	ψ_{Exact}	$\lvert\psi_{Exact} - \psi_{\gamma\to 1}\rvert$
0.001254638763000	0.000001117584543	0.000001308934357	0.000001570172653	0.000001570172653	0.000000000000000
0.101129174886700	0.006122676854777	0.007166530610375	0.008325918635909	0.008325919577721	0.000000000941812
0.201003711010400	0.019418218019035	0.023603836946491	0.026666019442542	0.026666240348711	0.000000220906168
0.300878247134100	0.031484746279150	0.043334756608928	0.048111505601534	0.048116834785285	0.000005329183751
0.400752783257800	0.030696982846672	0.061036319673086	0.068228963187316	0.068279174877708	0.000050211690392
0.500627319381500	0.000290599485105	0.071036082541795	0.084363211690693	0.084644671357633	0.000281459666940
0.600501855505200	-0.083425425506384	0.066156469168632	0.094938828262322	0.096071511108056	0.001132682845734
0.700376391628900	-0.252369011246053	0.036539232922323	0.098738351758051	0.102356182685923	0.003617830927871
0.800250927752600	-0.547109976159803	-0.031366532084263	0.094157400401123	0.103894386346209	0.009736985945086
0.900125463876300	-1.016048632516380	-0.155431185364272	0.078473340108886	0.101422314144214	0.022948974035328
1.000000000000000	-1.713046741318692	-0.357850864616609	0.047204894551498	0.095827316614370	0.048622422062872

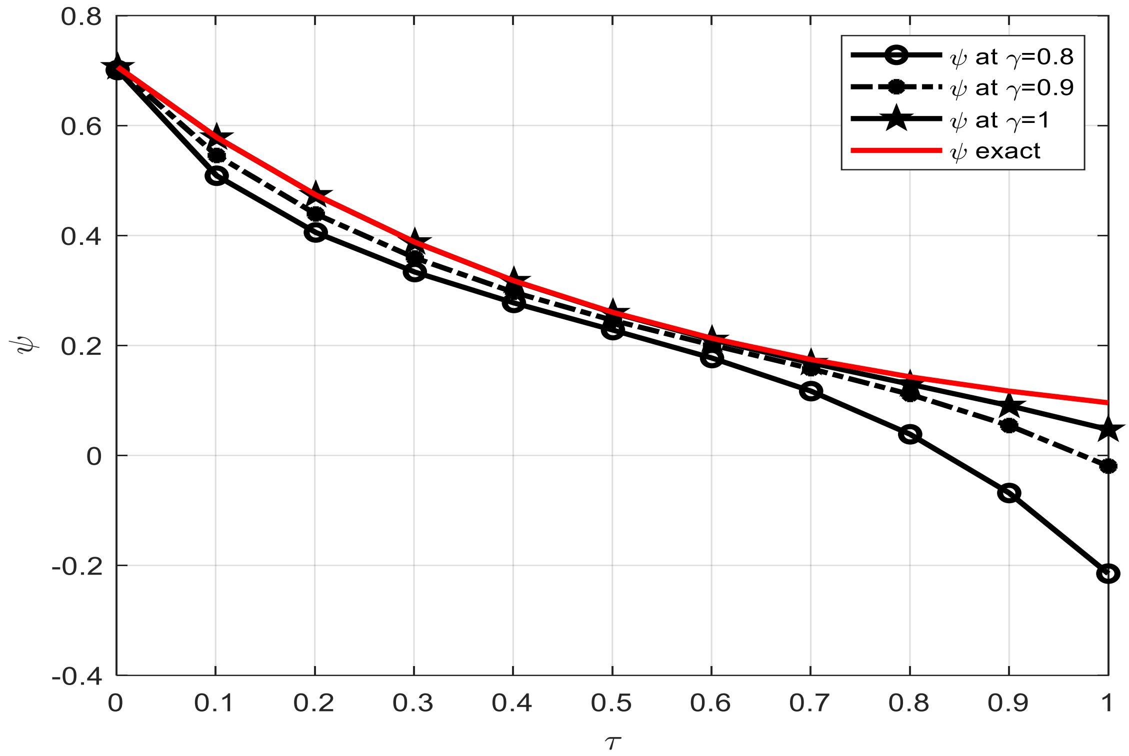

Figure 1. The approximate and exact solutions for different γ values when $x = y = 1$ of Eq.(3.1).

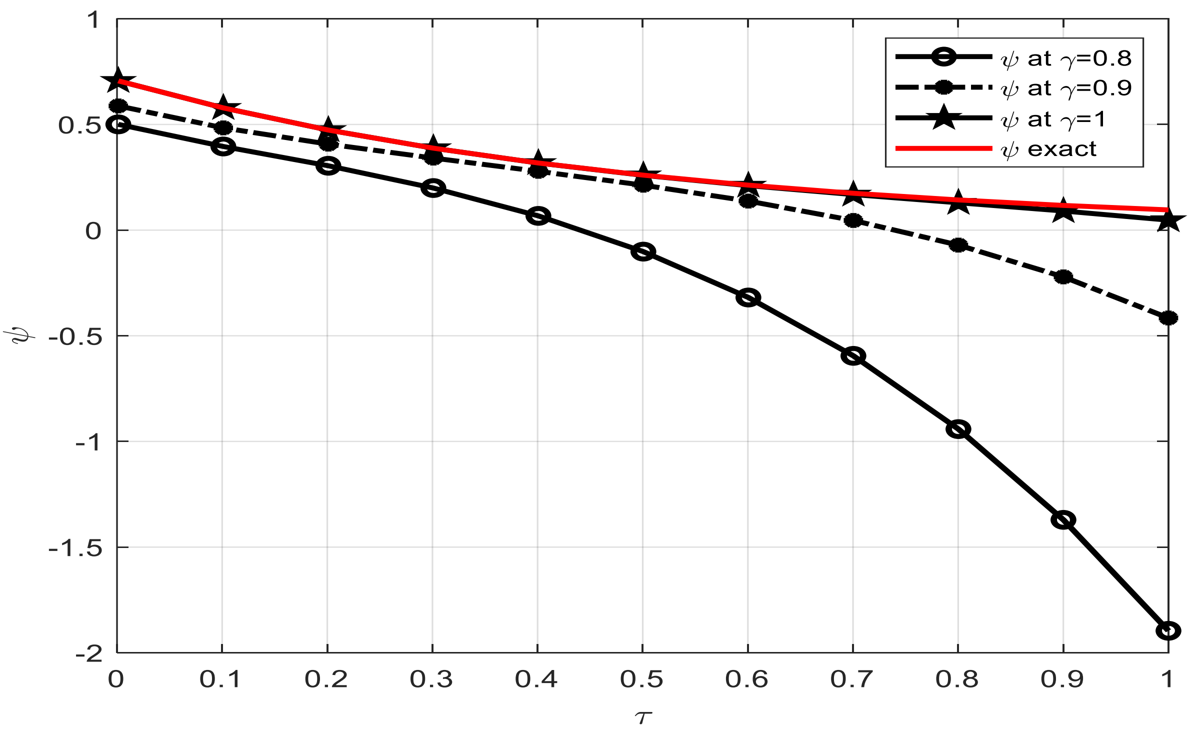

Figure 2. The approximate and exact solutions for different γ values when $x = y = 1$ of Eq.(3.12).

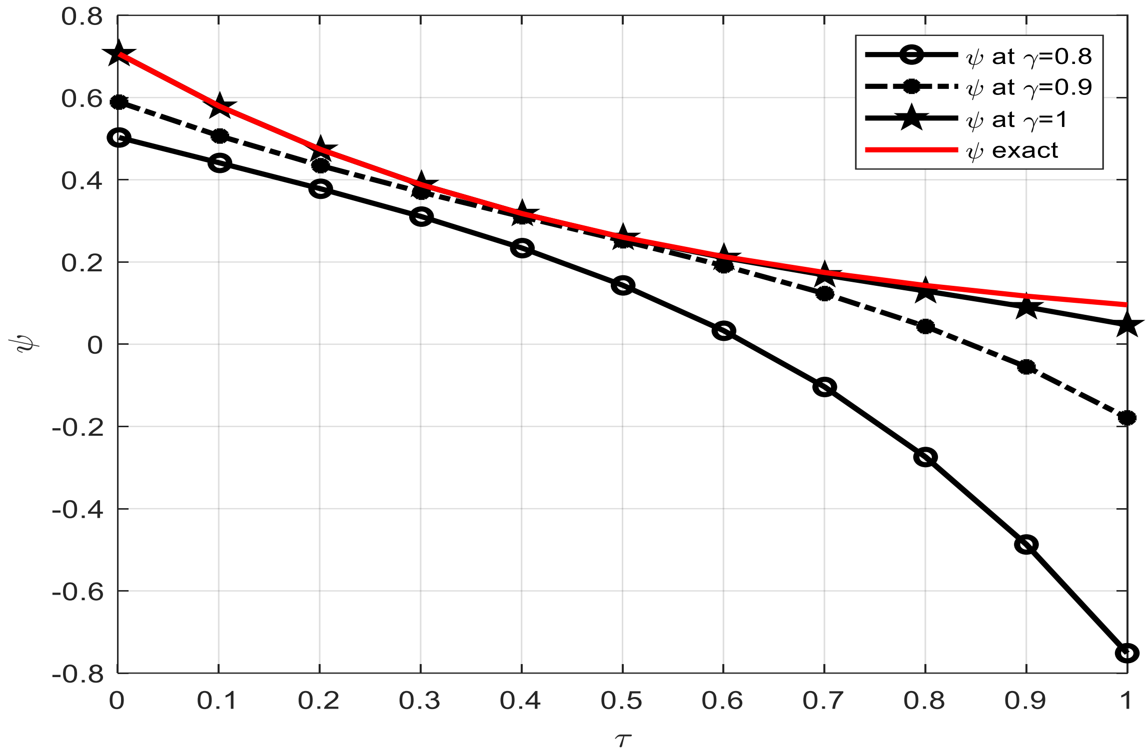

Figure 3. The approximate and exact solutions for different γ values when $x = y = 1$ of Eq.(3.18).

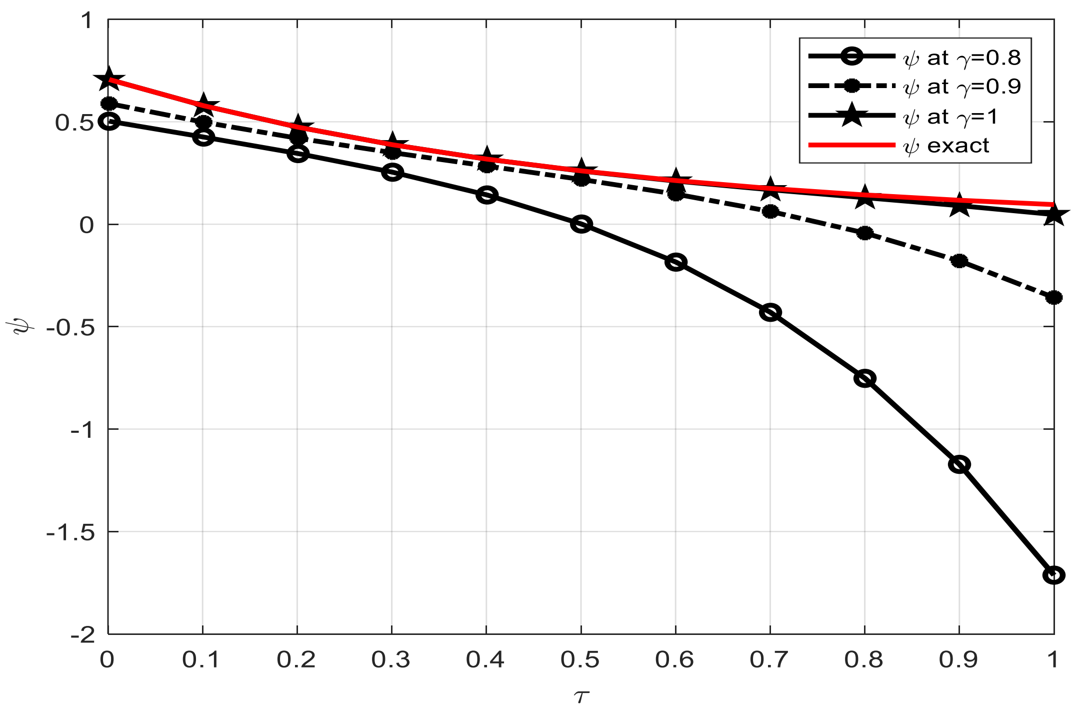

Figure 4. The approximate and exact solutions for different γ values when $x = y = 1$ of Eq.(3.24).

5- Conclusions

This work provides novel analytical and numerical solutions to the fractional heat-like problem utilizing the Laplace-Adomian decomposition approach. The approximate solutions found demonstrate that the strategy employed was successful and efficient. Tables 1, 2, 3, and 4, as well as Figures 1, 2, 3, and 4, demonstrate that the obtained solutions converge to the precise answer with great effectiveness and efficiency. As a result, this subject may be expanded to include various forms of fractional differential equations.

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