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Solving Fractional Differential Equations with Laplace Transform and Picard Iteration

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Abstract:

This paper explores approximate analytical solutions for a class of fractional differential equations involving the Caputo fractional derivative. The proposed method employs the Laplace transform in conjunction with the Picard iterative technique to derive solutions with improved accuracy and simplicity. The Caputo derivative's distinct formulation enables an intuitive representation of initial conditions, facilitating its application in various scientific and engineering problems. The study outlines the theoretical foundation of the approach, demonstrating its efficiency through illustrative examples. Results indicate that this methodology provides a reliable framework for addressing the complexities of fractional differential equations, offering insights into their behavior and practical applications.

Keywords: Laplace transform, Picard Iteration method, Caputo operator, successive approximation method.

1-Introduction

Picard's iterative method is one of the oldest and most numerous approximate methods used to solve many differential and integral equations as well as difference equations. This technique is characterized by the simplicity of its general idea, as the calculation of any new term depends on the previous term. Many researchers and interested people have studied fractional differential and integral equations because of their importance and applications in other sciences, for example, in thermodynamics, economics, astronomy, chemistry, biology, etc.

Here are some of these studies: In 2014, Ai-Min Yang, Cheng Zhang, Hossein Jafari, Carlo Cattani, and Ying Jiao studied the Fourier law of the one-dimensional heat conduction equation in fractal media. One-dimensional local fractional Volterra integral equation of the second kind, which is obtained from the transformation of the Fourier flux equation in discontinuous media, is also approximated in this study [1], Pankaj Kumar and Prakash Agrawal used the Picard iteration approach with Caputo-type fractional derivative to offer a numerical solution strategy for a family of fractional differential equations (FDEs) in 2006 [2], In 1992 Takeshi Taniguchi examined the circumstances in which a series of stochastic processes built using successive approximations converge uniformly to solutions of an Ito-type stochastic differential equation. He will introduce the local or global existence and uniqueness theorem for solutions of the aforementioned equation under broader circumstances[3], there are more studies on this approach see [4], [5], [6], [7], [8], [9], [10], [11], [12].

In this study, we will present analytical and numerical solutions for nonlinear fractional differential equations using the Picard approach with the Laplace transform. The paper will be presented as follows: In the second section, we will mention some basic concepts about the fractional derivative and the Laplace transform. In the third section, we will discuss the algorithm of the approach used with the differential equation in the case of the fractional derivative. We will apply the method to solve some equations using the approach mentioned in the fourth section. Finally, we will present the conclusion that includes the results and conclusions that we reached through this study.

2- Basic concepts

Definition 1. [13] A real function $\phi(\omega), \omega > 0$, is said to be in the space $C_\vartheta, \vartheta \in R$ if there exists a real number $q, (q > \vartheta)$, such that $\phi(\omega) = \omega^q \phi_1(\omega)$, where $\phi(\mu) \in [0, \infty)$ and it is said to be in the space C_ϑ^m if $\phi^{(m)} \in C_\vartheta, m \in N$.

Definition 2. [14] For $\varepsilon > 0$ the gamma function $\Gamma(\varepsilon)$ is defined by the integral

$$\Gamma(\varepsilon) = \int_0^\infty e^{-\tau} \tau^{\varepsilon-1} \, d\tau \tag{2.1}$$

The basic properties of the gamma function are that it satisfies the following as [15]

1. $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.
2. $\Gamma(\varepsilon + 1) = \varepsilon \, \Gamma(\varepsilon)$, $\varepsilon \in C$.
3. $\Gamma(\varepsilon) = (\varepsilon - 1)!$, $\varepsilon \in C$.

Definition 3. [16] The Mittag -Leffler function of one parameter can be defined in terms of a power series as

$$E_\varepsilon(\xi) = \sum_{k=0}^{\infty} \frac{\xi^k}{\Gamma(\varepsilon k + 1)}, \varepsilon > 0, \tag{2.2}$$

and the Mittag -Leffler function of two parameters is given by

$$E_{\varepsilon,\gamma}(\xi) = \sum_{k=0}^{\infty} \frac{\xi^k}{\Gamma(\varepsilon k + \gamma)}, \varepsilon > 0, \gamma > 0. \tag{2.3}$$

Properties For some specific values of ε and γ, the Mittag-Leffler function reduces to some familiar. For example, [17], [18], [19]

1. $E_{1,1}(\xi) = \sum_{k=0}^{\infty} \frac{\xi^k}{\Gamma(k+1)} = \sum_{k=0}^{\infty} \frac{\xi^k}{k!} = e^\xi.$
2. $E_{1,2}(\xi) = \sum_{k=0}^{\infty} \frac{\xi^k}{\Gamma(k+2)} = \frac{1}{\xi} \sum_{k=0}^{\infty} \frac{\xi^{k+1}}{(k+1)!} = \frac{e^\xi - 1}{\xi}.$
3. $E_{2,1}(\xi^2) = \sum_{k=0}^{\infty} \frac{\xi^{2k}}{\Gamma(2k+1)} = \sum_{k=0}^{\infty} \frac{\xi^{2k}}{(2k)!} = \text{Cosh}(\xi).$
4. $E_{2,2}(\xi^2) = \sum_{k=0}^{\infty} \frac{\xi^{2k}}{\Gamma(2k+2)} = \sum_{k=0}^{\infty} \frac{\xi^{2k+1}}{\xi(2k+1)!} = \frac{\text{Sinh}(\xi)}{\xi}.$
5. $E_{2,1}(-\xi^2) = \sum_{k=0}^{\infty} \frac{(-\xi^2)^k}{\Gamma(2k+1)} = \sum_{k=0}^{\infty} \frac{(-1)^k \xi^{2k}}{(2k)!} = \text{Cos}(\xi).$
6. $E_{2,2}(-\xi^2) = \sum_{k=0}^{\infty} \frac{(-\xi^2)^k}{\Gamma(2k+2)} = \sum_{k=0}^{\infty} \frac{(-1)^k \xi^{2k+1}}{\xi(2k+1)!} = \frac{\text{Sin}(\xi)}{\xi}.$

Definition 4. [16] The Riemann-Liouville integral operator of order $\varepsilon > 0$, of a Function $f(\xi) \in C_\mu$, $\mu \geq -1$ is defined as:

$$J_{\xi}^{\varepsilon}f(\xi) = \frac{1}{\Gamma(\varepsilon)}\int_{0}^{\xi} (\xi-\omega)^{\varepsilon-1}f(\omega)\, d\omega. \tag{2.4}$$

Definition 5. [20]The Caputo fractional derivative of order ε, where $n-1<\varepsilon<n$, $n \in N$, is defined by:

$$^{C}D_{\xi}^{\varepsilon}f(\xi) = J_{\xi}^{n-\varepsilon}D^{n}f(\xi) = \frac{1}{\Gamma(n-\varepsilon)}\int_{0}^{\xi}(\xi-\omega)^{n-\varepsilon-1}f^{(n)}(\omega)d\omega. \tag{2.5}$$

Definition 6. [21] Let $f(\xi)$ is an integrable function, then the Laplace transform of $f(\xi), \xi \geq 0$ is defined by

$$F(s) = L\{f(\xi)\} = \int_{0}^{\infty} e^{-s\xi}f(\xi)d\xi = \mathrm{Lim}_{A\to\infty}\int_{0}^{A} e^{-s\xi}f(\xi)d\xi. \tag{2.6}$$

Table 1. Laplace transforms of some important functions.

function	Laplace transform
k	$\frac{k}{s}$
ξ^{ε}	$\frac{\Gamma(\varepsilon+1)}{s^{\varepsilon+1}}$
$e^{\varepsilon\xi}$	$\frac{1}{s-\varepsilon}$
$\mathrm{Sin}(\varepsilon\xi)$	$\frac{\varepsilon}{s^2+\varepsilon^2}$
$\mathrm{Cos}(\varepsilon\xi)$	$\frac{s}{s^2+\varepsilon^2}$
$\mathrm{Sinh}(\varepsilon\xi)$	$\frac{\varepsilon}{s^2-\varepsilon^2}$
$\mathrm{Cosh}(\varepsilon\xi)$	$\frac{s}{s^2-\varepsilon^2}$
$f^{(n)}(\xi)$	$\frac{F(s)}{s^{-n}} - \sum_{k=0}^{n-1}\frac{f^{(k)}(0)}{s^{-n+k+1}}$
$J^{n}\, f(\xi)$	$\frac{F(s)}{s^{n}}$
$\omega^{\gamma-1}E_{\varepsilon,\gamma}(\lambda\omega^{\varepsilon})$	$\frac{s^{\varepsilon-\gamma}}{s^{\varepsilon}-\lambda}$

Lemma 2.1. [22]Laplace transform of Riemann-Liouville fractional integral of order $\varepsilon > 0$ is given by:

$$L\{J_{\xi}^{\varepsilon}f(\xi)\} = \frac{F(s)}{s^{\varepsilon}}. \tag{2.7}$$

Theorem 2.1. [23]Laplace transform of Caputo fractional derivative of order $\varepsilon > 0$ is given by:

$$L\{\,^{C}D_{\xi}^{\varepsilon}f(\xi)\} = s^{\varepsilon}F(s) - \sum_{k=0}^{n-1} s^{\varepsilon-k-1}f^{(k)}(0). \tag{2.8}$$

3- Analysis of the method

Let us consider the following partial differential equations Consider the following fractional differential equation

$$^{c}D_{\omega}^{\varepsilon}\phi(\xi,\omega) + R(\phi) + N(\phi) = g(\xi,\omega);\ \phi^{(k)}(\xi,0) = \phi_{0}^{k}(\xi). \tag{3.1}$$

Taking Laplace transform

$$S^{\varepsilon}\left\{L(\phi(\xi,\omega)) - \sum_{k=0}^{n-1}\frac{u^{(k)}(\xi,0)}{s^{n+1}}\right\} = L(g) - L(R(\phi)+N(\phi)). \tag{3.2}$$

Or equivalent

$$L(\phi(\xi,\omega)) = \sum_{k=0}^{n-1}\frac{\phi^{(k)}(\xi,0)}{s^{k+1}} + S^{-\varepsilon}L(g) - S^{-\varepsilon}L(R(\phi)+N(\phi)). \tag{3.3}$$

Taking inverse Laplace transform

$$\phi(\xi,\omega) = \sum_{k=0}^{n-1}\frac{\omega^k}{k!}\phi^{(k)}(\xi,0) + L^{-1}(\mathcal{S}^{-\varepsilon}L(g)) - L^{-1}\left(S^{-\varepsilon}L(R(\phi)+N(\phi))\right). \tag{3.4}$$

Suppose that the solution is an infinite series

$$\phi(\xi,\omega) = \sum_{i=0}^{\infty}\phi_i(\xi,\omega). \tag{3.5}$$

After placing Equation (3.5) in Equation (3.4), the following relationship will result:

$$\phi_0 + \sum_{i=0}^{\infty}\phi_{i+1} = \sum_{k=0}^{n-1}\frac{\omega^k}{k!}\phi^{(k)}(\xi,0) + L^{-1}\left(\frac{1}{s^{\varepsilon}}L(g)\right) - L^{-1}\left(\frac{1}{s^{\varepsilon}}L\left(R\left(\sum_{i=0}^{\infty}\phi_i\right) + N\left(\sum_{i=0}^{\infty}\phi_i\right)\right)\right). \tag{3.6}$$

Therefore, the iterative formula for calculating all terms is

$$\phi_0 = \sum_{k=0}^{n-1}\frac{\omega^k}{k!}\phi_0^k(\xi).$$

$$\phi_{n+1} = \sum_{k=0}^{n-1}\frac{\omega^k}{k!}\phi_0^k(\xi) + L^{-1}\left(\frac{1}{s^{\varepsilon}}L(g)\right) - L^{-1}\left(\frac{1}{s^{\varepsilon}}L(R(\phi_n)+N(\phi_n))\right).$$

The approximate solution is given by

$$\phi(\xi,\omega) = \lim_{n\to\infty}\phi_{n+1}. \tag{3.7}$$

4- Analysis solution

Example 1 Assume the following fractional differential equation with order $0 < \varepsilon \leq 1$

$$ {}^{C}D_t^{\varepsilon}\phi + \phi_{\xi} = \phi, \tag{4.1} $$

where $\phi(\xi, 0) = 1 + e^{\xi}$.

Through the algorithm of the method in the previous section, we arrive at:

$$ \sum_{i=0}^{\infty} \phi_{i+1} = 1 + e^{\xi} + L^{-1}\left(\frac{1}{s^{\varepsilon}} L\left(\left(\sum_{i=0}^{\infty} \phi_i\right) - \left(\sum_{i=0}^{\infty} \phi_i\right)_{\xi}\right)\right). $$

After a series of algebraic operations and using the Laplace transform and its inverse and Table (1.1), we obtain the approximate solution terms for Equation (4.1).

$$ \phi_0 = 1 + e^{\xi}. $$

$$ \phi_1 = 1 + e^{\xi} + \frac{\omega^{\varepsilon}}{\Gamma(1+\varepsilon)}. $$

$$ \phi_2 = 1 + e^{\xi} + \frac{\omega^{\varepsilon}}{\Gamma(1+\varepsilon)} + \frac{\omega^{2\varepsilon}}{\Gamma(1+2\varepsilon)}. $$

$$ \phi_3 = 1 + e^{\xi} + \frac{\omega^{\varepsilon}}{\Gamma(1+\varepsilon)} + \frac{\omega^{2\varepsilon}}{\Gamma(1+2\varepsilon)} + \frac{\omega^{3\varepsilon}}{\Gamma(1+3\varepsilon)}. $$

$$ \vdots $$

Thus, the approximate solution and the exact solution at $\varepsilon = 1$ of Equation (4.1) will be, respectively, as follows:

$$ \phi = 1 + e^{\xi} + \frac{\omega^{\varepsilon}}{\Gamma(1+\varepsilon)} + \frac{\omega^{2\varepsilon}}{\Gamma(1+2\varepsilon)} + \frac{\omega^{3\varepsilon}}{\Gamma(1+3\varepsilon)} + \cdots = 1 + e^{\xi} + E_{\varepsilon}(\omega^{\varepsilon}). $$

$$ \phi = e^{\xi} + e^{\omega}. $$

Example 2 Assume the following fractional differential equation with order $0 < \varepsilon \leq 1$

$$ {}^{C}D_t^{\varepsilon}\phi + \xi\phi_{\xi} = 3\phi, \tag{4.2} $$

where $\phi(\xi, 0) = \xi^2$.

Through the algorithm of the method in the previous section, we arrive at:

$$ \textstyle\sum_{i=0}^{\infty} \phi_{i+1} = x^2 + L^{-1}\left(\frac{1}{s^{\varepsilon}} L\left(3\left(\sum_{i=0}^{\infty} \phi_i\right) - \xi\left(\sum_{i=0}^{\infty} \phi_i\right)_{\xi}\right)\right). $$

After a series of algebraic operations and using the Laplace transform and its inverse and Table (1.1), we obtain the approximate solution terms for Equation (4.2).

$$ \phi_0 = \xi^2. $$

$$ \phi_1 = \xi^2 + \frac{\xi^2\omega^{\varepsilon}}{\Gamma(1+\varepsilon)}. $$

$$ \phi_2 = \xi^2 + \xi^2\left(\frac{\omega^{2\varepsilon}}{\Gamma(1+2\varepsilon)} + \frac{\omega^{\varepsilon}}{\Gamma(1+\varepsilon)}\right). $$

$$ \phi_3 = \xi^2 + \xi^2\left(\frac{\omega^{3\varepsilon}}{\Gamma(1+3\varepsilon)} + \frac{\omega^{2\varepsilon}}{\Gamma(1+2\varepsilon)} + \frac{\omega^{\varepsilon}}{\Gamma(1+\varepsilon)}\right). $$

$$ \vdots $$

Thus, the approximate solution and the exact solution at $\varepsilon = 1$ of Equation (4.2) will be, respectively, as follows:

$$\phi = \xi^2 + \xi^2\left(\frac{\omega^{3\varepsilon}}{\Gamma(1+3\varepsilon)} + \frac{\omega^{2\varepsilon}}{\Gamma(1+2\varepsilon)} + \frac{\omega^{\varepsilon}}{\Gamma(1+\varepsilon)}\right) + \cdots = x^2 E_\varepsilon(\omega^\varepsilon).$$

$$\phi = \xi^2 e^\omega.$$

Example 3 Assume the following fractional differential equation with order $0 < \varepsilon \leq 1$

$$ {}^{C}D_t^\varepsilon \phi - 6\phi\phi_\xi + \phi_{\xi\xi\xi} = 0, \tag{4.3}$$

where $\phi(\xi, 0) = 6\xi$.

Through the algorithm of the method in the previous section, we arrive at:

$$\sum_{i=0}^{\infty} \phi_{i+1} = 6\xi + L^{-1}\left(\frac{1}{s^\varepsilon} L\left(-\left(\sum_{i=0}^{\infty} \phi_i\right)_{\xi\xi\xi} + 6\left(\sum_{i=0}^{\infty} \phi_i\right)\left(\sum_{i=0}^{\infty} \phi_i\right)_\xi\right)\right).$$

After a series of algebraic operations and using the Laplace transform and its inverse and Table (1.1), we obtain the approximate solution terms for Equation (4.3).

$$\phi_0 = 6\xi.$$

$$\phi_1 = 6\xi + 216\frac{\xi\omega^\varepsilon}{\Gamma(1+\varepsilon)}.$$

$$\phi_2 = 6\xi + 216\xi\left(\frac{\omega^\varepsilon}{\Gamma(1+\varepsilon)} + 72\frac{\omega^{2\varepsilon}}{\Gamma(2\varepsilon+1)} + 1296\frac{\Gamma(\varepsilon+1/2)2^{2\varepsilon}\omega^{3\varepsilon}}{\Gamma(1+\varepsilon)\sqrt{\pi}\Gamma(1+3\varepsilon)}\right).$$

$$\vdots$$

Thus, the approximate solution and the exact solution at $\varepsilon = 1$ of Equation (4.3) will be, respectively, as follows:

$$\phi = 6\xi + 216\xi\left(\frac{\omega^\varepsilon}{\Gamma(1+\varepsilon)} + 72\frac{\omega^{2\varepsilon}}{\Gamma(2\varepsilon+1)} + 1296\frac{\Gamma(\varepsilon+1/2)2^{2\varepsilon}\omega^{3\varepsilon}}{\Gamma(1+\varepsilon)\sqrt{\pi}\Gamma(1+3\varepsilon)}\right) + \cdots.$$

$$\phi = \frac{6\xi}{1-36\omega}.$$

Example 4 Assume the following fractional differential equation with order $1 < \varepsilon \leq 2$

$$ {}^{C}D_t^\varepsilon \phi - \phi_{\xi\xi} + \phi^2 = \xi^2\omega^2, \tag{4.4}$$

where $\phi(\xi, 0) = 0, \phi_\omega(\xi, 0) = \xi$.

Through the algorithm of the method in the previous section, we arrive at:

$$\sum_{i=0}^{\infty} \phi_{i+1} = \xi\omega + L^{-1}\left(\frac{1}{s^\varepsilon} L\left(\xi^2\omega^2 - \left(\sum_{i=0}^{\infty} \phi_i\right)^2 + \left(\sum_{i=0}^{\infty} \phi_i\right)_{\xi\xi}\right)\right).$$

After a series of algebraic operations and using the Laplace transform and its inverse and Table (1.1), we obtain the approximate solution terms for Equation (4.4).

$$\phi_0 = \xi\omega.$$

$$\phi_1 = \xi\omega.$$

$$\phi_2 = \xi\omega.$$

$$\vdots$$

Thus, the approximate solution and the exact solution at $\varepsilon = 1$ of Equation (4.3) will be, respectively, as follows:

$$\phi = \xi\omega.$$

$$\phi = \xi\omega.$$

Example 5 Assume the following fractional system differential equation with order $0 < \varepsilon \leq 1$

$$^{C}D_t^{\varepsilon}\phi + \psi_\xi = 0, \tag{4.5}$$

$$^{C}D_t^{\gamma}\psi + \phi_\xi = 0, \tag{4.6}$$

where $\phi(\xi, 0) = e^{\xi}$ and $\psi(\xi, 0) = e^{-\xi}$.

Through the algorithm of the method in the previous section, we arrive at:

$$\sum_{i=0}^{\infty} \phi_{i+1} = e^{\xi} - L^{-1}\left(\frac{1}{s^{\varepsilon}} L\left(\left(\sum_{i=0}^{\infty} \psi_i\right)_\xi\right)\right).$$

$$\sum_{i=0}^{\infty} \psi_{i+1} = e^{-\xi} - L^{-1}\left(\frac{1}{s^{\varepsilon}} L\left(\left(\sum_{i=0}^{\infty} \phi_i\right)_\xi\right)\right).$$

After a series of algebraic operations and using the Laplace transform and its inverse and Table (1.1), we obtain the approximate solution terms for Equation (4.5) and Equation (4.6).

$$\phi_0 = e^{\xi}.$$

$$\psi_0 = e^{-\xi}.$$

$$\phi_1 = e^{\xi} - \frac{e^{\xi}\omega^{\varepsilon}}{\Gamma(1+\varepsilon)}.$$

$$\psi_1 = e^{-\xi} + \frac{e^{-\xi}\omega^{\gamma}}{\Gamma(1+\gamma)}.$$

$$\phi_2 = e^{\xi} + e^{\xi}\left(\frac{\omega^{2\varepsilon}}{\Gamma(1+2\varepsilon)} - \frac{\omega^{\varepsilon}}{\Gamma(1+\varepsilon)}\right).$$

$$\psi_2 = e^{-\xi} + e^{-\xi}\left(\frac{\omega^{2\gamma}}{\Gamma(1+2\gamma)} + \frac{\omega^{\gamma}}{\Gamma(1+\gamma)}\right).$$

$$\phi_3 = e^{\xi} + e^{\xi}\left(-\frac{\omega^{\varepsilon}}{\Gamma(1+\varepsilon)} - \frac{\omega^{3\varepsilon}}{\Gamma(1+3\varepsilon)} + \frac{\omega^{2\varepsilon}}{\Gamma(1+2\varepsilon)}\right).$$

$$\psi_3 = e^{-\xi} + e^{-\xi}\left(\frac{\omega^{3}}{\Gamma(1+3\gamma)} + \frac{\omega^{2\gamma}}{\Gamma(1+2\gamma)} + \frac{\omega^{\gamma}}{\Gamma(1+\gamma)}\right).$$

$$\vdots$$

Thus, the approximate solution and the exact solution at $\varepsilon = \gamma = 1$ of Equation (4.5) and Equation (4.6) will be, respectively, as follows:

$$\phi = e^{\xi} + e^{\xi}\left(-\frac{\omega^{\varepsilon}}{\Gamma(1+\varepsilon)} - \frac{\omega^{3\varepsilon}}{\Gamma(1+3\varepsilon)} + \frac{\omega^{2\varepsilon}}{\Gamma(1+2\varepsilon)}\right) + \cdots = e^{\xi}E_{\varepsilon}(-\omega^{\varepsilon}).$$

$$\psi = e^{-\xi} + e^{-\xi}\left(\frac{\omega\omega^{3\gamma}}{\Gamma(1+3\gamma)} + \frac{\omega^{2\gamma}}{\Gamma(1+2\gamma)} + \frac{\omega^{\gamma}}{\Gamma(1+\gamma)}\right) + \cdots = e^{-\xi}E_{\gamma}(\omega^{\gamma}).$$

$$\phi = e^{\xi}e^{-\omega}\ .$$

$$\psi = e^{-\xi}e^{\omega}.$$

5- Numercal Solution

In this section we will present numerical solutions to the equations discussed in the previous section. We obtained the numerical solutions using Picard's method and with the help of MATLAB. The tables show the solutions at different fractional orders of the differential equations.

Table 2. Numerical solutions of Equation 4.1 at ε different values of using the Picard Iteration method.

ξ	ω	$\phi_{\varepsilon=0.5}$	$\phi_{\varepsilon=0.6}$	$\phi_{\varepsilon=0.7}$	$\phi_{\varepsilon=0.8}$	$\phi_{\varepsilon=0.9}$	$\phi_{\varepsilon=1}$	ϕ_{Exact}	$\lvert\phi_1 - \phi_E\rvert$
0.0010	0.0020	2.0535	2.0284	2.0153	2.0085	2.0049	2.0030	2.0030	0.0000
0.1120	0.1129	2.6391	2.4988	2.4002	2.3291	2.2769	2.2380	2.2380	0.0000
0.2230	0.2238	3.0870	2.8965	2.7543	2.6469	2.5646	2.5005	2.5006	0.0001
0.3340	0.3347	3.5296	3.3040	3.1276	2.9894	2.8804	2.7935	2.7940	0.0006
0.4450	0.4456	3.9830	3.7327	3.5283	3.3629	3.2290	3.1200	3.1218	0.0018
0.5560	0.5564	4.4541	4.1878	3.9610	3.7714	3.6141	3.4837	3.4881	0.0045
0.6670	0.6673	4.9476	4.6730	4.4292	4.2185	4.0394	3.8879	3.8974	0.0095
0.7780	0.7782	5.4672	5.1916	4.9360	4.7077	4.5085	4.3367	4.3547	0.0180
0.8890	0.8891	6.0164	5.7466	5.4847	5.2426	5.0254	4.8342	4.8657	0.0315
1.0000	1.0000	6.5989	6.3415	6.0789	5.8269	5.5943	5.3849	5.4366	0.0516

Table 3. Numerical solutions of Equation 4.2 at ε different values of using the Picard Iteration method.

ξ	ω	$\phi_{\varepsilon=0.5}$	$\phi_{\varepsilon=0.6}$	$\phi_{\varepsilon=0.7}$	$\phi_{\varepsilon=0.8}$	$\phi_{\varepsilon=0.9}$	$\phi_{\varepsilon=1}$	ϕ_{Exact}	$\lvert\phi_1 - \phi_E\rvert$
0.0010	0.0020	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.1120	0.1129	0.0191	0.0173	0.0161	0.0152	0.0145	0.0140	0.0140	0.0000
0.2230	0.2238	0.0914	0.0819	0.0748	0.0695	0.0654	0.0622	0.0622	0.0000
0.3340	0.3347	0.2380	0.2128	0.1931	0.1777	0.1655	0.1558	0.1559	0.0001
0.4450	0.4456	0.4797	0.4302	0.3897	0.3569	0.3304	0.3088	0.3092	0.0004
0.5560	0.5564	0.8379	0.7556	0.6855	0.6268	0.5782	0.5379	0.5393	0.0014
0.6670	0.6673	1.3343	1.2122	1.1037	1.0100	0.9302	0.8629	0.8671	0.0042
0.7780	0.7782	1.9914	1.8246	1.6699	1.5317	1.4111	1.3072	1.3181	0.0109
0.8890	0.8891	2.8323	2.6191	2.4121	2.2207	2.0491	1.8980	1.9228	0.0249
1.0000	1.0000	3.8806	3.6233	3.3606	3.1086	2.8760	2.6667	2.7183	0.0516

Table 4. Numerical solutions of Equation 4.4 at ε different values of using the Picard Iteration method.

ξ	ω	$\phi_{\varepsilon=0.5}$	$\phi_{\varepsilon=0.6}$	$\phi_{\varepsilon=0.7}$	$\phi_{\varepsilon=0.8}$	$\phi_{\varepsilon=0.9}$	$\phi_{\varepsilon=1}$	ϕ_{Exact}	$\lvert\phi_1 - \phi_E\rvert$
0.0010	0.0020	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0
0.1120	0.1129	0.0126	0.0126	0.0126	0.0126	0.0126	0.0126	0.0126	0
0.2230	0.2238	0.0499	0.0499	0.0499	0.0499	0.0499	0.0499	0.0499	0
0.3340	0.3347	0.1118	0.1118	0.1118	0.1118	0.1118	0.1118	0.1118	0
0.4450	0.4456	0.1983	0.1983	0.1983	0.1983	0.1983	0.1983	0.1983	0
0.5560	0.5564	0.3094	0.3094	0.3094	0.3094	0.3094	0.3094	0.3094	0
0.6670	0.6673	0.4451	0.4451	0.4451	0.4451	0.4451	0.4451	0.4451	0

0.7780	0.7782	0.6055	0.6055	0.6055	0.6055	0.6055	0.6055	0.6055	0
0.8890	0.8891	0.7904	0.7904	0.7904	0.7904	0.7904	0.7904	0.7904	0
1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0

Table 5. Numerical solutions of Equation 4.5 at ε different values of using the Picard Iteration method.

ξ	ω	$\phi_{\varepsilon=0.5}$	$\phi_{\varepsilon=0.6}$	$\phi_{\varepsilon=0.7}$	$\phi_{\varepsilon=0.8}$	$\phi_{\varepsilon=0.9}$	$\phi_{\varepsilon=1}$	ϕ_{Exact}	$\lvert\phi_1 - \phi_E\rvert$
0.0010	0.0030	0.9420	0.9675	0.9824	0.9908	0.9954	0.9980	0.9980	0.0000
0.1120	0.1138	0.7938	0.8420	0.8878	0.9297	0.9667	0.9982	0.9982	0.0000
0.2230	0.2246	0.7870	0.8269	0.8695	0.9133	0.9568	0.9984	0.9984	0.0000
0.3340	0.3353	0.8083	0.8375	0.8725	0.9118	0.9543	0.9987	0.9987	0.0000
0.4450	0.4461	0.8467	0.8631	0.8882	0.9197	0.9569	0.9989	0.9989	0.0000
0.5560	0.5569	0.9005	0.9007	0.9137	0.9349	0.9635	0.9991	0.9991	0.0000
0.6670	0.6677	0.9706	0.9498	0.9478	0.9565	0.9736	0.9994	0.9993	0.0000
0.7780	0.7784	1.0601	1.0114	0.9907	0.9840	0.9871	0.9996	0.9996	0.0001
0.8890	0.8892	1.1740	1.0881	1.0431	1.0177	1.0038	1.0000	0.9998	0.0002
1.0000	1.0000	1.3187	1.1837	1.1069	1.0583	1.0241	1.0005	1.0000	0.0005

Table 6. Numerical solutions of Equation 4.6 at γ different values of using the Picard Iteration method.

Ξ	Ω	$\Psi_{\Gamma=0.5}$	$\Psi_{\Gamma=0.6}$	$\Psi_{\Gamma=0.7}$	$\Psi_{\Gamma=0.8}$	$\Psi_{\Gamma=0.9}$	$\Psi_{\Gamma=1}$	Ψ_{EXACT}	$\lvert\Psi_1 - \Psi_E\rvert$
0.0010	0.0030	1.0639	1.0639	1.0181	1.0093	1.0046	1.0020	1.0020	0.0000
0.1120	0.1138	1.3690	1.3690	1.1480	1.0838	1.0368	1.0018	1.0018	0.0000
0.2230	0.2246	1.4991	1.4991	1.2080	1.1199	1.0532	1.0016	1.0016	0.0000
0.3340	0.3353	1.5874	1.5874	1.2493	1.1447	1.0644	1.0013	1.0013	0.0000
0.4450	0.4461	1.6522	1.6522	1.2800	1.1632	1.0728	1.0011	1.0011	0.0000
0.5560	0.5569	1.7004	1.7004	1.3038	1.1775	1.0793	1.0009	1.0009	0.0000
0.6670	0.6677	1.7359	1.7359	1.3226	1.1888	1.0843	1.0007	1.0007	0.0000
0.7780	0.7784	1.7607	1.7607	1.3374	1.1978	1.0884	1.0004	1.0004	0.0000
0.8890	0.8892	1.7762	1.7762	1.3489	1.2050	1.0916	1.0002	1.0002	0.0000
1.0000	1.0000	1.7836	1.7836	1.3576	1.2106	1.0941	0.9999	1.0000	0.0001

Through the numerical solutions presented in this section, it is clear that the approximate solution approaches the exact solution when the order of the fractional differential equation approaches the integer order. Therefore, the method used in this study is considered an effective and efficient method that can be relied upon in solving other types of linear and nonlinear differential and integral equations.

6- Conclusions

We used VIT with ABFO to evaluate the fractional-order three-dimensional Navier–Stokes equations in this paper. The VIT result closely resembles the precise solution to the provided issues. The convergence of the fractional-order answers to integer-order solutions was confirmed by a graphical examination of the results. Furthermore, the proposed method is clear, simple, and low-cost to implement; it may be extended to solve additional fractional-order partial differential equations.

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