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Common Fixed Point Findings Using C-Class Function in Ǵ-Space via Applications

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Abstract

This paper aims to establish several coincidence and common fixed point theorems for self-mappings that satisfy contractive conditions, using the notion of the C-class function of symmetrical G-metric spaces. We also present some examples to demonstrate the validity of our results. Finally, we apply our fixed-point result to solve an integral equation.

Keywords: Fixed points, Ǵ-space, C-class functions, Common point.

1 Introduction

The Banach contraction principle is one of the first and most significant conclusions in fixed-point theory in [1] clearly proved it, using it to establish the existence of a solution to an integral equation. This principle is one of the field's most important findings.

This notion states that if a space (X, d) is a complete metric space, then a contraction map $T: X \rightarrow X$ possesses a single fixed point. Many authors have refined, extended and deepened the scope of this idea in nonlinear analysis due to its applications in a wide variety of scientific fields, both within and outside of mathematics. Certain generalizations, such as [2–6] and others, reduce the contractive nature of the map.

Nadler [7] expanded the Banach fixed point theorem in 1969 to include set-valued contractive mappings in addition to single-valued ones. Jungck [8] first introduced the concept of commuting maps in 1976. The 1984 introduction of the modifying distance function by M. S. Khan et al. [9] is a major development in metric spaces and may affect many mathematical applications. In addition to this, he generalized the fixed-point theorem of Banach.

In 2006, Z. Mustafa and B. Sims [10] introduced $\mathrm{G}-$metric, a new category of generalized metric space. This category was a generalization of a metric space (X, d). In fact, The $\mathrm{G}-$metric structure has been the subject of much research by several scholars who have tested various fixed-point theorems for self-mappings for instance, we direct readers to References ([11-18]).

As a major generalization of the Banach contraction principle, Ansari [19] presented the concept of the C-class function and achieved several fixed-point results on the basis of this generalization. Following that, a great number of authors were interested in obtaining common fixed-point theorems for C-class functions (see [20–22]). A few writers' recent research has led to the discovery of fixed points and common fixed points for C-class functions [23–26].

For contractive self-mappings, this study uses the C-class function of symmetrical G-metric spaces to prove many coincidences and common fixed point theorems. We also provide examples to support our findings. Finally, we solve an integral equation using our fixed-point result.

2 Materials and Methods

Mustafa and Sims [10] provided the subsequent definitions and supplementary findings in $\acute{G}$-metric domains, which we will need:

Definition 2.1. [10] Let E be a non-empty set and $\acute{G}: E \times E \times E \rightarrow \mathbb{R}^{+}$ be a function satisfying:

i. $\acute{G}(\pi, \nu, \varpi)=0$; if and only if $\pi=\nu=\varpi$.
ii. $0<\acute{G}(\pi, \pi, \nu)$; for all $\pi, \nu \in E$ with $\pi \neq \nu$.
iii. $\acute{G}(\pi, \pi, \nu) \leq \acute{G}(\pi, \nu, \varpi)$ for all $\pi, \nu, \varpi \in E$ in which $\varpi \neq \nu$.
iv. $\acute{G}(\pi, \nu, \varpi)=\acute{G}(\pi, \varpi, \nu)=\acute{G}(\nu, \varpi, \pi)=\cdots$.
v. $\acute{G}(\pi, \nu, \varpi) \leq \acute{G}(\pi, a, a)+\acute{G}(a, \nu, \varpi)$ for all $\pi, \nu, \varpi, a \in E$.

The pair $(E, \acute{G})$ is referred to as a $\acute{G}-$metric space and we will denoted by $\acute{G}-$space, and the function $\acute{G}$ is referred to as a $\acute{G}-$metric on E.

Definition 2.2. [10] Consider that $(E, \acute{G})$ is a $\acute{G}-$space, If $\acute{G}(\pi, \nu, \nu)=\acute{G}(\nu, \pi, \pi)$ for any $\pi, \nu \in E$, then $(E, \acute{G})$ is termed symmetric.

Example 2.3. [10] These are a few instances of $\acute{G}-$space,

i. Assume that (E, d) is a metric space, Describe $\acute{G}: E \times E \times E \rightarrow[0,1)$ via
$$\acute{G}(\pi, v, z)=d(\pi, v)+d(v, z)+d(\pi, \varpi), \qquad \text{for all } \pi, v, \varpi \in X,$$
It is evident that $(E, \acute{G})$ is a symmetric G-space.

ii. Assume that $E=\{a, b\}$, Define the function G via
$$\acute{G}(a, a, a)=\acute{G}(b, b, b)=0,$$
$$\acute{G}(a, a, b)=1,$$

$$\acute{G}(a,b,b) = 2.$$

Applying the symmetry in the variables, expand $\acute{G}$ to $E \times E \times E$. Thus, $(E, \acute{G})$ is a $\acute{G}$ –space, But $\acute{G}(a,a,b) \neq \acute{G}(a,b,b)$.

Proposition 2.4. [10] Consider the $\acute{G}$ −space E. Then, for every π, ν, ϖ, a ∈ E, it concludes that

i. İf $\acute{G}(\pi,\nu,\varpi) = 0$; then $\pi = \nu = \varpi$.
ii. $\acute{G}(\pi,\nu,\varpi) \leq \acute{G}(\pi,\pi,\nu) + \acute{G}(\pi,\pi,\varpi)$.
iii. $\acute{G}(\pi,\nu,\nu) \leq 2\acute{G}(\nu,\pi,\pi)$.
iv. $|\acute{G}(\pi,\nu,\varpi) - \acute{G}(\pi,\nu,a)| \leq \acute{G}(\pi,a,\varpi)$.
v. $|\acute{G}(\pi,\nu,\nu) - \acute{G}(\nu,\pi,\pi)| \leq \max\{\acute{G}(\nu,\pi,\pi), \acute{G}(\pi,\nu,\nu)\}$.

Definition 2.5. [10] Consider the space $(E, \acute{G})$ to be a $\acute{G}$ −space, and let $\{\pi_n\}$ represent a sequence of points in E. Then

i. $\{\pi_n\}$ is $\acute{G}$-convergent to x if

$$\lim_{n,m\to\infty} \acute{G}(\pi,\pi_n,\pi_m) = 0\,;$$

That is, for all $\varepsilon > 0$, there is $k \in \mathbb{N}$ in which: $\acute{G}(\pi,\pi_n,\pi_m) < \epsilon$, for all $n,m \geq k$,
We call π the limit of the sequence and write $\pi_n \to \pi$ or $\lim_{n\to\infty} \pi_n = \pi$.

ii. The sequence $\{\pi_n\}$ is said to be $\acute{G}$-Cauchy, if given any $\epsilon > 0$, there is $k \in \mathbb{N}$ in which:

$$\acute{G}(\pi_n,\pi_m,\pi_l) < \epsilon, \text{ for all } n,m,l \geq k,$$

It means that, if $\acute{G}(\pi_n,\pi_m,\pi_l) \to 0$ as $n,m,l \to \infty$.

iii. $(E, \acute{G})$ is referred to as complete $\acute{G}$-space if every $\acute{G}$ −Cauchy sequence in $(E, \acute{G})$is $\acute{G}$ −convergent in E.

We need this extra information to establish a context and background for the concepts we are currently addressing, Arslan Hojat Ansari's 2014 [19] introduction of C-class functions broadens the comprehension and application of contractive conditions in $\acute{G}$ −spaces.

Definition 2.6. [19] A continuous function $F: [0,+\infty) \times [0,+\infty) \to \mathbb{R}$ is referred to as a C-class if the next criteria are met for every $s,t \in [0,+\infty)$

C_1: $F(s,t) \leq s$.
C_2: $F(s,t) = s$, indicates that either $s = 0$ or $t = 0$.

If necessary, an additional condition on F that $F(0,0) = 0$ could be applied in certain circumstances, All C-class functions will have their class indicated by the letter C.

Example 2.7. [19] The following instances demonstrate that the class C is not empty:

i. $F(s,t) = s - t$ and $F(s,t) = s$ then, $t = 0$.
ii. $F(s,t) = ms$; for some $m \in (0,1)$ and $F(s,t) = s$ then, $s = 0$.

iii. $F(s,t) = \frac{s}{1+t}$, $F(s,t) = s$ then, $s = 0$ or $t = 0$.

Definition 2.8. [9] A function $\psi: [0, +\infty) \to [0, +\infty)$ refers to a changing distance, If the subsequent conditions are fulfilled:

i. ψ is a continuous and non-decreasing;
ii. $\psi(t) = 0$, if and only if $t = 0$.

Let us assume that Ψ denotes the type of the altering distance functions.

Definition 2.9. [19] A function $\varphi: [0, \infty) \to [0, \infty)$ refers to an Ultra-altering distance functions, In situations where the following characteristics manifest:

i. φ is a continuous;
ii. $\varphi(0) \geq 0$ and $\varphi(t) > 0, t \neq 0$.

Let us assume that Φ denote the class of the functions for Ultra-altering distance.

Definition 2.10. [19] A sequence (ψ, φ, F) is considered monotone if, for all $\pi, y \in [0,1]$, $x \leq y$ implies $F(\psi(\pi), \varphi(\pi)) \leq F(\psi(y), \varphi(y))$ and ψ and φ are real numbers, where $\psi \in \Psi$;$\varphi \in \Phi$ and $F \in C$.

Example 2.11. [19] Considering $F(s,t) = s - t$, $\varphi(\pi) = \sqrt{\pi}$

$$\psi(\pi)=\begin{cases} \sqrt{\pi} & \text{if} \quad 0 \leq \pi \leq 1 \\ \pi^2 & \text{if} \quad \pi > 1 \end{cases}$$

then (ψ, φ, F) is a monotone sequence.

Definition 2.12. [27] Consider that S and T be two self-mappings on a non-empty set E. Then

i. When $T\pi = \pi$, we say that π, which is a point in E, is a fixed point of T.
ii. A point $\pi \in E$ is referred to as a coincidence point of S and T if $S\pi = T\pi$, and we will refer to it as $\omega = S\pi = T\pi$, which is a point of coincidence of S and T.
iii. The point $\pi \in E$ is considered a common fixed point of S and T if $\pi = S\pi = T\pi$.

In this paper, the authors extend the results of previous work (see Kumar [28]) by using the concept of a symmetric Ǵ-space to establish coincidence and common fixed point theorems. They also apply their main theory to derive results for metric distances. We have used C- class functions as a key aspect of our approach, as these functions are known to play a central role in the study of fixed point theory and related topics. This work contributes to understanding the coincidence and common fixed points in these specific types of spaces.

3 Main Results

In this section, we present the key findings from our research on Ǵ-space. The results provide valuable insights into coincidence and common fixed-points.

Theorem 3.1. Consider that $(E, Ǵ)$ is a symmetric complete Ǵ-space and $S, T: E \to E$, If

i. $S(E) \subseteq T(E)$;

ii. $T(E)$ is closed;
iii. S is T- non-decreasing;
iv. There is $\pi_0 \in E$ with $T\pi_0 \leq S\pi_0$;
v. if $\{Tx_n\} \subset X$ is a non-decreasing sequence $(w, r, t, \leq)$ with $Tx_n \to Tz$ in $T(E)$, then $Tu \leq T(Tu)$ and $T\pi_n \leq Tu$, for all $n \in N$;
vi. There is C-class function F in which for every $(\pi, y) \in E \times E$ with $T\pi \leq Ty$, we have

$$\psi\big(\acute{G}(S\pi, Sv, S\varpi)\big) \leq F(\psi\left(H(S,T,\pi,v,\varpi)\right), \varphi\big(H(S,T,\pi,v,\varpi)\big)\Big). \qquad \textbf{(3,1)}$$

where

$$H(S,T,\pi,v,\varpi) = \max\{\acute{G}(T\pi, Tv, T\varpi), \acute{G}(Tv, S\pi, T\varpi), \acute{G}(Tv, Sv, T\varpi)\},$$

$F: [0,+\infty)^2 \to \mathbb{R}$ is C-class function

$\psi: [0,+\infty) \to [0,+\infty)$ is an altering distance function

And $\varphi: [0,+\infty) \to [0,+\infty)$ is an ultra-altering distance function

Then, S and T possess a point of coincidence. Additionally, if S and T commute at the points of coincidence. At that point, S and T possess a common fixed point.

Prior to proving Theorem 3.1, we first demonstrate the following lemmas, which are necessary for the subsequent part of the proof.

Lemma 3.2. Consider that the space $(E,\acute{G})$is a symmetric $\acute{G}$-space, and that the criteria of Theorem 3.1 are satisfied of S and T. If $\{\pi_n\}$ is a sequence in E in which $T\pi_{n+1} = S\pi_n$ for all $n \in \mathbb{N}$, and $T\pi_n \neq T\pi_{n+1}$; for all $n \in \mathbb{N}$, then

$$\lim_{n\to\infty} \acute{G}(T\pi_n, T\pi_n, T\pi_{n+1}) = 0,$$

Proof. By Theorem 3.1, (iii) and (iv) implications allow us to derive

$$T\pi_0 \leq T\pi_1 \leq T\pi_2 \leq \cdots \leq T\pi_n \leq T\pi_{n+1},$$

By Theorem 3.1 and (vi) meaning is that

$$\psi\big(\acute{G}(S\pi_{n-1}, S\pi_{n-1}, S\pi_n)\big) \leq F\Big(\psi\big(H(S,T,\pi_{n-1},\pi_{n-1},\pi_n)\big), \varphi\big(H(S,T,\pi_{n-1},\pi_{n-1},\pi_n)\big)\Big)$$

That is,

$$\psi\big(\acute{G}(T\pi_n, T\pi_n, T\pi_{n+1})\big) \leq F\Big(\psi\big(H(S,T,\pi_{n-1},\pi_{n-1},\pi_n)\big), \varphi\big(H(S,T,\pi_{n-1},\pi_{n-1},\pi_n)\big)\Big)$$

Where

$$H(S,T,\pi_{n-1},\pi_{n-1},\pi_n) = \max\{\acute{G}(T\pi_{n-1}, T\pi_{n-1}, T\pi_n), \acute{G}(T\pi_{n-1}, S\pi_{n-1}, T\pi_n), \acute{G}(T\pi_{n-1}, S\pi_{n-1}, T\pi_n)\}$$

Again, using the assumption of lemma 3.2, we have

$$H(S,T,\pi_{n-1},\pi_{n-1},\pi_n) = \max\{\acute{G}(T\pi_{n-1}, T\pi_{n-1}, T\pi_n), \acute{G}(T\pi_{n-1}, T\pi_n, T\pi_n), \acute{G}(T\pi_{n-1}, T\pi_n, T\pi_n)\}$$

$$= \max\{\acute{G}(T\pi_{n-1}, T\pi_{n-1}, T\pi_n), \acute{G}(T\pi_{n-1}, T\pi_n, T\pi_n)\}$$

Given that $(E, \acute{G})$ is a symmetric $\acute{G}$-space, it follows that

$$H(S, T, \pi_{n-1}, \pi_{n-1}, \pi_n) = \acute{G}(T\pi_{n-1}, T\pi_{n-1}, T\pi_n)$$

From the condition C_1, we have

$$\psi\big(\acute{G}(T\pi_n, T\pi_n, T\pi_{n+1})\big) \le \psi\big(\acute{G}(T\pi_{n-1}, T\pi_{n-1}, T\pi_n)\big)$$

By the non-decreasing of ψ, it follows that

$$\acute{G}(T\pi_n, T\pi_n, T\pi_{n+1}) \le \acute{G}(T\pi_{n-1}, T\pi_{n-1}, T\pi_n),$$

The inequality presented above provides evidence that $\{\acute{G}(T\pi_{n-1}, T\pi_{n-1}, T\pi_n)\}$ is a monotonic decreasing sequence of non-negative real's, and as a result, it must be convergent. Therefore, there is a sequence and hence it must be convergent. So, then there is $L \ge 0$, in which

$$\psi(\lim_{n\to+\infty} \acute{G}(T\pi_n, T\pi_n, T\pi_{n+1})) \le F\left(\psi(\lim_{n\to+\infty} \acute{G}(T\pi_{n-1}, T\pi_{n-1}, T\pi_n)), \varphi(\lim_{n\to+\infty} \acute{G}(Tx_{n-1}, Tx_{n-1}, T\pi_n))\right)$$

Then, we have

$$\psi(L) \le F\big(\psi(L), \varphi(L)\big) \le \psi(L).$$

Thus $\psi(L) = 0$ and we conclude that

$$\lim_{n\to+\infty} \acute{G}(T\pi_n, T\pi_n, T\pi_{n+1}) = 0.$$

Lemma 3.3. Assume that S and T meet the requirements of Theorem 3,1 and that $(E, \acute{G})$ is a symmetric $\acute{G}$-space. For every $n \in \mathbb{N}$, let $\{\pi_n\}$ be a sequence in E in which $T\pi_{n+1} \ne T\pi_n$. Hence, the set $\{T\pi_n\}$ is bounded.

Proof. if the sequence $\{T\pi_n\}$ does not have a boundary, So, there is $\{\pi_{n_j}\} \subseteq \{\pi_n\}$ in which $x_1 = 1$ and for each $k \in \mathbb{N}$, n_{j+1} is the minimum integer satisfying

$$\acute{G}\left(T\pi_{n_{j+1}}, T\pi_{n_j}, T\pi_{n_j}\right) > 1$$

and

$$\acute{G}\left(T\pi_k, T\pi_{n_j}, T\pi_{n_j}\right) \le 1$$

for $n_j \le k \le n_{j+1} - 1$, Utilizing the triangle inequality, we are able to obtain

$$\begin{aligned} 1 &< \acute{G}\left(T\pi_{n_{j+1}}, T\pi_{n_j}, T\pi_{n_j}\right) \\ &\le \acute{G}\left(T\pi_{n_{j+1}}, T\pi_{n_{j+1}-1}, T\pi_{n_{j+1}-1}\right) + \acute{G}\left(T\pi_{n_{j+1}-1}, T\pi_{n_j}, T\pi_{n_j}\right) \end{aligned}$$

$$\leq \acute{G}\left(T\pi_{n_{j+1}}, T\pi_{n_{j+1}-1}, T\pi_{n_{j+1}-1} \right) + 1.$$

Using Lemma 3,2 and allowing $j \to \infty$, we are able to obtain

$$\acute{G}\left(T\pi_{n_{j+1}}, T\pi_{n_j}, T\pi_{n_j} \right) = 1$$

Through the utilization of the triangle inequality, we have

$$1 < \acute{G}\left(T\pi_{n_{j+1}}, T\pi_{n_j}, T\pi_{n_j} \right)$$

$$\leq \acute{G}\left(T\pi_{n_{j+1}-1}, T\pi_{n_j-1}, T\pi_{n_j-1} \right)$$

$$\leq \acute{G}\left(T\pi_{n_{j+1}-1}, T\pi_{n_j}, T\pi_{n_j} \right) + \acute{G}\left(T\pi_{n_j}, T\pi_{n_j-1}, T\pi_{n_j-1} \right)$$

$$\leq 1 + \acute{G}\left(T\pi_{n_j}, T\pi_{n_j-1}, T\pi_{n_j-1} \right)$$

We obtain by applying Lemma 3.2 and allowing $j \to \infty$.

$$\lim_{j\to\infty} \acute{G}\left(T\pi_{n_{j+1}-1}, T\pi_{n_j-1}, T\pi_{n_j-1} \right) = 1. \quad \textbf{(3.2)}$$

The triangle inequality forces us to do it again:

$$\left| \acute{G}\left(T\pi_{n_{j+1}-1}, T\pi_{n_j}, T\pi_{n_j}\right) - \acute{G}\left(T\pi_{n_j}, T\pi_{n_j+1}, T\pi_{n_j+1}\right) \right| \leq \acute{G}\left(T\pi_{n_{j+1}-1}, T\pi_{n_j+1}, T\pi_{n_j+1}\right).$$

Using Lemma 3.2, and allowing $j \to \infty$, we are able to obtain

$$\lim_{j\to\infty} \acute{G}\left(T\pi_{n_{j+1}-1}, T\pi_{n_j}, T\pi_{n_j}\right) = 1. \quad \textbf{(3.3)}$$

When we apply the same reasoning, we get

$$\left| \acute{G}\left(T\pi_{n_j-1}, T\pi_{n_{j+1}}, T\pi_{n_{j+1}}\right) - \acute{G}\left(T\pi_{n_j-1}, T\pi_{n_{j+1}-1}, T\pi_{n_{j+1}-1}\right) \right| \leq \acute{G}\left(T\pi_{n_{j+1}}, T\pi_{n_{j+1}-1}, T\pi_{n_{j+1}-1}\right)$$

Letting $j \to \infty$ and using Lemma 3.2, we obtain

$$\lim_{j\to\infty} \acute{G}\left(T\pi_{n_j-1}, T\pi_{n_{j+1}}, T\pi_{n_{j+1}} \right) = 1. \quad \textbf{(3.4)}$$

From (3.2), (3.3), (3.4) and lemma 3.2, we have

$$H\left(S, T, \pi_{n_{j+1}-1}, \pi_{n_j-1}, \pi_{n_j-1}\right) = 1. \quad \textbf{(3.5)}$$

Invoking (3.1), (3.2), (3.3), (3.4), and (3.5), as well as condition C_2 from C-class function

$$\psi\left(\mathcal{G}\left(T\pi_{n_{j+1}}, T\pi_{n_j}, T\pi_{n_j}\right)\right)$$
$$\leq F\left(\psi\left(H\left(S, T, \pi_{n_{j+1}-1}, \pi_{n_j-1}, \pi_{n_j-1}\right), \varphi\left(H\left(S, T, \pi_{n_{j+1}-1}, \pi_{n_j-1}, \pi_{n_j-1}\right)\right)\right)\right)$$

So , as **j** $\to \infty$, we have

$$\psi(1) \leq F(\psi(1), \varphi(1)) \leq \psi(1),$$

Which leads to a contradiction because $1 > 0$.

Lemma 3.4. Assume that S and T meet the requirements of Theorem 3.1 and that $(E, \mathcal{G})$ is a symmetric $\mathcal{G}$-space. For every $n \in \mathbb{N}$, let $\{\pi_n\}$ be a sequence in E in which $T\pi_{n+1} = S\pi_n$.
Hence, the sequence $\{T\pi_n\}$ is Cauchy.

Proof. Letting $K_n = \sup\{\mathcal{G}(T\pi_p, T\pi_q, T\pi_q) : p, q \geq n\}$. Lemma 3.3 indicates that the sequence $\{T\pi_n\}$ is bounded. Hence, for any $n \in \mathbb{N}$, $K_n < \infty$, indicating that $\{K_n\}$ is a bounded and monotonic sequence and is therefore convergent. Consequently, there is a value $K \geq 0$. such that:

$$\lim_{n\to\infty} K_n = K.$$

We shall prove $K = 0$, Let us assume conversely that $K > 0$.

By the definition of K_n , for each $j \in N$, there is in which $n_j, m_j \in N$ in which $m_j > n_j \geq j$ and

$$K_j - \frac{1}{j} < \mathcal{G}\left(T\pi_{m_j}, T\pi_{n_j}, T\pi_{n_j}\right) \leq K_J$$

Therefore,

$$\lim_{j\to\infty} \mathcal{G}\left(T\pi_{m_j}, T\pi_{n_j}, T\pi_{n_j}\right) = K. \quad \textbf{(3.6)}$$

We obtain by applying triangle inequality and Lemma 3.3,

$$\mathcal{G}\left(T\pi_{m_j}, T\pi_{n_j}, T\pi_{n_j}\right) \leq \mathcal{G}\left(T\pi_{m_j-1}, T\pi_{n_j-1}, T\pi_{n_j-1}\right)$$
$$\leq \mathcal{G}\left(T\pi_{m_{j-1}}, T\pi_{m_j}, T\pi_{m_j}\right) + \mathcal{G}\left(T\pi_{m_j}, T\pi_{n_j}, T\pi_{n_j}\right) + \mathcal{G}\left(T\pi_{n_j}, T\pi_{n_{j-1}}, T\pi_{n_{j-1}}\right)$$

By utilizing equation (3.6), Lemma 3.3, and allowing $j \to 1$, we are able to achieve the following:

$$\lim_{j\to\infty} \mathcal{G}\left(T\pi_{m_{j-1}}, T\pi_{n_{j-1}}, T\pi_{n_{j-1}}\right) = K. \quad \textbf{(3.7)}$$

Continuing in the same manner, we are able to demonstrate that

$$\lim_{j\to\infty} \mathcal{G}\left(T\pi_{m_{j-1}}, T\pi_{n_j}, T\pi_{n_j}\right) = K. \quad \textbf{(3.8)}$$

And

$$\lim_{j\to\infty} \acute{G}\left(T\pi_{n_{j-1}}, T\pi_{m_j}, T\pi_{m_j} \right) = K. \tag{3.9}$$

Using (3.7), (3.8), (3.9) and Lemma 3.3, we get

$$\lim_{j\to\infty} H \left(S, T, \pi_{m_{j-1}}, \pi_{n_{j-1}}, \pi_{n_{j-1}} \right) = K. \tag{3.10}$$

The condition of the C-class function, along with the equations (3.1), (3.6), and (3.10), allows us to derive

$$\psi\left(\acute{G}\left(T\pi_{m_j}, T\pi_{n_j}, T\pi_{n_j}\right)\right) \leq F\left(\psi\left(H\left(S, T, \pi_{m_{j-1}}, \pi_{n_{j-1}}, \pi_{n_{j-1}}\right)\right), \varphi\left(H\left(S, T, \pi_{m_{j-1}}, \pi_{n_{j-1}}, \pi_{n_{j-1}}\right)\right)\right)$$

So , as $j \to \infty$ we get

$$\psi(K) \leq F\big(\psi(K), \varphi(K)\big) \leq \psi(K).$$

It means that $\psi(K) = 0$. This contradiction shows that $K = 0$. So

$$\lim_{j\to\infty} K_j = 0.$$

Hence, $\{T\pi_n\}$ is a Cauchy sequence.

Now, in order to demonstrate Theorem 3.1

Proof. Due to Lemma 3.4, $\{T\pi_n\}$ is a Cauchy sequence and by $\acute{G}$-completeness of E, Then $\{T\pi_n\}$ is a converge to $u \in E$ as $n \to +\infty$, and

$$T\pi_n \to Tu, \text{ when } n \to \infty. \tag{3.11}$$

Let us assume that $\acute{G}(Su, Tu, Tu) > 0$. Applying (3.11) and letting $n \to \infty$

$$\begin{aligned} H(S, T, \pi_n, u, u) &= \max\{\acute{G}(T\pi_n, Tu, Tu), \acute{G}(Tu, S\pi_n, Tu), \acute{G}(Tu, Su, Tu)\} \\ &= \max\{\acute{G}(Tu, Tu, Tu), \acute{G}(Tu, Su, Tu), \acute{G}(Tu, Su, Tu)\} \\ &= \acute{G}(Su, Tu, Tu) > 0. \end{aligned}$$

Using (3.1), (3.11) and C_1, one can get

$$\begin{aligned} \psi\big(\acute{G}(Su, T\pi_{n+1}, T\pi_{n+1})\big) &= \psi\big(\acute{G}(Su, Tu, Tu)\big) \\ &\leq F(\psi(H(S, T, u, \pi_n, \pi_n)), \varphi(H(S, T, u, \pi_n, \pi_n))) \\ &\leq \psi\big(\acute{G}(Su, Tu, Tu)\big). \end{aligned}$$

This paradox demonstrates that the value of $\acute{G}(Su, Tu, Tu) = 0$.

So, u is a coincident point of S and T, Let $Su = Tu = v$, Due to the fact that S and T commute at their coincident point u. Therefore, $Sv = S(Tu) = T(Su) = Tv$, By (v), we have

$$Tu \leq T(Tu) = Tv,$$

$$H(S,T,v,u,u) = \max\{ Ǵ(Tv,Tu,Tu), Ǵ(Tu,Sv,Tu), Ǵ(Tu,Su,Tu)\}$$

$$= \max\{ Ǵ(Tv,v,v), Ǵ(v,Sv,v), Ǵ(v,v,v)\}$$

$$= \max\{ Ǵ(Tv,v,v), Ǵ(v,Tv,v)\},$$

Since, $(Ǵ, E)$ is symmetric Ǵ-space. Therefore, $H(S,T,v,u,u) = Ǵ(Tv,v,v)$.

Using (3.1) and C_1, we get

$$\psi\left(Ǵ(Tv,Tu,Tu)\right) = \psi\left(Ǵ(Tv,v,v)\right)$$

$$\leq F\left(\psi(Ǵ(Tv,v,v)), \varphi(Ǵ(Tv,v,v))\right)$$

$$\leq \psi\left(Ǵ(Sv,v,v)\right).$$

It means that, $\psi(Ǵ(Tv,v,v)) = 0$. which implies that $Sv = Tv = v$. So, v is common fixed point of S and T. Subsequently, we offer an example that illustrates the practicality of Theorem 3.1. $\boxdot$

Example 3.5. Consider that $E = [0,1]$ associated with the Ǵ-metric represented by

$$Ǵ(\pi, v, \varpi) = \max\{ |\pi - v|, |v - \varpi|, |\varpi - \pi| \} \text{ for every } \pi, v, \varpi \in E,$$

Let us consider $\varpi \leq v \leq \pi$ without losing generality. Thus, $Ǵ(\pi, v, \varpi) = |\pi - \varpi|$. Define the mappings $S, T: E \to E$, by $S\pi = \frac{\pi}{25}$ and $T\pi = \frac{\pi}{5}$ for each $\pi \in E$. It is an obvious fact that the conditions (i) to (v) of Theorem 3,1 are $\pi_0 = 0$, Consider that $\psi(t) = t$ and $F: E \times E \to \mathbb{R}$ be given by:

$$F(t,s) = \frac{9}{10}t, \text{ for } t \in [0, +\infty).$$

Indeed for all $\pi \neq v \neq \varpi$, we have

$$F\left(\psi(H(S,T,\pi,v,\varpi)), \varphi(H(S,T,\pi,v,\varpi))\right) = \frac{9}{10}\psi(H(S,T,\pi,v,\varpi))$$

$$= \frac{9}{10}\left(H(S,T,\pi,v,\varpi)\right). \quad \mathbf{(3.12)}$$

Where,

$$H(S,T,\pi,v,\varpi) = \max\{Ǵ(T\pi,Tv,T\varpi), Ǵ(Tv,S\pi,T\varpi), Ǵ(Tv,Sv,T\varpi)\}$$

$$= Ǵ(T\pi,Tv,T\varpi).$$

So,

$$\psi(Ǵ(S\pi,Sv,S\varpi)) = \left|\frac{\pi}{25} - \frac{\varpi}{25}\right| \leq \frac{1}{4}\left|\frac{\pi}{5} - \frac{\varpi}{5}\right| = \frac{1}{4}Ǵ(T\pi,Tv,T\varpi). \quad \mathbf{(3.13)}$$

Because of (3.12) and (3.13), we are able to derive

$$\psi\big(\acute{G}(S\pi, Sv, S\varpi)\big) \leq F\Big(\psi\big(H(S,T,\pi,v,\varpi)\big), \varphi\big(H(S,T,\pi,v,\varpi)\big)\Big).$$

As a result, by Theorem 3.1 presumptions are all true. As a result, there is a coincident point between S and T, which is $0 \in E$. Additionally, S and T commute at 0, meaning that 0 is the only fixed point that S and T have in common. Using C-class mapping, we can infer multiple conclusions of coincidence and common fixed point from Theorem 3.1.

Corollary 3.6. Consider that $(E, \acute{G})$ is a symmetric complete $\acute{G}$-space and $S, T: E \to E$, If

i. $S(E) \subseteq T(E)$;
ii. $T(E)$ is closed;
iii. S is T-non-decreasing;
iv. There is $\pi_0 \in E$ per $T\pi_0 \leq S\pi_0$;
v. If $\{T\pi_n\} \subset E$ is a non-decreasing sequence (w. r. t. $\leq$) per $T\pi_n \to T\varpi$ in $T(E)$, then $Tu \leq T(Tu)$ and $T\pi_n \leq Tu$, for all $n \in \mathbb{N}$;

vi. There is C-class function F in which for every $(\pi, v) \in E \times E$ with $T\pi \leq Tv$, we have

$$\psi\big(\acute{G}(S\pi, Sv, S\varpi)\big) \leq F\Big(\psi\big(\acute{G}(T\pi, Tv, T\varpi)\big), \varphi\big(\acute{G}(T\pi, Tv, T\varpi)\big)\Big),$$

Corollary 3.7. Consider that $(E, \acute{G})$ is a symmetric complete $\acute{G}$-space and $S: E \to E$. Suppose that

i. There is $\pi_0 \in E$ in which $\pi_0 \leq S\pi_0$;
ii. $(\pi, v) \in E \times E$, $\pi \leq v$ implies that $S\pi \leq Sv$;
iii. If $\{\pi_n\} \subset E$ is a non-decreasing sequence (w, r, t, $\leq$) per $\pi_n \to u$ in E. Then, $\pi_n \leq u$, for all $n \in$ N;
iv. There is C-class function F in which for every $(\pi, v) \in E \times E$ per $\pi \leq v$, we have

$$\acute{G}(S\pi, Sv, S\varpi) \leq \acute{G}(\pi, v, \varpi).$$

Then, $\{S\pi_0\}$ converges to fixed point of S.

Proof. By using T is identity mapping, one can prove Corollary 3.7. from Theorem 3.1.

4 An Application to the Integral Equation of Fredholm

In this section, we wish to investigate the existence of a unique solution to a nonlinear integral equations of fredholm, Through the application of the primary result that we obtained from Corollary 3.7.

Consider the following Fredholm integral equation of the second type:

$$\pi(t) = f(t) + \lambda \int_0^1 K(t, s, \pi(s))ds, \qquad \text{for } t, s \in [0,1], \qquad 4,1$$

Where f and K are known functions and λ is a constant.

Consider that $E = C([0,1])$ is the set of all continuous function defined on $[0,1]$, and Define $\acute{G}: E \times E \times E \to \mathbb{R}^+$ by:

$$\acute{G}(\pi, v, \varpi) = \sup_{t\in[0,1]} |\pi(t) - v(t)| + \sup_{t\in[0,1]} |v(t) - \varpi(t)| + \sup_{t\in[0,1]} |\pi(t) - \varpi(t)|,$$

Obviously, $(E, \acute{G})$ is a complete $\acute{G}$-space, Define the mappings $S: E \to E$ by

$$S\pi(t) = f(t) + \int_0^1 K(t, s, \pi(s))ds, \text{ for } t, s \in [0,1]$$

Suppose that $f: [0,1] \to \mathbb{R}$ and $K: [0,1] \times [0,1] \times \mathbb{R} \to \mathbb{R}$ are continuous function, There is a continuous function $\rho: [0,1] \times [0,1] \to [0, \infty)$ such that $|K(s, t, \alpha) - K(s, t, \beta)| \leq \rho(s, t)|\alpha - \beta|$ for each $\alpha, \beta \in \mathbb{R}$ and $\sup_{t\in[0,1]} \int_0^1 \rho(s, t)ds < q$ for some $q \in (0,1)$,

Then the integral equation (4.1) has a solution $u \in E$,

To prove for $\pi, v \in E$, we have

$$\begin{aligned}
\acute{G}(S\pi, Sv, Sv) &= 2 \sup_{t\in[0,1]} |S\pi(t) - Sv(t)| \\
&= 2 \sup_{t\in[0,1]} \left| \int_0^1 K(t, s, \pi(s))ds - \int_0^1 K(t, s, v(s))ds \right| \\
&= 2 \sup_{t\in[0,1]} \left| \int_0^1 \Big(K(t, s, \pi(s)) - K(t, s, v(s))\Big) ds \right| \\
&\leq 2 \sup_{t\in[0,1]} \int_0^1 |K(t, s, \pi(s)) - K(t, s, v(s))| \, ds \\
&\leq 2 \sup_{t\in[0,1]} \int_0^1 \rho(t, s)|\pi(s) - v(s)| \, ds \\
&\leq 2 \sup_{t\in[0,1]} |\pi(t) - v(t)| \sup_{t\in[0,1]} \int_0^1 \rho(t, s)ds \\
&\leq q \, \acute{G}(\pi, v, v) \\
&\leq \acute{G}(\pi, v, v),
\end{aligned}$$

Thus, Corollary 3.7 is application to S which guarantees the existence and the uniqueness of the fixed point $u \in E$, Thus, u is the unique solution of the integral equation 4.1. □

5 CONCLUSION

In this paper, we investigate the existence of a coincident point of generalized metric space and formulation of a unique common fixed point. Specifically speaking, we established the results for new contraction via new kind of C-class function in three variables. Additionally, an illustrative example and corollaries are provided to demonstrate the main results. Moreover, as an application, we employ the achieved result to earn the existence criteria of the solution of a kind of non-linear Fredholm integral equation.

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