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Some New Results on Partial Fuzzy Metric Spaces

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Abstract

In this work, we introduce a different interpretation of the notion of a partial fuzzy metric, which we refer to as a partial fuzzy co-metric. We define a partial fuzzy co-metric from a t-conorm and compare it with partial fuzzy metric, in contrast to the conventional approach to the theory of partial fuzzy metric spaces, which is based on the use of a t-norm. Here, we limit the scope of our analysis to Sedghi's definition of partial fuzzy metrics. Additionally, we proposed and compared the ideas of strong partial fuzzy co-metric spaces and strong partial fuzzy metric spaces. We also presented a few examples of these novel ideas.

Keywords: Partial fuzzy co-metric spaces, strong partial fuzzy metric spaces, strong Partial fuzzy co-metric space, t-conorm.

1. Introduction

The idea of fuzzy sets was initiated by Zadeh [1] in 1965, and topological researchers have been studying multiple versions of fuzzy metric spaces. This metric were introduced by Kramosi and Michalek specifically in 1975 [2]. A more robust concept than fuzzy metric was presented by George and Veeramani [3]. In 1994, Matthews [4] presented the concept of a partial metric, a generalized metric that has not always zero self-distance. Numerous writers have contributed to the study of partial metric space from a mathematical perspective since the definition of this notion (see, for example [5]-[7]). Furthermore, technical applications like color image filtering (see [8]) and perceptual color difference (see [9] and [10]) have effectively employed fuzzy metrics. The concept of partial fuzzy metric spaces was introduced in recent years by Yue and Gu [11], Sedghi et al. [12], and Gregori et al. [13] in several interpretations to combine the two aforementioned generalizations of classical metric, partial metric and fuzzy one into a single idea. In order to make the distance more closely align with the concept of a metric.

In 2012, Noori F. AL-Mayhi and I. H. Radhi [14] introduced the fuzzy metric that is built on t-conorm depending the fuzzy metric that is built upon t-norm. Olga G. et al., in 2020, defined a fuzzy metric in [15] so that the distance better corresponded to the concept of a metric. This update makes use of a t-conorm rather than a t-norm.

In this work, we evaluate the idea of partial fuzzy co-metric space by employing the t-conorm instead of the t-norm in the sense of Sedghi's definition of partial fuzzy metric space. In addition, we defined the terms strong partial fuzzy and strong partial fuzzy co-metric. Furthermore, we provided some examples of these new concepts.

2. Preliminaries

In this section, we define the previous and basic definitions and properties. We will consider that [0,1]=K throughout this paper.

Definition 2.1 [4] Let $\hbar: \Theta \times \Theta \to R^+$ be a mapping on a nonempty set Θ, the pair $(\Theta, \hbar)$ is said to be partial metric space if $\hbar$ satisfies the following conditions for all $\kappa, \varpi, \alpha \in \Theta$,

1. $\hbar(\kappa,\kappa) \le \hbar(\kappa,\varpi)$,
2. $\hbar(\kappa,\kappa) = \hbar(\kappa,\varpi) = \hbar(\varpi,\varpi)$ if and only if $\kappa = \varpi$,
3. $\hbar(\kappa,\varpi) = \hbar(\varpi,\kappa)$,
4. $\hbar(\kappa,\alpha) \le \hbar(\kappa,\varpi) + \hbar(\varpi,\alpha) - \hbar(\varpi,\varpi)$.

Remark 2.2 [4] In partial metric space if $\hbar(\kappa,\varpi) = 0$, then $\kappa = \varpi$ for all $\kappa, \varpi \in \Theta$, but the converse is not true. If $\hbar(\kappa,\kappa) = 0$, then the partial metric $\hbar$ is an ordinary metric on Θ.

Example 2.3 [4] Let $\hbar: R^+ \times R^+ \to R^+$ be a mapping defined as $\hbar(\kappa,\varpi) = max\,\{\kappa,\varpi\}$ for each $\kappa, \varpi \in R^+$, then $(R^+, \hbar)$ is an partial metric space.

Definition 2.4 [16]: Let $\circ : K \times K \to K$ be a binary operation, we say that $\circ$ is a continuous t-norm if it satisfies the axioms:

1. $\xi \circ \eta = \eta \circ \xi$, $\forall\, \xi, \eta \in K$,
2. $(\xi \circ \eta) \circ \rho = \xi \circ (\eta \circ \rho)$, $\forall \xi, \eta, \rho \in K$,
3. $\circ$ is continuous,
4. $\xi \circ 1 = \xi \quad \forall \xi \in K$,
5. $\xi \circ \eta \leq \upsilon \circ \rho$ whenever $\xi \leq \upsilon$ and $\eta \leq \rho$, $\forall \xi, \eta, \upsilon, \rho \in K$.

Examples 2.5 [16]:

$\xi \circ \eta = \xi . \eta$, $\xi \circ \eta = min\,\{\xi, \eta\}$ are continuous t-norms.

Definition 2.6 [16]: Let $\odot$: $K \times K \to K$ be a binary operation, we say that $\odot$ is a continuous t-conorm if it is satisfies the axioms:

1. $\xi \odot \eta = \eta \odot \xi$, $\forall\, \xi, \eta \in K$,
2. $(\xi \odot \eta) \odot \rho = \xi \odot (\eta \odot \rho)$, $\forall \xi, \eta, \rho \in K$
3. $\odot$ is continuous,
4. $\xi \odot 0 = \xi \quad \forall \xi \in K$,
5. $\xi \odot \eta \leq \upsilon \odot \rho$ whenever $\xi \leq \upsilon$ and $\eta \leq \rho$, for all $\xi, \eta, \upsilon, \rho \in K$.

Examples 2.7 [16]:

$\xi \odot \eta = \xi + \eta - \xi\eta$, $\xi \odot \eta = max\,\{\xi, \eta\}$ and $\xi \odot \eta = \xi + \eta$ are examples of continuous t-conorms.

Definition 2.8 [16]: Let $\circ$ is t-norm and $\odot$ is t-conorm. $\circ$ and $\odot$ are said to be dual if satisfying the following axioms:

1. $\xi \circ \eta = 1 - \big((1 - \xi) \odot (1 - \eta)\big)$ for all $\xi, \eta \in K$.
2. $\xi \odot \eta = 1 - \big((1 - \xi) \circ (1 - \eta)\big)$ for all $\xi, \eta \in K$.

Definition 2.9 [3] A fuzzy metric space (F.M.S for simply) is a triple $(\Theta, L, \circ)$, if Θ is a nonempty set, $\circ$ is continuous t-norm and $L: \Theta^2 \times (0, \infty) \to K$ is a F.S. satisfying the conditions, $\forall \kappa, \varpi, \alpha \in \Theta$ and $s, t > 0$,

1. $L(\kappa, \varpi, t) > 0$,
2. $L(\kappa, \varpi, t) = 1 \Leftrightarrow \kappa = \varpi$,
3. $L(\kappa, \varpi, t) = L(\varpi, \kappa, t)$,

4. $L(\kappa, \alpha, t+s) \geq L(\kappa, \varpi, t) \circ L(\varpi, \alpha, s)$,
5. The map $L(\kappa, \varpi, t): (0, \infty) \rightarrow K$ is continuous.

Definition 2.10 [4]: Let $\Theta \neq \emptyset$, $\circ$ is a continuous t-norm and $R_\hbar: \Theta \times \Theta \times (0, \infty) \rightarrow K$ be a mapping. The triple $(\Theta, R_\hbar, \circ)$ is said to be fuzzy partial metric space (P.F.M.S for simply) if $R_\hbar$ satisfy the following conditions for all $\kappa, \varpi, \alpha \in \Theta$ and $t, s > 0$:

1. $R_\hbar(\kappa, \varpi, t) = R_\hbar(\kappa, \kappa, t) = R_\hbar(\varpi, \varpi, t)$ if and only if $\kappa = \varpi$,
2. $R_\hbar(\kappa, \kappa, t) \geq R_\hbar(\kappa, \varpi, t) > 0$,
3. $R_\hbar(\kappa, \varpi, t) = R_\hbar(\varpi, \kappa, t)$,
4. $R_\hbar(\kappa, \varpi, max\,\{t, s\}) \circ R_\hbar(\alpha, \alpha, max\,\{t, s\}) \geq R_\hbar(\kappa, \alpha, t) \circ R_\hbar(\alpha, \varpi, s)$,
5. $R_\hbar(\kappa, \varpi, .)$ is continuous on $(0, \infty)$.

3. Main results

In this section, we defined some new definition of strong P.F.M.S, P.F.co-metric, strong P.F.co-metric and introduced some examples for these definitions.

Definition 3.1: Let $(\Theta, R_\hbar, \circ)$ be a P.F.M.S, if $R_\hbar$ satisfies the additional condition for the definition **2.10**.

(6) $R_\hbar(\kappa, \alpha, t) \circ R_\hbar(\varpi, \varpi, t) \geq R_\hbar(\kappa, \varpi, t) \circ R_\hbar(\varpi, \alpha, t)$, then $(\Theta, R_\hbar, \circ)$ is said to be strong P.F.M.S.

Example 3.2: Let $\Theta = R^+$, $\xi \circ \eta = \xi . \eta$ for all $\xi, \eta \in K$ and $R_\hbar: \Theta \times \Theta \times (0, \infty) \rightarrow K$ defined by

$R_\hbar(\kappa, \varpi, t) = \frac{\{\kappa, \varpi\} + t}{\{\kappa, \varpi\} + t}$, for all $\kappa, \varpi \in X, t > 0$, then $(X, R_\hbar, \circ)$ is strong P.F.M.S.

Solution:

1) If $\kappa = \varpi$, then $\{\kappa, \varpi\} = \{\kappa, \varpi\}$, $\frac{\{\kappa, \varpi\} + t}{\{\kappa, \varpi\} + t} = 1$

Therefore,

$R_\hbar(\kappa, \kappa, t) = R_\hbar(\kappa, \varpi, t) = R_\hbar(\varpi, \varpi, t) = 1$.

If $R_\hbar(\kappa, \kappa, t) = R_\hbar(\kappa, \varpi, t) = R_\hbar(\varpi, \varpi, t)$

Since $R_\hbar(\kappa, \kappa, t) = R_\hbar(\varpi, \varpi, t) = 1$, then $R_\hbar(\kappa, \varpi, t) = 1$

$\Longrightarrow \{\kappa, \varpi\} = \{\kappa, \varpi\}$, that is $\kappa = \varpi$

2) Since $R_\hbar(\kappa, \kappa, t) = 1$, and $R_\hbar(\kappa, \varpi, t) \leq 1$ for all $\kappa, \varpi \in X$,

$R_{\hbar}(\kappa,\varpi,t) \leq R_{\hbar}(\kappa,\kappa,t)$

3) Clearly $R_{\hbar}(\kappa,\varpi,t) = R_{\hbar}(\varpi,\kappa,t)$.

4) To prove the condition (4), for all $\kappa,\varpi,\alpha \in \Theta = R^{+}$, $t,s > 0$, we have 6 cases:

case1: If $\kappa < \varpi, \kappa < \alpha$ and $\varpi > \alpha$,

$$R_{\hbar}(\kappa,\varpi,max\,\{t,s\}) \circ R_{\hbar}(\alpha,\alpha,\{t,s\}\,) = \frac{\kappa+\{t,s\}}{\varpi+\{t,s\}}.1 > \frac{\kappa+t}{\alpha+t}.\frac{\alpha+s}{\varpi+s} = R_{\hbar}(\kappa,\alpha,t) \circ R_{\hbar}(\alpha,\varpi,s).$$

case 2: If $\kappa < \varpi, \kappa < \alpha$ and $\varpi < \alpha$,

$$R_{\hbar}(\kappa,\varpi,max\,\{t,s\}) \circ R_{\hbar}(\alpha,\alpha,\{t,s\}\,) = \frac{\kappa+\{t,s\}}{\varpi+\{t,s\}}.1 > \frac{\kappa+t}{\alpha+t}.\frac{\varpi+s}{\alpha+s} = R_{\hbar}(\kappa,\alpha,t) \circ R_{\hbar}(\alpha,\varpi,s).$$

case 3: If $\kappa < \varpi, \kappa > \alpha$ and $\varpi > \alpha$,

$$R_{\hbar}(\kappa,\varpi,max\,\{t,s\}) \circ R_{\hbar}(\alpha,\alpha,\{t,s\}\,) = \frac{\kappa+\{t,s\}}{\varpi+\{t,s\}}.1 > \frac{\alpha+t}{\kappa+t}.\frac{\alpha+s}{\varpi+s} = R_{\hbar}(\kappa,\alpha,t) \circ R_{\hbar}(\alpha,\varpi,s).$$

case 4: If $\kappa > \varpi, \kappa < \alpha$ and $\varpi < \alpha$,

$$R_{\hbar}(\kappa,\varpi,max\,\{t,s\}) \circ R_{\hbar}(\alpha,\alpha,\{t,s\}\,) = \frac{\varpi+\{t,s\}}{\kappa+\{t,s\}}.1 > \frac{\kappa+t}{\alpha+t}.\frac{\varpi+s}{\alpha+s} = R_{\hbar}(\kappa,\alpha,t) \circ R_{\hbar}(\alpha,\varpi,s\}).$$

case 5: If $\kappa > \varpi, \kappa > \alpha$ and $\varpi > \alpha$,

$$R_{\hbar}(\kappa,\varpi,max\,\{t,s\}) \circ R_{\hbar}(\alpha,\alpha,\{t,s\}\,) = \frac{\varpi+\{t,s\}}{\kappa+\{t,s\}}.1 > \frac{\alpha+t}{\kappa+t}.\frac{\alpha+s}{\varpi+s} = R_{\hbar}(\kappa,\alpha,t) \circ R_{\hbar}(\alpha,\varpi,s).$$

case 6: If $\kappa > \varpi, \kappa > \alpha$ and $\varpi < \alpha$,

$$R_{\hbar}(\kappa,\varpi,max\,\{t,s\}) \circ R_{\hbar}(\alpha,\alpha,\{t,s\}\,) = \frac{\varpi+\{t,s\}}{\kappa+\{t,s\}}.1 = \frac{\alpha+t}{\kappa+t}.\frac{\varpi+s}{\alpha+s} = R_{\hbar}(\kappa,\alpha,t) \circ R_{\hbar}(\alpha,\varpi,s).$$

Therefore, for all cases, we deduce that the condition (4).

5) $R_{\hbar}(\kappa,\varpi,.)\colon (0,\infty) \to K$ is continuous.

6) By the same way of proof of condition (4).

Therefore, $(\Theta, R_{\hbar}, \circ)$ is strong P.F.M.S.

Theorem 3.3: Let $\Theta \neq \emptyset$, , $\circ$ be a continuous t-norm such that $\xi \circ \eta \geq \xi \circ \upsilon$ whenever $\eta \geq \upsilon$ for all $\xi, \eta, \upsilon \in K$, then $R_\hbar: \Theta^2 \times (0, \infty) \to K$ is strong P.F.M function iff it is satisfy the conditions for all $\kappa, \varpi, \alpha \in \Theta, t > 0$,

1) $R_\hbar(\kappa, \varpi, t) = R_\hbar(\kappa, \kappa, t) = R_\hbar(\varpi, \varpi, t)$ if and only if $\kappa = \varpi$,
2) $R_\hbar(\kappa, \kappa, t) \geq R_\hbar(\kappa, \varpi, t)$,
3) $R_\hbar(\kappa, \varpi, t) \circ R_\hbar(\alpha, \alpha, t) \geq R_\hbar(\kappa, \alpha, t) \circ R_\hbar(\varpi, \alpha, t)$,
4) $R_\hbar(\kappa, \varpi, .)$ is continuous on $(0, \infty)$.

Proof: The first direction from the definition.

To prove the second direction, we consider the conditions hold.

The conditions of strong P.F.M 1,2 and 5 satisfy from 1, 2 and 4, the condition 3,

$R_\hbar(\kappa, \varpi, t) = R_\hbar(\kappa, \varpi, t) \circ 1 \geq R_\hbar(\kappa, \varpi, t) \circ R_\hbar(\kappa, \kappa, t) \geq R_\hbar(\kappa, \kappa, t) \circ R_\hbar(\varpi, \kappa, t)$

$\Rightarrow R_\hbar(\kappa, \varpi, t) \geq R_\hbar(\varpi, \kappa, t)$ ……(1)

Also, $R_\hbar(\varpi, \kappa, t) \geq R_\hbar(\varpi, \kappa, t) \circ R_\hbar(\varpi, \varpi, t) \geq R_\hbar(\varpi, \varpi, t) \circ R_\hbar(\kappa, \varpi, t)$

$\Rightarrow R_\hbar(\varpi, \kappa, t) \geq R_\hbar(\kappa, \varpi, t)$ ……(2)

From (1) and (2), we have $R_\hbar(\kappa, \varpi, t) = R_\hbar(\varpi, \kappa, t)$

Now, from (3) $R_\hbar(\kappa, \varpi, t) \circ R_\hbar(\alpha, \alpha, t) \geq R_\hbar(\kappa, \alpha, t) \circ R_\hbar(\varpi, \alpha, t)$

$\geq R_\hbar(\kappa, \alpha, t) \circ R_\hbar(\alpha, \varpi, t)$

To prove the condition (4), let $s, t > 0$, from (3)

$R_\hbar(\kappa, \varpi, max\,\{t, s\}) \circ R_\hbar(\alpha, \alpha, \{t, s\}\,) \geq R_\hbar(\kappa, \alpha, max\,\{t, s\}) \circ R_\hbar(\varpi, \alpha, max\,\{t, s\}) \geq R_\hbar(\kappa, \alpha, t) \circ R_\hbar(\alpha, \varpi, s)$.

Therefore, $R_\hbar$ is strong P.F.M.

Definition 3.4: Let $\Theta \neq \emptyset$, $\odot$ be a continuous t-conorm. A mapping $CR_\hbar: \Theta^2 \times (0, \infty) \to K$ is called a partial fuzzy co-metric (P.F.co-M) on Θ if $CR_\hbar$ satisfy the axioms, for all $\kappa, \varpi, \alpha \in \Theta$ and $t, s > 0$,

1) $CR_\hbar(\kappa, \varpi, t) = CR_\hbar(\kappa, \kappa, t) = CR_\hbar(\varpi, \varpi, t)$ if and only if $\kappa = \varpi$,
2) $CR_\hbar(\kappa, \kappa, t) \leq CR_\hbar(\kappa, \varpi, t)$,
3) $CR_\hbar(\kappa, \varpi, t) = CR_\hbar(\varpi, \kappa, t)$,
4) $CR_\hbar(\kappa, \varpi, max\,\{t, s\}) \odot CR_\hbar(\alpha, \alpha, max\,\{t, s\}) \leq CR_\hbar(\kappa, \alpha, t) \odot CR_\hbar(\alpha, \varpi, s)$,
5) $CR_\hbar(\kappa, \varpi, .): (0, \infty) \to K$ is continuous.

Lemma 3.5: $CR_{\hbar}(\kappa, \varpi, .)$ is non-increasing with respect to t for all $\kappa, \varpi \in \Theta, t > 0$, if the continuous t-conorm $\odot$ satisfy the condition, for all $\xi, \eta, \upsilon \in K, \xi \odot \eta \leq \xi \odot \upsilon \Longrightarrow \eta \leq \upsilon$.

Proof:

From(4) of definition 3.4 for all $\kappa, \varpi, \alpha \in \Theta$ and $s, t > 0$, we have

$CR_{\hbar}(\kappa, \varpi, max\,\{t, s\}) \odot CR_{\hbar}(\alpha, \alpha, max\,\{t, s\}) \leq CR_{\hbar}(\kappa, \alpha, t) \odot CR_{\hbar}(\alpha, \varpi, s)$

Let $t < s$, then taking $\alpha = \varpi$,

$CR_{\hbar}(\kappa, \varpi, s) \odot CR_{\hbar}(\varpi, \varpi, s) \leq CR_{\hbar}(\kappa, \varpi, t) \odot CR_{\hbar}(\varpi, \varpi, s)$

$\Longrightarrow CR_{\hbar}(\kappa, \varpi, s) \leq CR_{\hbar}(\kappa, \varpi, t)$ by condition.

Then $CR_{\hbar}(\kappa, \varpi, .)$ is non-increasing.

Example 3.6: Let $(\Theta, \hbar)$ be a P.M.S. Denote $\xi \odot \eta = \xi + \eta$ for all $\xi, \eta \in K$ and let $CR_{\hbar} = \frac{\hbar(\kappa,\varpi)}{\hbar(\kappa,\varpi)+t}$, then $(\Theta, CR_{\hbar}, \odot)$ is a P.F.co-M.S and we call that P.F. co-M induced by P.M $\hbar$ as the standard P.F. co-M.

proof:

1) $\kappa = \varpi \Leftrightarrow \hbar(\kappa, \kappa) = \hbar(\kappa, \varpi) = \hbar(\varpi, \varpi)$

$\Leftrightarrow \frac{\hbar(\kappa,\kappa)}{\hbar(\kappa,\kappa)+t} = \frac{\hbar(\kappa,\varpi)}{\hbar(\kappa,\varpi)+t} = \frac{\hbar(\varpi,\varpi)}{\hbar(\varpi,\varpi)+t}$

$\Leftrightarrow CR_{\hbar}(\kappa, \kappa, t) = CR_{\hbar}(\kappa, \varpi, t) = CR_{\hbar}(\varpi, \varpi, t)$

2) Since $\hbar(\kappa, \kappa) \leq \hbar(\kappa, \varpi)$

$\Longrightarrow \frac{\hbar(\kappa,\kappa)}{\hbar(\kappa,\kappa)+t} \leq \frac{\hbar(\kappa,\varpi)}{\hbar(\kappa,\varpi)+t} \Longrightarrow CR_{\hbar}(\kappa, \kappa, t) \leq CR_{\hbar}(\kappa, \varpi, t)$

3) clearly $CR_{\hbar}(\kappa, \varpi, t) = CR_{\hbar}(\varpi, \kappa, t)$

4) Since $\hbar(\kappa, \varpi) + \hbar(\alpha, \alpha) \leq \hbar(\kappa, \alpha) + \hbar(\alpha, \varpi)$

$\Longrightarrow \frac{\hbar(\kappa,\varpi)}{\hbar(\kappa,\varpi)+\{t,s\}} + \frac{\hbar(\alpha,\alpha)}{\hbar(\alpha,\alpha)+\{t,s\}} \leq \frac{\hbar(\kappa,\alpha)}{\hbar(\kappa,\alpha)+t} + \frac{\hbar(\alpha,\varpi)}{\hbar(\alpha,\varpi)+t}$

$\Longrightarrow CR_{\hbar}(\kappa, \varpi, max\,\{t, s\}) \odot CR_{\hbar}(\alpha, \alpha, max\,\{t, s\}) \leq CR_{\hbar}(\kappa, \alpha, t) \odot CR_{\hbar}(\alpha, \varpi, t)$

5) $CR_{\hbar}(\kappa, \varpi, .) : (0, \infty) \rightarrow K$ is continuous.

So, $(\Theta, CR_{\hbar}, \odot)$ is P.F. co-M.S.

Theorem 3.7: Let $(\Theta, R_{\hbar}, \circ)$ be a P.F.M.S. Let $N = 1 - R_{\hbar}$, then $(\Theta, N, \odot)$ is P.Fco-M.S.

Proof:

1) $\kappa = \varpi \Leftrightarrow R_{\hbar}(\kappa, \kappa, t) = R_{\hbar}(\kappa, \varpi, t) = R_{\hbar}(\varpi, \varpi, t)$

$\Leftrightarrow 1 - R_{\hbar}(\kappa, \kappa, t) = 1 - R_{\hbar}(\kappa, \varpi, t) = 1 - R_{\hbar}(\varpi, \varpi, t)$

$\Leftrightarrow N(\kappa, \kappa, t) = N(\kappa, \varpi, t) = N(\varpi, \varpi, t)$.

2) $R_{\hbar}(\kappa, \kappa, t) \geq R_{\hbar}(\kappa, \varpi, t)$

$\Longrightarrow 1-R_{\hbar}(\kappa,\kappa,t)\leq 1-R_{\hbar}(\kappa,\varpi,t)$

$\Longrightarrow N(\kappa,\kappa,t)\leq N(\kappa,\varpi,t)$.

3) $N(\kappa,\varpi,t)=1-CR_{\hbar}(\kappa,\varpi,t)=1-R_{\hbar}(\varpi,\kappa,t)=N(\varpi,\kappa,t)$

4) $N(\kappa,\varpi,\{t,s\})\odot N(\alpha,\alpha,\{t,s\})$

$=[1-R_{\hbar}(\kappa,\varpi,\{t,s\})]\odot[1-R_{\hbar}(\alpha,\alpha,\{t,s\})]$

$=1-[R_{\hbar}(\kappa,\varpi,\{t,s\})\circ(R_{\hbar}(\kappa,\varpi,\{t,s\})]$

$\leq 1-[R_{\hbar}(\kappa,\alpha,t)\circ R_{\hbar}(\alpha,\varpi,t)]$

$=\big(1-R_{\hbar}(\kappa,\alpha,t)\big)\odot\big(1-R_{\hbar}(\alpha,\varpi,t)\big)=N(\kappa,\alpha,t)\odot N(\alpha,\varpi,t)$

5) $N(\kappa,\varpi,.):(0,\infty)\to K$ is continuous.

Then, $(\Theta,N,\odot)$ is P.F.co-M.S.

Definition 3.8: Let $(\Theta,CR_{\hbar},\odot)$ is P.F.co-M.S and $\{\kappa_n\}$ be a sequence in Θ, we call that $\{\kappa_n\}$ is:

1) Fuzzy converge to a point $\kappa\in\Theta$ if $CR_{\hbar}(\kappa_n,\kappa,t)=CR_{\hbar}(\kappa,\kappa,t)$ for all $t>0$.

2) Fuzzy Cauchy sequence in Θ if $CR_{\hbar}(\kappa_n,\kappa_m,t)$ exists (fuzzy 0-Cauchy if $CR_{\hbar}(\kappa_n,\kappa_m,t)=0$).

Definition 3.9: A P.F.co-M.S is called complete (0-complete) if every F. Cauchy (F. 0-Cuachy) sequence belong to Θ is F. converges in it.

Theorem 3.10: Let $(\Theta,CR_{\hbar},\odot)$ is P.F.co-M.S, then every sequence in Θ has a unique fuzzy convergence if $\odot$ satisfy the condition $\xi\odot\eta\leq\xi\odot\upsilon\to\eta\leq\upsilon$ for all $\xi,\eta,\upsilon\in K$ and $CR_{\hbar}(\kappa_n,\kappa_n,t)=CR_{\hbar}(\kappa,\kappa,,t)=CR_{\hbar}(\varpi,\varpi,t)$.

Proof: Suppose that $\{\kappa_n\}$ be a fuzzy converge sequence in Θ to two distinct points κ and ϖ, that is

$CR_{\hbar}(\kappa_n,\kappa,t)=CR_{\hbar}(\kappa,\kappa,t)$ and $CR_{\hbar}(\kappa_n,\kappa,t)=CR_{\hbar}(\varpi,\varpi,t)$

$CR_{\hbar}(\kappa,\varpi,t)\odot CR_{\hbar}(\kappa_n,\kappa_n,t)\leq CR_{\hbar}(\kappa,\kappa_n,t)\odot CR_{\hbar}(\kappa_n,\varpi,t)$

By taking the limit as $n\to\infty$,

$\Longrightarrow CR_{\hbar}(\kappa,\varpi,t)\odot CR_{\hbar}(\kappa,\kappa,t)\leq CR_{\hbar}(\kappa,\kappa,t)\odot CR_{\hbar}(\varpi,\varpi,t)$

$\Longrightarrow CR_{\hbar}(\kappa,\varpi,t)\leq CR_{\hbar}(\varpi,\varpi,t)$ and since $CR_{\hbar}(\kappa,\varpi,t)\geq CR_{\hbar}(\varpi,\varpi,t)$.

$\Longrightarrow CR_{\hbar}(\kappa,\varpi,t)=CR_{\hbar}(\varpi,\varpi,t)=CR_{\hbar}(\kappa,\kappa,t)$, and so $\kappa=\varpi$.

Definition 3.11: Let $(\Theta,CR_{\hbar},\odot)$ is P.F.co-M.S. We call that $CR_{\hbar}$ is strong P.F.co-M.S if it is satisfy the additional condition:

6) $CR_{\hbar}(\kappa,\varpi,t\})\odot CR_{\hbar}(\alpha,\alpha,t\})\leq CR_{\hbar}(\kappa,\alpha,t)\odot CR_{\hbar}(\alpha,\varpi,t)$ for all $\kappa,\varpi,\alpha\in\Theta$ and $t>0$.

Example 3.12: Let $\Theta = R^{+}$, $\xi \odot \eta = \xi + \eta$, such that $\eta \leq \upsilon$ whenever $\xi \odot \eta \leq \xi \odot \upsilon$ for all $\xi, \eta, \upsilon \in K$, and $CR_{\hbar}: \Theta^2 \times (0, \infty) \to K$ defined by

$CR_{\hbar}(\kappa, \varpi, t) = 1 - \frac{\{\kappa,\varpi\} + t}{\{\kappa,\varpi\} + t}$, for all $\kappa, \varpi \in \Theta, t > 0$, then $(\Theta, CR_{\hbar}, \odot)$ is strong P.F.co-M.S.

Proof:

1) If $\kappa = \varpi \Longrightarrow 1 - \frac{\{\kappa,\varpi\} + t}{\{\kappa,\varpi\} + t} = 0$

$\Longrightarrow CR_{\hbar}(\kappa, \kappa, t) = CR_{\hbar}(\kappa, \varpi, t) = CR_{\hbar}(\varpi, \varpi, t)$

If $CR_{\hbar}(\kappa, \kappa, t) = CR_{\hbar}(\kappa, \varpi, t) = CR_{\hbar}(\varpi, \varpi, t)$

Since $CR_{\hbar}(\kappa, \kappa, t) = CR_{\hbar}(\varpi, \varpi, t) = 0 \Longrightarrow CR_{\hbar}(\kappa, \varpi, t) = 0$

$\Longrightarrow 1 - \frac{\{\kappa,\varpi\} + t}{\{\kappa,\varpi\} + t} = 0 \Longrightarrow \frac{\{\kappa,\varpi\} + t}{\{\kappa,\varpi\} + t} = 1 \Longrightarrow \kappa = \varpi$

2) $CR_{\hbar}(\kappa, \varpi, t) \geq CR_{\hbar}(\kappa, \kappa, t) = 0$.

3) $CR_{\hbar}(\kappa, \varpi, t) = CR_{\hbar}(\varpi, \kappa, t)$

6) As in example (3.2) there are 6 cases to comparable among κ, ϖ and α and from these cases we deduce that

$$CR_{\hbar}(\kappa, \varpi, t\}) \odot CR_{\hbar}(\alpha, \alpha, t\}) \leq CR_{\hbar}(\kappa, \alpha, t) \odot CR_{\hbar}(\alpha, \varpi, t)$$

4) From (6) for $t, s > 0$, and the condition of $\odot$

$$CR_{\hbar}(\kappa, \varpi, max\,\{t, s\}) \odot CR_{\hbar}(\alpha, \alpha, \{t, s\}\,) \leq CR_{\hbar}(\kappa, \alpha, \{t, s\}\,) \odot CR_{\hbar}(\alpha, \varpi, \{t, s\}\,) \leq CR_{\hbar}(\kappa, \alpha, t\}) \odot CR_{\hbar}(\alpha, \varpi, s\}\,)$$

5) $CR_{\hbar}(\kappa, \varpi, .)$ is continuous.

Theorem 3.13: Let $\Theta \neq \emptyset$, $\odot$ be a continuous t-conorm such that $\xi \odot \eta \leq \xi \odot \upsilon$ whenever $\eta \leq \upsilon$, then $CR_{\hbar}: \Theta^2 \times (0, \infty) \to K$ is strong P.F.co-M function iff it is satisfy the conditions for all $\kappa, \varpi, \alpha \in \Theta, t > 0$,

1) $CR_{\hbar}(\kappa, \kappa, t) = CR_{\hbar}(\kappa, \varpi, t) = CR_{\hbar}(\varpi, \varpi, t)$ if and only if $\kappa = \varpi$,

2) $CR_{\hbar}(\kappa, \kappa, t) \leq CR_{\hbar}(\kappa, \varpi, t)$,

3) $CR_{\hbar}(\kappa, \varpi, t) \odot CR_{\hbar}(\alpha, \alpha, t) \leq CR_{\hbar}(\kappa, \alpha, t) \odot CR_{\hbar}(\varpi, \alpha, t)$,

4) $CR_{\hbar}(\kappa, \varpi, .) : (0, \infty) \to K$ is continuous.

Proof: If M is strong P.F.co-M, then by its definition the conditions hold.

On the other hand, if the conditions valid, we prove that $CR_{\hbar}$ is strong P.F.co-M, the conditions 1,2 and 5 satisfy from the conditions 1,2 and 4.

To prove the condition 3 of definition strong P.F.co-M, from the third condition and the condition of $\odot$,

$CR_{\hbar}(\kappa,\varpi,t)\odot CR_{\hbar}(\kappa,\kappa,t) \leq CR_{\hbar}(\kappa,\kappa,t)\odot CR_{\hbar}(\varpi,\kappa,t)$ (1)

$\Longrightarrow CR_{\hbar}(\kappa,\varpi,t) \leq CR_{\hbar}(\varpi,\kappa,t)$

Also, $CR_{\hbar}(\varpi,\kappa,t)\odot CR_{\hbar}(\varpi,\varpi,t) \leq CR_{\hbar}(\varpi,\varpi,t)\odot CR_{\hbar}(\kappa,\varpi,t)$

$\Rightarrow CR_{\hbar}(\varpi,\kappa,t) \leq CR_{\hbar}(\kappa,\varpi,t)$ (2)

From (1) and (2), we have $CR_{\hbar}(\kappa,\varpi,t) = CR_{\hbar}(\varpi,\kappa,t)$

Now, from (3) $CR_{\hbar}(\kappa,\varpi,t)\odot CR_{\hbar}(\alpha,\alpha,t) \leq CR_{\hbar}(\kappa,\alpha,t)\odot CR_{\hbar}(\varpi,\alpha,t)$

$= CR_{\hbar}(\kappa,\alpha,t)\odot CR_{\hbar}(\alpha,\varpi,t)$

To prove the condition (4), let $s,t > 0$, from (3)

$CR_{\hbar}(\kappa,\varpi,\{t,s\})\odot CR_{\hbar}(\alpha,\alpha,\{t,s\})$

$\leq CR_{\hbar}(\kappa,\alpha,max\,\{t,s\})\odot CR_{\hbar}(\varpi,\alpha,max\,\{t,s\}) \leq CR_{\hbar}(\kappa,\alpha,t)\odot CR_{\hbar}(\alpha,\varpi,s)$.

Therefore, $CR_{\hbar}$ is strong P.F.co-M.

Theorem 3.14: Let $\{\kappa_n\}$ and $\{\varpi_n\}$ be two sequences in strong P.F.co-M.S $(\Theta, CR_{\hbar},\odot)$ such that $\eta \leq \upsilon$ whenever $\xi\odot\eta \leq \xi\odot\upsilon$ for all $\xi,\eta,\upsilon \in K$,$CR_{\hbar}(\kappa_n,\kappa,t) = \lim_{n\to\infty} CR_{\hbar}(\kappa_n,\kappa_n,,t) = CR_{\hbar}(\kappa,\kappa,t)$ and $CR_{\hbar}(\varpi_n,\varpi,t) = \lim_{n\to\infty} CR_{\hbar}(\varpi_n,\varpi_n,,t) = CR_{\hbar}(\varpi,\varpi,t)$, then

$\lim_{n\to\infty} CR_{\hbar}(\kappa_n,\varpi_n,,t) = CR_{\hbar}(\kappa,\varpi,t)$.

Proof: As $CR_{\hbar}(\kappa_n,\varpi_n,t)\odot CR_{\hbar}(\kappa,\kappa,t) \leq CR_{\hbar}(\kappa_n,\kappa,t)\odot CR_{\hbar}(\kappa,\varpi_n,t)$

$\Rightarrow CR_{\hbar}(\kappa_n,\varpi_n,t)\odot CR_{\hbar}(\kappa,\kappa,t)\odot CR_{\hbar}(\varpi,\varpi,t)$

$\leq CR_{\hbar}(\kappa_n,\kappa,t)\odot CR_{\hbar}(\kappa,\varpi_n,t)\odot CR_{\hbar}(\varpi,\varpi,t)$

$\leq CR_{\hbar}(\kappa_n,\kappa,t)\odot CR_{\hbar}(\kappa,\varpi,t)\odot CR_{\hbar}(\varpi,\varpi_n,t)$

$\Rightarrow \lim_{n\to\infty} CR_{\hbar}(\kappa_n,\varpi_n,t)\odot CR_{\hbar}(\kappa,\kappa,t)\odot CR_{\hbar}(\varpi,\varpi,t)$

$\leq \lim_{n\to\infty} CR_{\hbar}(\kappa_n,\kappa,t)\odot CR_{\hbar}(\kappa,\varpi,t)\odot \lim_{n\to\infty} CR_{\hbar}(\varpi,\varpi_n,t)$

$= CR_{\hbar}(\kappa,\kappa,t)\odot CR_{\hbar}(\kappa,\varpi,t)\odot CR_{\hbar}(\varpi,\varpi,t)$

$\Rightarrow \lim_{n\to\infty} CR_{\hbar}(\kappa_n,\varpi_n,t) \leq CR_{\hbar}(\kappa,\varpi,t)$……(1)

Also, as $CR_{\hbar}(\kappa,\varpi,t)\odot CR_{\hbar}(\kappa_n,\kappa_n,t) \leq CR_{\hbar}(\kappa,\kappa_n,t)\odot CR_{\hbar}(\kappa_n,\varpi,t)$

$\Rightarrow CR_{\hbar}(\kappa,\varpi,t)\odot CR_{\hbar}(\kappa_n,\kappa_n,t)\odot CR_{\hbar}(\varpi_n,\varpi_n,t)$

$\leq CR_{\hbar}(\kappa,\kappa_n,t)\odot CR_{\hbar}(\kappa_n,\varpi,t)\odot CR_{\hbar}(\varpi_n,\varpi_n,t)$

$\leq CR_{\hbar}(\kappa,\kappa_n,t)\odot CR_{\hbar}(\kappa_n,\varpi_n,t)\odot CR_{\hbar}(\varpi,\varpi_n,t)$

$\Rightarrow CR_{\hbar}(\kappa,\varpi,t)\odot \lim_{n\to\infty} CR_{\hbar}(\kappa_n,\kappa_n,t)\odot \lim_{n\to\infty} CR_{\hbar}(\varpi_n,\varpi_n,t)$

$\leq \lim_{n\to\infty} CR_{\hbar}(\kappa,\kappa_n,t)\odot \lim_{n\to\infty} CR_{\hbar}(\kappa_n,\varpi_n,t)\odot \lim_{n\to\infty} CR_{\hbar}(\varpi,\varpi_n,t)$

$\Rightarrow CR_\hbar(\kappa,\varpi,t) \odot CR_\hbar(\kappa,\kappa,t) \odot CR_\hbar(\varpi,\varpi,t)$

$\leq CR_\hbar(\kappa,\kappa,t) \odot \lim_{n\to\infty} CR_\hbar(\kappa_n,\varpi_n,t) \odot CR_\hbar(\varpi,\varpi,t)$

$\Rightarrow CR_\hbar(\kappa,\varpi,t) \leq \lim_{n\to\infty} CR_\hbar(\kappa_n,\varpi_n,t)$…..(2)

From (1) and (2), we deduce that $\lim_{n\to\infty} CR_\hbar(\kappa_n,\varpi_n,t) = CR_\hbar(\kappa,\varpi,t)$.

Theorem 3.15: Let $(\Theta, CR_\hbar, \circ)$ is strong P.F.M.S. If define $B = 1 - CR_\hbar$, then $(\Theta, B, \odot)$ is strong P.F.co-M.S.

The proof is similar to Theorem (3.7).

4. Conclusion

In this article, we revise the notion of a P.F.M by using t-conorms rather than t-norms, naming it a P.F. co-metric, which is analogous to revising the concept of a F.M.by using t-conorms instead of t-norms. This notion was expressed by Noori et al. in 2012 and Alexander Šostak in 2018. We also discussed the concepts of strong P.F.M. and strong P.F. co-metric, as well as some problems and examples that relate to them.

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