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CARTESIAN PRODUCT OF INTUITIONISTIC FUZZY RECTANGULAR *n*-NORMED SPACES

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Abstract:

This paper present a new concept of intuitionistic fuzzy rectangular *n*-normed spaces, with some fundamental definitions. Subsequently, we present the Cartesian product of intuitionistic fuzzy rectangular n-normed spaces, study its effect on the properties of intuitionistic fuzzy rectangular *n*-normed spaces, and prove that the Cartesian product of intuitionistic fuzzy rectangular *n*-normed spaces is also an intuitionistic fuzzy rectangular *n*-normed spaces. Finally, we establish the completeness of the Cartesian product of complete intuitionistic fuzzy rectangular *n*-normed spaces and provide some theorems related to these spaces.

Keywords: *Rectangular n-normed space, fuzzy rectangular n-normed space, intuitionistic fuzzy rectangular n-normed space, the Cartesian product of intuitionistic fuzzy rectangular n-normed spaces.*

1-Introduction:

In 1986, K. Atanassov [1] presented the concept of the intuitionistic fuzzy set as a generalization of fuzzy set. Later, in an intuitionistic fuzzy set, M. J. Mohammed and G. A. Ataa [2] created an intuitionistic fuzzy topology space and established some features. In 2020, N. H. Sharif and M. J. Mohammed [13] presented a study on b-intuitionistic fuzzy normed spaces with some characterizations, building on the research form studies in [7,12]. The theory of *2*-normed and *n*-normed linear spaces was first initially presented by S. Gähler [5, 6]. Subsequently, A. Narayan and S.Vijayabalaji [10] established and expanded the theory of fuzzy *n*-normed space, building upon the work of S. Gähler [6] and A. Katsaras [8]. S.Vijayabalaji and N. Thillaigovindan et al. [14] introduced the concept of intuitionistic fuzzy *n*-normed linear space, and they also established some fundamental results .On the other hand, A. Branciari proposed the idea of rectangular metric space in 2000 [3]. Following this, H. H. Muteer and M. J. Mohammed [9] presented the idea of intuitionistic fuzzy rectangular b-normed spaces. Recently, M. R. Bader and M. J. Mohammed [4] introduced the concept of fuzzy rectangular *n*-normed space and discussed some of their properties.

In this paper, we present the definition of an intuitionistic fuzzy rectangular *n*-normed space, as well as the Cartesian product of these spaces. Also, we study its effect on the properties of intuitionistic fuzzy rectangular *n*-normed spaces, proving some related theorems.

2- Preliminaries

In this paragraph, we review some fundamental ideas and preliminaries regarding fuzzy rectangular *n*-normed space.

Definition 2.1 [4]

Let X be a vector space of dimension $d \geq n$, $n \in$ N(natural numbers). A rectangular *n*-norm on X is a function $\|.,\ldots,.\|$ on $X\times X\times \cdots \times X = X^n$ satisfying the following for $\eta_1, \eta_2,\ldots, \eta_n, \mathfrak{h}, z \in X$.

1) $\|\eta_1, \eta_2,\ldots, \eta_n\| = 0 \Leftrightarrow \eta_1, \eta_2,\ldots, \eta_n$ are linearly dependent,
2) $\|\eta_1, \eta_2,\ldots, \eta_n\|$ is invariant under any permutation,
3) $\|\lambda\eta_1, \lambda\eta_2,\ldots, \lambda\eta_n\| = |\lambda| \, \|\eta_1, \eta_2,\ldots, \eta_n\|$ for any $\lambda \in R$,
4) $\|\eta_1, \eta_2,\ldots, \eta_n + \mathfrak{h} + z\| \leq \|\eta_1, \eta_2,\ldots, \eta_n\| + \|\eta_1, \eta_2, \ldots, \mathfrak{h}\| + \|\eta_1, \eta_2, \ldots, z\|$.

$\|.,..,.\|$ is said to be a rectangular *n*-norm on X and the pair $(X, \|.,\ldots,.\|)$ is said to be a rectangular *n*-normed space.

Definition 2.2 [11]

A continuous t-norm $*$ is a binary operation on the interval [0,1], which satisfies the following axioms:

1) For each $e \in [0,1]$ implies that $e * 1 = e$;
2) $*$ is associative and commutative;
3) $*$ is continuous;
4) For each $e, s, z, d \in [0,1]$ and $e \leq z$ and $s \leq d$ implies that $e * s \leq z * d$.

Definition 2.3 [11]

A continuous t-conorm $\lozenge$ is a binary operation on the interval [0,1] which satisfies the following axioms:

1) For each $e \in [0,1]$ implies that $e \lozenge 0 = e$;
2) $\lozenge$ is associative and commutative;
3) $\lozenge$ is continuous;
4) For each $e, s, z, d \in [0,1]$ and $e \leq z$ and $s \leq d$ implies that $e \lozenge s \leq z \lozenge d$.

Definition 2.4 [4]

Let X be a vector space, $*$ be a continuous t-norm. Then the 3-tuple $(X, \Upsilon, *)$ is called a fuzzy rectangular *n*-normed space (for short, *FR-n-NS*) on X, if Υ is a fuzzy set on $X^n\times(0, \infty)$ satisfies the following for all $\eta_1, \eta_2,\ldots, \eta_n, \mathfrak{h}, z \in X$ and $\ell, ƚ, ɞ > 0$

1) $\Upsilon(\eta_1, \eta_2,\ldots, \eta_n, \ell) = 0$, for all $\ell \in R$ with $\ell \leq 0$,
2) $\Upsilon(\eta_1, \eta_2,\ldots, \eta_n, \ell) = 1 \Leftrightarrow \eta_1, \eta_2,\ldots, \eta_n$ are linearly dependent,
3) $\Upsilon(\eta_1, \eta_2,\ldots, \eta_n, \ell)$ is invariant under any permutation of $\eta_1, \eta_2,\ldots, \eta_n$,
4) $\Upsilon(\lambda\eta_1, \lambda\eta_2,\ldots, \lambda\eta_n, \ell) = \Upsilon(\eta_1, \eta_2,\ldots, \eta_n, \frac{\ell}{|\lambda|})$, if $\lambda \in F\backslash\{0\}$,
5) $\Upsilon(\eta_1, \eta_2,\ldots, \eta_n + \mathfrak{h} + z, \ell + ƚ + ɞ) \geq \Upsilon(\eta_1, \eta_2,\ldots, \eta_n, \ell)$ $\Upsilon(\eta_1, \eta_2, \ldots, \mathfrak{h}, ƚ) * \Upsilon(\eta_1, \eta_2, \ldots, z, ɞ)$
6) $\Upsilon(\eta_1, \eta_2,\ldots, \eta_n, \ell)$ is a non-decreasing function of $\ell \in R$ and $\lim_{\ell\to\infty} \Upsilon(\eta_1, \eta_2,\ldots, \eta_n, \ell)= 1$,

Hence, (Υ) is said to be a fuzzy rectangular *n*-norm on X.

Definition 2.5[4]

Let (X, ϒ, $*$) be a *FR-n-NS*. Then:

(i) A sequence $\{\eta_n\}$ in X is said to be convergent to η, if given φ > 0, ℓ > 0, 0 < φ < 1 there is $n_0 \in N$ in which

$$\Upsilon(\eta_1, \eta_2, \ldots, \eta_{n-1}, \eta_n - \eta, \ell) > 1 - \varphi \quad \text{for all } n \geq n_0.$$

(ii) A sequence $\{\eta_n\}$ in X is said to be Cauchy sequence if, a given φ > 0 with 0 < φ < 1 and ℓ > 0 there is $n_0 \in N$ in which

$$\Upsilon(\eta_1, \eta_2, \ldots, \eta_{n-1}, \eta_n - \eta_\kappa, \ell) > 1 - \nu \text{ for all } n, \kappa \geq n_0.$$

(iii) A *FR-n-NS* (X, ϒ, $*$) is said to be complete if, every Cauchy sequence converges.

3-CARTESIAN PRODUCT OF INTUITIONISTIC FUZZY RECTANGULAR *n*-NORMED SPACES

In this section, we present the definition of an intuitionistic fuzzy rectangular *n*-normed space, and also we define the Cartesian product of two-intuitionistic fuzzy rectangular *n*-normed space and prove some results related to it.

Definition 3.1:

Let X be a vector space , $*$ be a continuous t-norm, ◊ be a continuous t-conorm, a function ϒ, Ꜧ: $X^n \times (0, \infty) \to [0,\infty]$ is called intuitionistic fuzzy rectangular *n*-norm if it satisfying the following for all $(\eta_1, \eta_2,\ldots, \eta_n, \mathfrak{h}, z) \in X$ and ℓ, ƚ, ɕ > 0:

1) $\Upsilon(\eta_1, \eta_2,\ldots, \eta_n, \ell) + Ꜧ(\eta_1, \eta_2,\ldots, \eta_n, \ell) \leq 1$,
2) $\Upsilon(\eta_1, \eta_2,\ldots, \eta_n, \ell) = 0$, for all $\ell \in R$ with $\ell \leq 0$,
3) $\Upsilon(\eta_1, \eta_2,\ldots, \eta_n, \ell) = 1 \Leftrightarrow \eta_1, \eta_2,\ldots, \eta_n$ are linearly dependent,
4) $\Upsilon(\eta_1, \eta_2,\ldots, \eta_n, \ell)$ is invariant under any permutation of $\eta_1, \eta_2,\ldots, \eta_n$,
5) $\Upsilon(\lambda\eta_1, \lambda\eta_2,\ldots, \lambda\eta_n, \ell) = \Upsilon(\eta_1, \eta_2,\ldots, \eta_n, \frac{\ell}{|\lambda|})$, if $\lambda \in F\backslash\{0\}$,
6) $\Upsilon(\eta_1, \eta_2,\ldots, \eta_n + \mathfrak{h} + z, \ell + ƚ + ɕ) \geq \Upsilon(\eta_1, \eta_2,\ldots, \eta_n, \ell)$ $*\Upsilon(\eta_1, \eta_2, \ldots, \mathfrak{h}, ƚ) * \Upsilon(\eta_1, \eta_2, \ldots, z, ɕ)$
7) $\Upsilon(\eta_1, \eta_2,\ldots, \eta_n, \ell)$ is a non-decreasing function of $\ell \in R$ and $\lim_{\ell\to\infty} \Upsilon(\eta_1, \eta_2,\ldots, \eta_n, \ell) = 1$,
8) $Ꜧ(\eta_1, \eta_2,\ldots, \eta_n, \ell) = 1$,
9) $Ꜧ(\eta_1, \eta_2,\ldots, \eta_n, \ell) = 0 \Leftrightarrow \eta_1, \eta_2,\ldots, \eta_n$ are linearly dependent,
10) $Ꜧ(\eta_1, \eta_2,\ldots, \eta_n, \ell)$ is invariant under any permutation of $\eta_1, \eta_2,\ldots, \eta_n$,
11) $Ꜧ(\lambda\eta_1, \lambda\eta_2,\ldots, \lambda\eta_n, \ell) = Ꜧ(\eta_1, \eta_2,\ldots, \eta_n, \frac{\ell}{|\lambda|})$, if $\lambda \in F\backslash\{0\}$,
12) $Ꜧ(\eta_1, \eta_2,\ldots, \eta_n + \mathfrak{h} + z, \ell + ƚ + ɕ) \leq Ꜧ(\eta_1, \eta_2,\ldots, \eta_n, \ell)$ ◊ $Ꜧ(\eta_1, \eta_2, \ldots, \mathfrak{h}, ƚ)$ ◊ $Ꜧ(\eta_1, \eta_2, \ldots, z, ɕ)$,
13) $Ꜧ(\eta_1, \eta_2,\ldots, \eta_n, \ell)$ is a non- increasing function of $\ell \in R$ and $\lim_{\ell\to\infty} Ꜧ(\eta_1, \eta_2,\ldots, \eta_n, \ell) = 0$.

Hence, (X, ϒ, Ꜧ, $*$, ◊) is called an intuitionistic fuzzy rectangular *n*-normed space (for short, *IFR-n-NS*).

Example 3.2:

Let (X, ‖.,…,.‖) be a rectangular *n*-normed space. Define e $*$ s = e.s and e ◊ s = min{1, e + s} for each e, s ∈ [0, l]. Defined as follows:

$$\Upsilon(\eta_1, \eta_2,\ldots, \eta_n, \ell) = \frac{\ell}{\ell + ||\eta_1,\eta_2,\ldots,\eta_n||}, \quad Ꜧ(\eta_1, \eta_2,\ldots, \eta_n, \ell) = \frac{||\eta_1,\eta_2,\ldots,\eta_n||}{\ell + ||\eta_1,\eta_2,\ldots,\eta_n||},$$

ℓ > 0 , $(\eta_1, \eta_2,\ldots, \eta_n) \in X$, so (X, ϒ, Ꜧ, $*$, ◊) is an *IFR-n-NS*. Hence (X, ϒ, Ꜧ, $*$, ◊) is said to be a standard intuitionistic fuzzy rectangular *n*-normed space(for short, *St-IFR-n-NS*) induced by a rectangular *n*- normed space (X, ‖.,…,.‖).

Definition 3.3:

Let $(X, \Upsilon, Ꞩ, *, \lozenge)$ be an *IFR-n-NS*. Then:

(i) A sequence $\{\eta_n\}$ in X is said to be convergent to η, if for each $\varphi \in (0,1)$ and $\ell > 0$ there is $n_0 \in N$ in which

$$\Upsilon(\eta_1, \eta_2,\ldots, \eta_{n-1}, \eta_n - \eta, \ell) > 1 - \varphi \text{ and } Ꞩ(\eta_1, \eta_2,\ldots, \eta_{n-1}, \eta_n - \eta, \ell) < \varphi \text{ ,for all } n \geq n_0.$$

(Or equivalently,

$$\lim_{\ell\to\infty} \Upsilon(\eta_1, \eta_2,\ldots, \eta_{n-1}, \eta_n - \eta, \ell) = 1 \text{ and } \lim_{\ell\to\infty} Ꞩ(\eta_1, \eta_2,\ldots, \eta_{n-1}, \eta_n - \eta, \ell) = 0).$$

(ii) A sequence $\{\eta_n\}$in X is said to be Cauchy if, for all each $\varphi \in (0,1)$ and $\ell > 0$ there is $n_0 \in N$ in which

$$\Upsilon(\eta_1, \eta_2,\ldots, \eta_{n-1}, \eta_n - \eta_\kappa, \ell) > 1 - \varphi \text{ and } Ꞩ(\eta_1, \eta_2,\ldots, \eta_{n-1}, \eta_n - \eta_\kappa, \ell) < \varphi \text{ ,for all } n, \kappa \geq n_0.$$

(Or equivalently,

$$\lim_{\ell\to\infty} \Upsilon(\eta_1, \eta_2,\ldots, \eta_{n-1}, \eta_n - \eta_\kappa, \ell) = 1 \text{ and } \lim_{\ell\to\infty} Ꞩ(\eta_1, \eta_2,\ldots, \eta_{n-1}, \eta_n - \eta_\kappa, \ell) = 0.$$

(ii) An *IFR-n-NS* $(X, \Upsilon, Ꞩ)$ is said to be complete if, every Cauchy sequence converges.

Definition 3.4:

Let $(X, \Upsilon_1, Ꞩ_1, *, \lozenge)$ and $(U, \Upsilon_2, Ꞩ_2, *, \lozenge)$ be two *IFR-n-NS*. The Cartesian product of $(X, \Upsilon_1, Ꞩ_1, *, \lozenge)$ and $(U, \Upsilon_2, Ꞩ_2, *, \lozenge)$ is the product space $(X \times U, \Upsilon, Ꞩ, *, \lozenge)$, where $X \times U$ is the Cartesian product of the sets $X^n \times U^n$ and $\Upsilon, Ꞩ$ are a function

$\Upsilon:((X^n \times U^n) \times (0,\infty) \to [0,1])$ and

$Ꞩ:((X^n \times U^n) \times (0,\infty) \to [0,1])$ are given by:

$\Upsilon:(\eta_1, \eta_2,\ldots, \eta_n, \vartheta_1, \vartheta_2, \ldots, \vartheta_n), \ell) = \Upsilon_1(\eta_1, \eta_2,\ldots, \eta_n, \ell) * \Upsilon_2(\vartheta_1, \vartheta_2,\ldots, \vartheta_n, \ell)$ and

$Ꞩ:(\eta_1, \eta_2,\ldots, \eta_n, \vartheta_1, \vartheta_2, \ldots, \vartheta_n), \ell) = Ꞩ_1(\eta_1, \eta_2,\ldots, \eta_n, \ell) \lozenge Ꞩ_2(\vartheta_1, \vartheta_2,\ldots, \vartheta_n, \ell)$.

For all $(\eta_1, \eta_2,\ldots, \eta_n, \vartheta_1, \vartheta_2, \ldots, \vartheta_n) \in X^n \times U^n$ and $\ell > 0$.

Next we show that if X and Y are *IFR-n-NSs*, then their Cartesian product will also be an *IFR-n-NS.*

Theorem 3.5:

Let $(X, \Upsilon_1, Ꞩ_1, *, \lozenge)$ and $(U, \Upsilon_2, Ꞩ_2, *, \lozenge)$ be an *IFR-n-NSs*. Then $(X^n \times U^n, \Upsilon, Ꞩ, *, \lozenge)$ is an *IFR-n-NS.*

Proof:

Since $(X, \Upsilon_1, Ꞩ_1, *, \lozenge)$ and $(U, \Upsilon_2, Ꞩ_2, *, \lozenge)$ be an *IFR-n-NSs*

(1)

Since $\Upsilon_1(\eta_1, \eta_2,\ldots, \eta_n, \ell) + Ꞩ_1(\vartheta_1, \vartheta_2, \ldots, \vartheta_n, \ell) \leq 1$

and $\Upsilon_2(\eta_1, \eta_2,\ldots, \eta_n, \ell) + Ꞩ_2(\vartheta_1, \vartheta_2, \ldots, \vartheta_n, \ell) \leq 1$

$\Rightarrow \Upsilon((\eta_1, \eta_2,\ldots, \eta_n, \vartheta_1, \vartheta_2,\ldots, \vartheta_n), \ell) + Ꞩ((\eta_1, \eta_2,\ldots, \eta_n, \vartheta_1, \vartheta_2,\ldots, \vartheta_n), \ell) \leq 1.$

(2)

Since $\Upsilon_1(\eta_1, \eta_2,\ldots, \eta_n, \ell) = 0$ and $\Upsilon_2(\vartheta_1, \vartheta_2,\ldots, \vartheta_n, \ell) = 0$, for all $\ell > 0$

$\Rightarrow \Upsilon((\eta_1, \eta_2,\ldots, \eta_n, \vartheta_1, \vartheta_2,\ldots, \vartheta_n), \ell) = 0.$

(3)

Since $\Upsilon_1(\eta_1, \eta_2,\ldots, \eta_n, \ell) = 1 \Leftrightarrow \eta_1, \eta_2,\ldots, \eta_n$ are linearly dependent

and $\Upsilon_2(\vartheta_1, \vartheta_2, \ldots, \vartheta_n, \ell) = 1 \Leftrightarrow \vartheta_1, \vartheta_2, \ldots, \vartheta_n$ are linearly dependent

$\Rightarrow \Upsilon((\eta_1, \eta_2,\ldots, \eta_n, \vartheta_1, \vartheta_2,\ldots, \vartheta_n), \ell) = 1$

$\Leftrightarrow (\eta_1, \eta_2,\ldots, \eta_n, \vartheta_1, \vartheta_2,\ldots, \vartheta_n)$ are linearly dependent.

(4)

Since $\Upsilon_1(\lambda\eta_1, \lambda\eta_2,\ldots, \lambda\eta_n, \ell) = \Upsilon_1(\eta_1, \eta_2,\ldots, \eta_n, \frac{\ell}{|\lambda|})$

and $\Upsilon_2(\lambda\vartheta_1, \lambda\vartheta_2,\ldots, \lambda\vartheta_n, \ell) = \Upsilon_2(\vartheta_1, \vartheta_2,\ldots, \vartheta_n, \frac{\ell}{|\lambda|})$

$\Rightarrow \Upsilon(\lambda(\eta_1, \eta_2,\ldots, \eta_n, \vartheta_1, \vartheta_2,\ldots, \vartheta_n), \ell)$

$= \Upsilon_1(\lambda\eta_1, \lambda\eta_2,\ldots, \lambda\eta_n, \ell) * \Upsilon_2(\lambda\vartheta_1, \lambda\vartheta_2,\ldots, \lambda\vartheta_n, \ell)$

$= \Upsilon_1(\eta_1, \eta_2,\ldots, \eta_n, \frac{\ell}{|\lambda|}) * \Upsilon_2(\vartheta_1, \vartheta_2,\ldots, \vartheta_n), \frac{\ell}{|\lambda|})$

$= \Upsilon((\eta_1, \eta_2,\ldots, \eta_n, \vartheta_1, \vartheta_2,\ldots, \vartheta_n), \frac{\ell}{|\lambda|})$.

(5)

Since $\Upsilon_1(\eta_1, \eta_2,\ldots, \eta_n + \mathfrak{h} + z, \ell + \mathfrak{t} + \mathfrak{e})$

$\geq \Upsilon_1(\eta_1, \eta_2,\ldots, \eta_n, \ell) * \Upsilon_1(\eta_1, \eta_2,\ldots, \mathfrak{h}, \mathfrak{t}) * \Upsilon_1(\eta_1, \eta_2,\ldots, z, \mathfrak{e})$ and

$\Upsilon_2(\vartheta_1, \vartheta_2,\ldots, \vartheta_n + \mathfrak{J} + w, \ell + \mathfrak{t} + \mathfrak{e})$

$\geq \Upsilon_2(\vartheta_1, \vartheta_2,\ldots, \vartheta_n, \ell) * \Upsilon_2(\vartheta_1, \vartheta_2,\ldots, \mathfrak{J}, \mathfrak{t}) * \Upsilon_2(\vartheta_1, \vartheta_2,\ldots, w, \mathfrak{e})$

$\Rightarrow \Upsilon((\eta_1, \eta_2,\ldots, \eta_n, \vartheta_1, \vartheta_2,\ldots, \vartheta_n) + (\eta_1, \eta_2,\ldots, \mathfrak{h}, \vartheta_1, \vartheta_2,\ldots, \mathfrak{J})$

$+ (\eta_1, \eta_2,\ldots, z, \vartheta_1, \vartheta_2,\ldots, w), (\ell + \mathfrak{t} + \mathfrak{e}))$

$\Rightarrow \Upsilon(\eta_1, \eta_2,\ldots, \eta_n + \mathfrak{h} + z, \vartheta_1, \vartheta_2,\ldots, \vartheta_n + \mathfrak{J} + w, (\ell + \mathfrak{t} + \mathfrak{e}))$

$= \Upsilon_1(\eta_1, \eta_2,\ldots, \eta_n + \mathfrak{h} + z, \ell + \mathfrak{t} + \mathfrak{e}) * \Upsilon_2(\vartheta_1, \vartheta_2, \ldots, \vartheta_n + \mathfrak{J} + w, \ell + \mathfrak{t} + \mathfrak{e})$

$\geq \Upsilon_1(\eta_1, \eta_2,\ldots, \eta_n, \ell) * \Upsilon_1(\eta_1, \eta_2, \ldots, \mathfrak{h}, \mathfrak{t}) * \Upsilon_1(\eta_1, \eta_2, \ldots, z, \mathfrak{e}) *$

$\Upsilon_2(\vartheta_1, \vartheta_2, \ldots, \vartheta_n, \ell) * \Upsilon_2(\vartheta_1, \vartheta_2, \ldots, \mathfrak{J}, \mathfrak{t}) * \Upsilon_2(\vartheta_1, \vartheta_2,\ldots, w, \mathfrak{e})$

$\geq \Upsilon_1(\eta_1, \eta_2,\ldots, \eta_n, \ell) * \Upsilon_2(\vartheta_1, \vartheta_2, \ldots, \vartheta_n, \ell) * \Upsilon_1(\eta_1, \eta_2, \ldots, \mathfrak{h}, \mathfrak{t}) *$

$\Upsilon_2(\vartheta_1, \vartheta_2, \ldots, \mathfrak{J}, \mathfrak{t}) * \Upsilon_1(\aleph_1, \aleph_2, \ldots, z, \mathfrak{e}) * \Upsilon_2(\vartheta_1, \vartheta_2, \ldots, w, \mathfrak{e})$

$= \Upsilon((\eta_1, \eta_2,\ldots, \eta_n, \vartheta_1, \vartheta_2,\ldots, \vartheta_n), \ell) * \Upsilon((\eta_1, \eta_2, \ldots, \mathfrak{h}, \vartheta_1, \vartheta_2,\ldots, \mathfrak{J}), \mathfrak{t}) *$

$\Upsilon((\eta_1, \eta_2, \ldots, z, \vartheta_1, \vartheta_2,\ldots, w), \mathfrak{e})$.

(6)

Since $\Upsilon_1(\eta_1, \eta_2,\ldots, \eta_n, \ell): (0,\infty) \to [0,1]$ is continuous in ℓ

and $\Upsilon_2(\vartheta_1, \vartheta_2, \ldots, \vartheta_n, \ell): (0, \infty) \to [0,1]$ is continuous in ℓ

$\Rightarrow \Upsilon((\eta_1, \eta_2,\ldots, \eta_n, \vartheta_1, \vartheta_2,\ldots, \vartheta_n), \ell): (0,\infty) \to [0,1]$ is continuous in ℓ.

(7)

Since $\lim_{\ell\to\infty} \Upsilon_1(\eta_1, \eta_2,\ldots, \eta_n, \ell) = 1$ and

$\lim_{\ell\to\infty} \Upsilon_2(\vartheta_1,\vartheta_2,\ldots, \vartheta_n, \ell) = 1$

$\Rightarrow \lim_{\ell\to\infty} \Upsilon((\eta_1, \eta_2,\ldots, \eta_n, \vartheta_1, \vartheta_2,\ldots, \vartheta_n), \ell) = 1$.

(8)

Since $\mathcal{H}_1(\eta_1, \eta_2,\ldots, \eta_n, \ell) = 1$ and $\mathcal{H}_2(\vartheta_1, \vartheta_2,\ldots, \vartheta_n, \ell) = 1$, for all $\ell > 0$

$\Rightarrow \mathcal{H}((\eta_1, \eta_2,\ldots, \eta_n, \vartheta_1, \vartheta_2,\ldots, \vartheta_n), \ell) = 1$.

(9)

Since $\mathcal{H}_1(\eta_1, \eta_2,\ldots, \eta_n, \ell) = 0 \Leftrightarrow \eta_1, \eta_2,\ldots, \eta_n$ are linearly dependent

and $\mathcal{H}_2(\vartheta_1, \vartheta_2, \ldots, \vartheta_n, \ell) = 0 \Leftrightarrow \vartheta_1, \vartheta_2, \ldots, \vartheta_n$ are linearly dependent

$\Rightarrow \mathcal{H}((\eta_1, \eta_2,\ldots, \eta_n, \vartheta_1, \vartheta_2,\ldots, \vartheta_n), \ell) = 0$

$\Leftrightarrow (\eta_1, \eta_2,\ldots, \eta_n, \vartheta_1, \vartheta_2,\ldots, \vartheta_n)$ are linearly dependent.

(10)

Since $\mathcal{H}_1(\lambda\eta_1, \lambda\eta_2,\ldots, \lambda\eta_n, \ell) = \mathcal{H}_1(\eta_1, \eta_2,\ldots, \eta_n, \frac{\ell}{|\lambda|})$

and $\mathcal{H}_2(\lambda\vartheta_1, \lambda\vartheta_2,\ldots, \lambda\vartheta_n, \ell) = \mathcal{H}_2(\vartheta_1, \vartheta_2,\ldots, \vartheta_n, \frac{\ell}{|\lambda|})$

$\Rightarrow \mathcal{H}(\lambda(\eta_1, \eta_2,\ldots, \eta_n, \vartheta_1, \vartheta_2,\ldots, \vartheta_n), \ell)$

$= \mathcal{H}_1(\lambda\eta_1, \lambda\eta_2,\ldots, \lambda\eta_n, \ell) \lozenge \mathcal{H}_2(\lambda\vartheta_1, \lambda\vartheta_2,\ldots, \lambda\vartheta_n, \ell)$

$= \mathcal{H}_1(\eta_1, \eta_2,\ldots, \eta_n, \frac{\ell}{|\lambda|}) \lozenge \mathcal{H}_2(\vartheta_1, \vartheta_2,\ldots, \vartheta_n), \frac{\ell}{|\lambda|})$

$= Ꞩ((η_1, η_2,…, η_n, ϑ_1, ϑ_2,…, ϑ_n), \frac{ℓ}{|λ|})$.

(11)

Since $Ꞩ_1(η_1, η_2,…, η_n + ɧ + z, ℓ + ƭ + ɕ)$
$≤ Ꞩ_1(η_1, η_2,…, η_n, ℓ) ◊ Ꞩ_1(η_1, η_2,…, ɧ, ƭ) ◊ Ꞩ_1(η_1, η_2,…, z, ɕ)$ and
$Ꞩ_2(ϑ_1, ϑ_2,…, ϑ_n + ȷ + w, ℓ + ƭ + ɕ)$
$≤ Ꞩ_2(ϑ_1, ϑ_2,…, ϑ_n, ℓ) ◊ Ꞩ_2(ϑ_1, ϑ_2,…, ȷ, ƭ) * ϒ_2(ϑ_1, ϑ_2,…, w, ɕ)$
$⇒ Ꞩ((η_1, η_2,…, η_n, ϑ_1, ϑ_2,…, ϑ_n) + (η_1, η_2,…, ɧ, ϑ_1, ϑ_2,…, ȷ)$
$+ (η_1, η_2,…, z, ϑ_1, ϑ_2,…, w), (ℓ + ƭ + ɕ))$
$⇒ Ꞩ(η_1, η_2,…, η_n + ɧ + z, ϑ_1, ϑ_2,…, ϑ_n + ȷ + w, (ℓ + ƭ + ɕ))$
$= Ꞩ_1(η_1, η_2,…, η_n + ɧ + z, ℓ + ƭ + ɕ) ◊ Ꞩ_2(ϑ_1, ϑ_2, …, ϑ_n + ȷ + w, ℓ + ƭ + ɕ)$
$≤ Ꞩ_1(η_1, η_2,…, η_n, ℓ) ◊ Ꞩ_1(η_1, η_2, …, ɧ, ƭ) ◊ Ꞩ_1(η_1, η_2, …, z, ɕ)$
$◊ Ꞩ_2(ϑ_1, ϑ_2, …, ϑ_n, ℓ) ◊ Ꞩ_2(ϑ_1, ϑ_2, …, ȷ, ƭ) ◊ Ꞩ_2(ϑ_1, ϑ_2,…, w, ɕ)$
$≤ Ꞩ_1(η_1, η_2,…, η_n, ℓ) ◊ Ꞩ_2(ϑ_1, ϑ_2, …, ϑ_n , ℓ) ◊ Ꞩ_1(η_1, η_2, …, ɧ, ƭ)$
$◊ Ꞩ_2(ϑ_1, ϑ_2, …, ȷ , ƭ) ◊ Ꞩ_1(ℵ_1, ℵ_2, …, z, ɕ) ◊ Ꞩ_2(ϑ_1, ϑ_2, …, w, ɕ)$
$= Ꞩ((η_1, η_2,…, η_n, ϑ_1, ϑ_2,…, ϑ_n), ℓ) ◊ Ꞩ((η_1, η_2, …, ɧ, ϑ_1, ϑ_2,…, ȷ), ƭ)$
$◊ Ꞩ((η_1, η_2, …, z, ϑ_1, ϑ_2,…, w), ɕ)$.

(12)

Since $Ꞩ_1(η_1, η_2,…, η_n, ℓ): (0,∞) → [0,1]$ is continuous in $ℓ$
and $Ꞩ_2(ϑ_1, ϑ_2, …, ϑ_n, ℓ) : (0, ∞) → [0,1]$ is continuous in $ℓ$
$⇒ Ꞩ((η_1, η_2,…, η_n, ϑ_1, ϑ_2,…, ϑ_n), ℓ): (0,∞) → [0,1]$ is continuous in $ℓ$.

(13) Since $\lim_{ℓ→∞} Ꞩ_1(η_1, η_2,…, η_n, ℓ) = 0$ and
$\lim_{ℓ→∞} Ꞩ_2(ϑ_1,ϑ_2,…, ϑ_n, ℓ) = 0$
$⇒ \lim_{ℓ→∞} Ꞩ((η_1, η_2,…, η_n, ϑ_1, ϑ_2,…, ϑ_n), ℓ) = 0$.

Therefore, it is a complete proof.

After that the following theorem proves that the converse of the above theorem (3.5) is true.

Theorem 3.6:

If $(X^n × U^n, ϒ, Ꞩ, *, ◊)$ is an *IFR-n-NS*, then $(X, ϒ_1, Ꞩ_1, *, ◊)$ and $(U, ϒ_2, Ꞩ_2, *, ◊)$ be an *IRF-n-NSs* by defining

$ϒ_1(η_1, η_2,…, η_n, ℓ) = ϒ((η_1, η_2,…, η_n, 0), ℓ)$ and

$Ꞩ_1(η_1, η_2,…, η_n, ℓ) = Ꞩ((η_1, η_2,…, η_n, 0), ℓ)$,

$ϒ_2(ϑ_1, ϑ_2, …, ϑ_n, ℓ) = ϒ((0, ϑ_1, ϑ_2,…, ϑ_n), ℓ)$ and

$Ꞩ_2(ϑ_1, ϑ_2, …, ϑ_n, ℓ) = Ꞩ((0, ϑ_1, ϑ_2, …, ϑ_n), ℓ)$

for all $η_1, η_2,…, η_n ∈ X$ and $ϑ_1, ϑ_2,…, ϑ_n ∈ U$ and $ℓ > 0$.

Proof:

(1)

$ϒ_1(η_1, η_2,…, η_n, ℓ) + Ꞩ_1(η_1, η_2,…, η_n, ℓ)$
$= ϒ((η_1, η_2,…, η_n, 0), ℓ) + Ꞩ((η_1, η_2,…, η_n, 0), ℓ) ≤ 1$
$⇒ ϒ_1(η_1, η_2,…, η_n, ℓ) + Ꞩ_1(η_1, η_2,…, η_n, ℓ) ≤ 1$.

(2)

$ϒ_1(η_1, η_2,…, η_n, ℓ) = ϒ((η_1, η_2,…, η_n, 0), ℓ) = 0$
for all $η_1, η_2,…, η_n ∈ X$
$⇒ ϒ_1(η_1, η_2,…, η_n, ℓ) = 0$ and

$Ħ_1(η_1, η_2,…, η_n, ℓ) = Ħ((η_1, η_2,…, η_n, 0), ℓ) = 1$
For all $η_1, η_2,…, η_n ∈ X ⇒ Ħ_1(η_1, η_2,…, η_n, ℓ) = 1$.
(3)
For all $ℓ > 0$, $1 = ϒ_1(η_1, η_2,…, η_n, ℓ) = ϒ((η_1, η_2,…, η_n, 0), ℓ)$
$⇔ η_1, η_2,…, η_n$ are linearly dependent
and $0 = Ħ_1(η_1, η_2,…, η_n, ℓ) = Ħ((η_1, η_2,…, η_n, 0), ℓ)$
$⇔ η_1, η_2,…, η_n$ are linearly dependent.
(4)
For all $ℓ > 0$,
$ϒ_1(λη_1, λη_2,…, λη_n, ℓ) = ϒ(λ(η_1, η_2,…, η_n, 0), ℓ)$
$ϒ((η_1, η_2,…, η_n, 0), \frac{ℓ}{|λ|}) = ϒ_1(η_1, η_2,…, η_n, \frac{ℓ}{|λ|})$ for all $λ ∈ F\backslash\{0\}$ and
$Ħ_1(λη_1, λη_2,…, λη_n, ℓ) = Ħ(λ(η_1, η_2,…, η_n, 0), ℓ)$
$Ħ((η_1, η_2,…, η_n, 0), \frac{ℓ}{|λ|}) = Ħ_1(η_1, η_2,…, η_n, \frac{ℓ}{|λ|})$ for all $λ ∈ F\backslash\{0\}$.
(5)
For all $η_1, η_2,…, η_n + ɧ + z ∈ X$ and $ℓ_1, ℓ_2, ℓ_3 > 0$. Then
$ϒ_1(η_1, η_2,…, η_n + ɧ + z, (ℓ_1 + ℓ_2 + ℓ_3))$
$= ϒ((η_1, η_2, …, η_n + ɧ + z, 0), (ℓ_1 + ℓ_2 + ℓ_3))$
$= ϒ((η_1, η_2, …, η_n, 0) + (η_1, η_2, …, ɧ, 0) + (η_1, η_2,…, z, 0), (ℓ_1 + ℓ_2 + ℓ_3))$
$≥ ϒ((η_1, η_2, …, η_n, 0), ℓ_1) * ϒ((η_1, η_2, …, ɧ, 0), ℓ_2) * ϒ((η_1, η_2, …, z, 0), ℓ_3)$
$≥ ϒ_1(η_1, η_2, …, η_n, ℓ_1) * ϒ_1(η_1, η_2, …, ɧ, ℓ_2) * ϒ_1(η_1, η_2, …, z, ℓ_3)$
$ϒ_1(η_1, η_2, …, η_n + ɧ + z, (ℓ_1 + ℓ_2 + ℓ_3))$
$≥ ϒ_1(η_1, η_2, …, η_n, ℓ_1) * ϒ_1(η_1, η_2, …, ɧ, ℓ_2) * ϒ_1(η_1, η_2, …, z, ℓ_3)$
and $Ħ_1(η_1, η_2,…, η_n + ɧ + z, (ℓ_1 + ℓ_2 + ℓ_3))$
$= Ħ((η_1, η_2, …, η_n + ɧ + z, 0), (ℓ_1 + ℓ_2 + ℓ_3))$
$= Ħ((η_1, η_2, …, η_n, 0) + (η_1, η_2, …, ɧ, 0) + (η_1, η_2,…, z, 0), (ℓ_1 + ℓ_2 + ℓ_3))$
$≤ Ħ((η_1, η_2, …, η_n, 0), ℓ_1) ◊ Ħ((η_1, η_2, …, ɧ, 0), ℓ_2) ◊ Ħ((η_1, η_2, …, z, 0), ℓ_3)$
$≤ Ħ_1(η_1, η_2, …, η_n, ℓ_1) ◊ Ħ_1(η_1, η_2, …, ɧ, ℓ_2) ◊ Ħ_1(η_1, η_2, …, z, ℓ_3)$
$Ħ_1(η_1, η_2, …, η_n + ɧ + z, (ℓ_1 + ℓ_2 + ℓ_3))$
$≤ Ħ_1(η_1, η_2, …, η_n, ℓ_1) ◊ Ħ_1(η_1, η_2, …, ɧ, ℓ_2) ◊ Ħ_1(η_1, η_2, …, z, ℓ_3)$.
(6)
$ϒ_1(η_1, η_2, …, η_n, ℓ) = ϒ((η_1, η_2, …, η_n, 0) ℓ)$ is a continuous in $ℓ$
and $Ħ_1((η_1, η_2, …, η_n, ℓ) = Ħ((η_1, η_2, …, η_n, 0), ℓ)$ is a continuous in $ℓ$.
(7)
$\lim_{ℓ→∞} ϒ_1(η_1, η_2, …, η_n, ℓ) = \lim_{ℓ→∞} ϒ((η_1, η_2, …, η_n, 0), ℓ) = 1$
and $\lim_{ℓ→∞} Ħ_1(η_1, η_2, …, η_n, ℓ) = \lim_{ℓ→∞} Ħ((η_1, η_2, …, η_n, 0), ℓ) = 0$.
Then $(X, ϒ_1, Ħ_1, *, ◊)$ is an *IFR-n-NS*.
Similarly, we can prove that $(U, ϒ_2, Ħ_2, *, ◊)$ is a *IFR-n-NS.*

In the following the theorem, we prove that if there is a convergent sequence in X and another convergent sequence in U, then their Cartesian product will also be convergent.

Theorem 3.7:

Let $\{η_n\}$ be a sequence in an *IFR-n- NS* $(X, ϒ_1, Ħ_1, *, ◊)$ converge to $η$ in X, $\{ϑ_n\}$ be a sequence in an *IFR-n- NS* $(U, ϒ_2, Ħ_2, *, ◊)$ converge to $ϑ$ in U, then $\{(η_n, ϑ_n)\}$ is a sequence in an *IFR-n- NS* $(X×U, ϒ, Ħ, *, ◊)$ converge to $(η, ϑ)$ in $X × U$.

Proof:

Let $\varphi \in (0,1)$ and $\ell > 0$. Since $\{\eta_n\}$ is a convergence sequence in X,
there is $n_1 \in N$ in which $\Upsilon_1(\eta_1, \eta_2, \ldots, \eta_{n-1}, \eta_n - \eta, \ell) > 1 - \varphi$
and $Ң_1(\eta_1, \eta_2, \ldots, \eta_{n-1}, \eta_n - \eta, \ell) < \varphi$, for all $n \geq n_1$.
Since $\{\vartheta_n\}$is a convergence sequence in U,
there is $n_2 \in N$ in which $\Upsilon_2(\vartheta_1, \vartheta_2, \ldots, \vartheta_n - \vartheta, \ell) > 1 - \varphi$
and $Ң_2(\vartheta_1, \vartheta_2, \ldots, \vartheta_n - \vartheta, \ell) < \varphi$, for all $n \geq n_2$.
Then, for all $\varphi \in (0,1)$ and $\ell > 0$, there is $n_0 \in N$,
where $n_0 = \max\{n_1, n_2\}$ in which
$\Upsilon(\eta_1, \eta_2, \ldots, \eta_{n-1}, \vartheta_1, \vartheta_2, \ldots, \vartheta_{n-1}, (\eta_n, \vartheta_n) - (\eta, \vartheta), \ell)$
$\geq \Upsilon_1(\eta_1, \eta_2, \ldots, \eta_{n-1}, \eta_n - \eta, \ell) * \Upsilon_2(\vartheta_1, \vartheta_2, \ldots, \vartheta_{n-1}, \vartheta_n - \vartheta, \ell)$
$> (1 - \varphi) * (1 - \varphi) > 1 - \varphi$
and $Ң(\eta_1, \eta_2, \ldots, \eta_{n-1}, \vartheta_1, \vartheta_2, \ldots, \vartheta_{n-1}, (\eta_n, \vartheta_n) - (\eta, \vartheta), \ell)$
$\leq Ң_1(\eta_1, \eta_2, \ldots, \eta_{n-1}, \eta_n - \eta, \ell) \Diamond Ң_2(\vartheta_1, \vartheta_2, \ldots, \vartheta_{n-1}, \vartheta_n - \vartheta, \ell)$
$< \varphi \Diamond \varphi < \varphi$.
Thus, $\{(\eta_n, \vartheta_n)\}$ converges to (η, ϑ).

After that the following theorem proves that the converse of the above theorem (3.7) is true.

Theorem 3.8:

Let (η_n, ϑ_n) be a sequence in an *IFR-n-NS* $(X \times U, \Upsilon, Ң, *, \Diamond)$, then $\{\eta_n\}$ is a sequence in an *IFR-n- NS* $(X, \Upsilon_1, Ң_1, *, \Diamond)$ converge to η in X and $\{\vartheta_n\}$ be a sequence in an *IFR-n- NS* $(U, \Upsilon_2, Ң_2, *, \Diamond)$ converge to ϑ in U.

Proof:

The prove of this Theorem is clear.

In the following the theorem, we prove that if there is a Cauchy sequence in X and another Cauchy sequence in U, then their Cartesian product will also be Cauchy.

Theorem 3.9:

Let $\{\eta_n\}$ be a Cauchy sequence in an *IFR-n- NS* $(X, \Upsilon_1, Ң_1, *, \Diamond)$ and $\{\vartheta_n\}$ be a Cauchy sequence in an *IFR-n-NS* $(U, \Upsilon_2, Ң_2, *, \Diamond)$, then $\{(\eta_n, \vartheta_n)\}$ is a Cauchy sequence in an *IFR-n- NS* $(X \times U, \Upsilon, Ң, *, \Diamond)$.

Proof:

By theorem (3.5), $(X \times U, \Upsilon, Ң, *, \Diamond)$ is an *IFR-n-NS*.
Since $\{\eta_n\}$ be a Cauchy sequence in an *IFR-n- NS* $(X, \Upsilon_1, Ң_1, *, \Diamond)$
Then for all $\varphi \in (0,1)$ and $\ell > 0$, there is $n_1 \in N$ in which
$\Upsilon_1(\eta_1, \eta_2, \ldots, \eta_n - \eta_\kappa, \ell) > 1 - \varphi$ and
$Ң_1(\eta_1, \eta_2, \ldots, \eta_n - \eta_\kappa, \ell) < \varphi$, for all $n, \kappa \geq n_1$.
Since $\{\vartheta_n\}$ be a Cauchy sequence in an *IFR-n- NS* $(U, \Upsilon_2, Ң_2, *, \Diamond)$.
Then for all $\varphi \in (0,1)$ and $\ell > 0$, there is $n_2 \in N$ in which
$\Upsilon_2(\vartheta_1, \vartheta_2, \ldots, \vartheta_n - \vartheta_\kappa, \ell) > 1 - \varphi$ and
$Ң_2(\vartheta_1, \vartheta_2, \ldots, \vartheta_n - \vartheta_\kappa, \ell) < \varphi$, for all $n, \kappa \geq n_2$.
Then for all $\varphi \in (0,1)$ and $\ell > 0$, there is $n_0 \in N$
where, $n_0 = \max\{n_1, n_2\}$, for all $n, \kappa \geq n_0$.
$\Upsilon(\eta_1, \eta_2, \ldots, \eta_{n-1}, \vartheta_1, \vartheta_2, \ldots, \vartheta_{n-1}, (\eta_n, \vartheta_n) - (\eta_\kappa, \vartheta_\kappa), \ell)$
$\geq \Upsilon_1(\eta_1, \eta_2, \ldots, \eta_{n-1}, \eta_n - \eta_\kappa, \ell) * \Upsilon_2(\vartheta_1, \vartheta_2, \ldots, \vartheta_{n-1}, \vartheta_n - \vartheta_\kappa, \ell)$

$> (1 - \varphi) * (1 - \varphi) > 1 - \varphi$ and
$Ḫ(\eta_1, \eta_2, \ldots, \eta_{n-1}, \vartheta_1, \vartheta_2, \ldots, \vartheta_{n-1}, (\eta_n, \vartheta_n) - (\eta_\kappa, \vartheta_\kappa), \ell)$
$\leq Ḫ_1(\eta_1, \eta_2, \ldots,\eta_{n-1}, \eta_n - \eta_\kappa, \ell) \Diamond Ḫ_2(\vartheta_1, \vartheta_2, \ldots,\vartheta_{n-1}, \vartheta_n - \vartheta_\kappa, \ell)$
$< \varphi \Diamond \varphi < \varphi$.
Thus, $\{(\eta_n, \vartheta_n)\}$ is a Cauchy sequence in $(X \times U, \Upsilon, Ḫ, *, \Diamond)$.

After that the following theorem proves that the converse of the above theorem (3.9) is true.

Theorem 3.10:
If $\{(\eta_n, \vartheta_n)\}$ is a Cauchy sequence in an *IFR-n- NS* $(X\times U, \Upsilon, Ḫ, *, \Diamond)$, then $\{\eta_n\}$ be a Cauchy sequence in an *IFR-n-NS* $(X, \Upsilon_1, Ḫ_1, *, \Diamond)$ and $\{\vartheta_n\}$ be a Cauchy sequence in an *IFR-n- NS* $(U, \Upsilon_2, Ḫ_2, *, \Diamond)$.

Proof:
The prove of this Theorem is clear**.**

The following can be proved using techniques in (3.10) and (3.7).

Theorem 3.11:
If $(X, \Upsilon_1, Ḫ_1, *, \Diamond)$ and $(U, \Upsilon_2, Ḫ_2, *, \Diamond)$ are complete an *IFR-n-NSs*, then the product $(X\times U, \Upsilon, Ḫ, *, \Diamond)$ is complete an *IFR-n-NS.*

Proof:
Let (η_n, ϑ_n) be a Cauchy sequence in $X\times U$
by theorem (3.10)
$\Rightarrow \{\eta_n\}$be a Cauchy sequence in $(X, \Upsilon_1, Ḫ_1, *, \Diamond)$
and $\{\vartheta_n\}$ is a Cauchy sequence in $(U, \Upsilon_2, Ḫ_2, *, \Diamond)$.
Since X and U are complete by definition
$\{\eta_n\}$ is a convergence sequence in X and $\{\vartheta_n\}$ is a convergence sequence in U
by theorem (3.7)
$\Rightarrow \{(\eta_n, \vartheta_n)\}$ is a convergence sequence in $X \times U$.

The following can be proved using techniques in (3.9) and (3.8).

Theorem 3.12:
If $(X \times U, \Upsilon, Ḫ, *, \Diamond)$ be a complete an *IFR-n-NS,* then $(X, \Upsilon_1, Ḫ_1, *, \Diamond)$ and $(U, \Upsilon_2, Ḫ_2, *, \Diamond)$ are complete an *IFR-n-NSs.*

Proof:

Let $\{\eta_n\}$ be a Cauchy sequence in X, $\{\vartheta_n\}$ be a Cauchy sequence in U
by theorem (3.9)
$\Rightarrow (\eta_n, \vartheta_n)$ is a Cauchy sequence in $X \times U$
since $X \times U$ complete, by definition
$\Rightarrow \{(\eta_n, \vartheta_n)\}$ is a convergence sequence in $X \times U$
by theorem (3.8)
$\Rightarrow \{\eta_n\}$ is a convergence sequence in X and $\{\vartheta_n\}$ is a convergence sequence in U.

4-Discussion

This study, we first presented the definition of intuitionistic fuzzy rectangular *n*-normed spaces. Further, we define the corresponding Cartesian product for these spaces. Additionally, we presented some related concepts and theorems concerning the Cartesian product. The results of this study will help researchers better understand how to handle the Cartesian product in intuitionistic fuzzy rectangular *n*-normed spaces.

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