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Analytical Solutions for Fractional Biological Population Model By Reduce Differential Transform Approach

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Abstract:
The fractional reduce differential transform approach is used in this article to solve the fractional-order biological population model (FOBPM). The fractional derivative is defined using the fractional derivative of Caputo. The suggested technique provides several solutions for FOBPM.

Keywords: fractional reduce differential transform Method, biological population model, Caputo derivative.

1-Introduction

Fractional differential equations play a vital role in various fields, particularly in engineering and mathematical physics. These equations are commonly used to model a wide range of phenomena, including the diffusion equation, the Klein-Gordon equation, the Laplace equation, the Schrödinger equation, and the nonlinear gas dynamics equation. Over the years, many techniques have been developed to solve linear and nonlinear partial differential equations that involve local fractional differential operators. These methods include: the Local Fractional Finite Difference Method (LFFDM) [1, 2, 3], the Local Fractional Adamian Analysis Method (LFADM) [3, 4], the Local Fractional Series Expansion Method (LFSEM) [5, 6], the Local Fractional Laplace Transform Method (LFLTM) [7, 8], the Local Fractional Sum Series Method (LFFSM) [9, 10], the Local Fractional Variable Iteration Method (LFVIM) [11], the Local Fractional Differential Transform Method (LFDTM) [10], and the Local Fractional Laplace Differential Method (LFLDM) [11].

In addition to partial differential equations, many physical problems are governed by systems of differential algebraic equations (DAEs). The solutions to these equations have garnered significant attention in recent years. While accurate solutions for linear differential algebraic equations have been achieved in many cases, finding accurate solutions for nonlinear equations remains a challenge. Consequently, a range of numerical methods has been proposed to approximate the solutions of differential algebraic equations [9].

This paper focuses on the application of the Fractional Differential Transform Method (FRDT) to solve the differential biological population model.

Definition 1. [10] Let $\gamma \geq 0$ is then the Riemann Liouville integral (RLI) of order γ is:

$$I^{\gamma}u(t)=\begin{cases}\frac{1}{\Gamma(\gamma)}\int_0^t (t-\tau)^{\gamma-1}u(t)d\tau, & \gamma>0, t>0\\ u(t), & \gamma=0\end{cases}$$

Properties of operator I^{γ} [11]:

1. $I^{\gamma}I^{\sigma}u(t)=I^{\gamma+\sigma}u(t)$.
2. $I^{\gamma}I^{\sigma}u(t)=I^{\sigma}I^{\gamma}u(t)$.

Definition 2. [12] Let $\gamma \geq 0$ then fractional Caputo derivative (FCD) of order γ is:

$$D^{\gamma}u(t)=\frac{1}{\Gamma(m-\gamma)}\int_0^t (t-\tau)^{m-\gamma-1}u^{(m)}(\tau)d\tau$$

For $m-1<\gamma<m$, $m \in N$, $t>0$ and $u \in C_{-1}^{m}$ [13]:

1. $D^{\gamma}t^{\sigma}=\frac{\Gamma(\sigma+1)}{\Gamma(\sigma-\gamma+1)}\,t^{\sigma-\gamma}$,
2. $D^{\gamma}D^{\sigma}u(t)=D^{\gamma+\sigma}u(t)$
3. $I^{\gamma}D^{\gamma}u(t)=u(t)-\sum_{k=0}^{m-1}u^{(k)}(0)\frac{t^k}{k!}$.

Definition 3. [14] The Mittage Leffler function (MLF) $E_{\gamma}(z)$ with $\gamma>0$ is

$$E_{\gamma}(z)=\sum_{m=0}^{\infty}\frac{z^{\gamma}}{\Gamma(m\gamma+1)}.$$

2-Fractional Reduced Differential Transform Method (FRDTM)

Definition 4. [15] If $\psi(x,t)$ is analytic and continuously differentiable with respect to the space variable x and time
variable t in the domain of interest, then the spectrum function

$$\psi_{\,i}(x)=\frac{1}{\Gamma(1+i\alpha)}\left[D_t^{i\alpha}\psi(x,t)\right]_{t=t_0}, \quad (1)$$

is referred as the fractional reduced differential transform function of $\psi(x,t)$ where α is a parameter which describes the order of the time-fractional derivative.

Definition 5. [16] The inverse fractional reduced differential transforms of $\psi_i(x)$ is defined as

$$\psi(x,t) = \sum_{i=0}^{\infty} \psi_i(x)(t-t_0)^{i\alpha} \tag{2}$$

In particular, for $t_o = 0$, becomes

$$\psi(x,t) = \sum_{i=0}^{\infty} \frac{1}{\Gamma(1+i\alpha)} \left[D_t^{i\alpha}\psi(x,t)\right]_{t=t_0} t^{i\alpha} \tag{3}$$

Table 2. Fundamental operations of the FRDTM

$w(x,t)$	$R_D\{w(x,t)\} = W_k(x)$
$w(x,t) = u(x,t)v(x,t)$	$W_k(x) = \sum_{r=0}^{k} U_r(x)V_{k-r}(x)$
$w(x,t) = D_t^{n\alpha}u(x,t)$	$W_k(x) = \frac{\Gamma(1+k\alpha+n\alpha)}{\Gamma(1+k\alpha)} U_{k+n}(x)$
$w(x,t) = D_x^m u(x,t)$	$W_k(x) = D_x^m U_k(x)$
$w(x,t) = x^m t^n$	$W_k(x) = x^m \delta(k-n)$
$w(x,t) = x^m t^n u(x,t)$	$W_k(x) = \{x^m U_{k-n}(x), k \geq n$

3-Reduced Differential Transform Method with biological population model

Let us consider a generalized non-linear biological population equation of the form:

$$^{C}D_t^{\gamma} w(x,y,t) = (w^2)_{xx} + (w^2)_{yy} + hw - hrw^2,\ 0 < \gamma \leqslant 1 \tag{4}$$

where $^{C}D_t^{\gamma}$ Caputo operator, h, r are real numbers. The initial condition is $\varphi(x,y,0) = \lambda(x,y)$.

Applying FRDTM on above equation, we find the following recurrence relation

$$W_{k+1}(x,y) = \frac{\Gamma(1+k\gamma)}{\Gamma(1+k\gamma+\gamma)} \begin{bmatrix} \frac{\partial^2}{\partial x^2}\left(\sum_{r=0}^{k} W_r(x,y)W_{k-r}(x,y)\right) + \frac{\partial^2}{\partial y^2}\left(\sum_{r=0}^{k} W_r(x,y)W_{k-r}(x,y)\right) \\ +hW_k(x,y) - hr\sum_{r=0}^{k} W_r(x,y)W_{k-r}(x,y) \end{bmatrix}.$$

Then, we obtain

$$W_0(x,y) = \varphi(x,y,0).$$

$$W_1(x,y) = \frac{1}{\Gamma(1+\gamma)} \begin{bmatrix} \frac{\partial^2}{\partial x^2}\left(W_0^2(x,y)\right) + \frac{\partial^2}{\partial y^2}\left(W_0^2(x,y)\right) \\ +hW_0(x,y) - hrW_0^2(x,y) \end{bmatrix}$$

$$W_2(x,y) = \frac{\Gamma(1+\gamma)}{\Gamma(1+2\gamma)} \begin{bmatrix} \frac{\partial^2}{\partial x^2}\left(2W_0(x,y)W_1(x,y)\right) \\ +\frac{\partial^2}{\partial y^2}\left(2W_0(x,y)W_1(x,y)\right) \\ +hW_1(x,y) - hr2W_0(x,y)W_1(x,y) \end{bmatrix}$$

$$W_3(x,y)=\frac{\Gamma(1+2\gamma)}{\Gamma(1+3\gamma)}\begin{bmatrix}\frac{\partial^2}{\partial x^2}\left(2W_0(x,y)W_2(x,y)+W_1^2(x,y)\right)\\+\frac{\partial^2}{\partial y^2}\left(2W_0(x,y)W_2(x,y)+W_1^2(x,y)\right)\\+hW_2(x,y)-hr\left(2W_0(x,y)W_2(x,y)+W_1^2(x,y)\right)\end{bmatrix}$$

$$W_4(x,y)=\frac{\Gamma(1+3\gamma)}{\Gamma(1+4\gamma)}\begin{bmatrix}\frac{\partial^2}{\partial x^2}\left(2W_0(x,y)W_3(x,y)+2W_1(x,y)W_2(x,y)\right)\\+\frac{\partial^2}{\partial y^2}\left(2W_0(x,y)W_3(x,y)+2W_1(x,y)W_2(x,y)\right)\\+hW_3(x,y)-hr\left(2W_0(x,y)W_3(x,y)+2W_1(x,y)W_2(x,y)\right)\end{bmatrix}$$

$$\vdots$$

Thus, the approximate solution of Eq.(4) is given by

$$\mathrm{w}(\mathrm{x},\mathrm{y},\mathrm{t})=W_0(x,y)+W_1(x,y)\mathrm{t}^{\gamma}+W_2(x,y)\mathrm{t}^{2\gamma}+W_3(x,y)\mathrm{t}^{3\gamma}+W_4(x,y)\mathrm{t}^{4\gamma}+\cdots.$$

4- Numerical Examples:

In this section, the equations in the examples will be solved by fractional reduced differential transform method.

Example 1. Consider the BPM at $h=1$.

$$^{C}D_t^{\gamma}\varphi(x,y,t)=\frac{\partial^2\varphi^2}{\partial x^2}+\frac{\partial^2\varphi^2}{\partial y^2}+\varphi-r\varphi, \tag{5}$$

where $0<\gamma\leq 1$ and subject to the initial condition

$$\varphi(x,y,0)=e^{\frac{\sqrt{2r}(x+y)}{4}}.$$

By alghorithm of the fractional reduced differential transform method, we get

$$W_0(x,y)=e^{\frac{\sqrt{2r}(x+y)}{4}}.$$

$$W_1(x,y)=e^{\frac{\sqrt{2r}(x+y)}{4}}\frac{1}{\Gamma(1+\gamma)}\mathrm{t}^{\gamma}$$

$$W_2(x,y)=e^{\frac{\sqrt{2r}(x+y)}{4}}\frac{1}{\Gamma(1+2\gamma)}\mathrm{t}^{2\gamma}$$

$$W_3(x,y)=e^{\frac{\sqrt{2r}(x+y)}{4}}\frac{1}{\Gamma(1+3\gamma)}\mathrm{t}^{3\gamma}$$

$$W_4(x,y)=e^{\frac{\sqrt{2r}(x+y)}{4}}\frac{1}{\Gamma(1+4\gamma)}\mathrm{t}^{4\gamma}$$

$$\vdots$$

Thus, the approximate solution of Eq.(5) is given by

$$\mathrm{w}(\mathrm{x},\mathrm{y},\mathrm{t})=e^{\frac{\sqrt{2r}(x+y)}{4}}+e^{\frac{\sqrt{2r}(x+y)}{4}}\frac{1}{\Gamma(1+\gamma)}\mathrm{t}^{\gamma}+e^{\frac{\sqrt{2r}(x+y)}{4}}\frac{1}{\Gamma(1+2\gamma)}\mathrm{t}^{2\gamma}+e^{\frac{\sqrt{2r}(x+y)}{4}}\frac{1}{\Gamma(1+3\gamma)}\mathrm{t}^{3\gamma}$$
$$+e^{\frac{\sqrt{2r}(x+y)}{4}}\frac{1}{\Gamma(1+4\gamma)}\mathrm{t}^{4\gamma}+\cdots=e^{\frac{\sqrt{2r}(x+y)}{4}}\sqrt{xy}\ E_{\gamma}(t^{\gamma}).$$

Table 2. Numerical solutions of Eq.(5) at different fractional orders, where $h=r=1$.

x,y,t	$\varphi_{\gamma=0.5}$	$\varphi_{\gamma=0.6}$	$\varphi_{\gamma=0.7}$	$\varphi_{\gamma=0.8}$	$\varphi_{\gamma=0.9}$	$\varphi_{\gamma=1}$	φ_E	$\lvert\varphi_{\gamma=1}-\varphi_E\rvert$
0.0030	1.0657	1.0359	1.0199	1.0111	1.0063	1.0038	1.0038	0.0000
0.1138	1.6114	1.4625	1.3577	1.2822	1.2267	1.1853	1.1853	0.0000
0.2246	2.0567	1.8433	1.6840	1.5637	1.4714	1.3996	1.3997	0.0001
0.3353	2.5233	2.2564	2.0477	1.8842	1.7551	1.6522	1.6528	0.0007
0.4461	3.0284	2.7156	2.4601	2.2532	2.0858	1.9495	1.9518	0.0023
0.5569	3.5810	3.2293	2.9296	2.6791	2.4712	2.2989	2.3048	0.0059

0.6677	4.1880	3.8047	3.4642	3.1701	2.9199	2.7084	2.7217	0.0133
0.7784	4.8557	4.4490	4.0718	3.7349	3.4409	3.1873	3.2139	0.0266
0.8892	5.5902	5.1694	4.7609	4.3832	4.0444	3.7462	3.7952	0.0491
1.0000	6.3981	5.9738	5.5407	5.1252	4.7417	4.3966	4.4817	0.0851

Example 2. Consider the BPM at $r = 0$

$$ {}^{AB}D_t^{\gamma}\varphi(x,y,t) = \frac{\partial^2\varphi^2}{\partial x^2} + \frac{\partial^2\varphi^2}{\partial y^2} + \mathrm{h}\varphi, \tag{6}$$

where $0 < \gamma \leq 1$ and subject to the initial condition

$$\varphi(x,y,0) = \sqrt{xy}\,.$$

By alghorithm of the fractional reduced differential transform method, we get

$$W_0(x,y) = \sqrt{xy}.$$
$$W_1(x,y) = \mathrm{h}\sqrt{xy}\frac{1}{\Gamma(1+\gamma)}\mathrm{t}^{\gamma}$$
$$W_2(x,y) = \mathrm{h}^2\sqrt{xy}\frac{1}{\Gamma(1+2\gamma)}\mathrm{t}^{2\gamma}$$
$$W_3(x,y) = \mathrm{h}^3\sqrt{xy}\frac{1}{\Gamma(1+3\gamma)}\mathrm{t}^{3\gamma}$$
$$W_4(x,y) = \mathrm{h}^4\sqrt{xy}\frac{1}{\Gamma(1+4\gamma)}\mathrm{t}^{4\gamma}$$
$$\vdots$$

Thus, the approximate solution of Eq.(5) is given by

$$\mathrm{w}(\mathrm{x},\mathrm{y},\mathrm{t}) = \sqrt{xy} + \mathrm{h}\sqrt{xy}\frac{1}{\Gamma(1+\gamma)}\mathrm{t}^{\gamma} + \mathrm{h}^2\sqrt{xy}\frac{1}{\Gamma(1+2\gamma)}\mathrm{t}^{2\gamma} + \mathrm{h}^3\sqrt{xy}\frac{1}{\Gamma(1+3\gamma)}\mathrm{t}^{3\gamma} + \mathrm{h}^4\sqrt{xy}\frac{1}{\Gamma(1+4\gamma)}\mathrm{t}^{4\gamma} + \cdots = \sqrt{xy}\ E_{\gamma}((ht)^{\gamma}).$$

Table 3. Numerical solutions of Eq.(6) at different fractional orders, where $h = r = 1$.

x,y,t	$\varphi_{\gamma=0.5}$	$\varphi_{\gamma=0.6}$	$\varphi_{\gamma=0.7}$	$\varphi_{\gamma=0.8}$	$\varphi_{\gamma=0.9}$	$\varphi_{\gamma=1}$	φ_E	$\lvert\varphi_{\gamma=1}-\varphi_E\rvert$
0.0030	0.0015	0.0015	0.0014	0.0014	0.0014	0.0014	0.0014	0.0000
0.1138	0.1722	0.1557	0.1444	0.1363	0.1304	0.1260	0.1260	0.0000
0.2246	0.4185	0.3708	0.3373	0.3127	0.2940	0.2796	0.2796	0.0000
0.3353	0.7412	0.6490	0.5833	0.5345	0.4970	0.4675	0.4675	0.0000
0.4461	1.1480	0.9978	0.8894	0.8082	0.7454	0.6956	0.6956	0.0000
0.5569	1.6492	1.4269	1.2645	1.1420	1.0467	0.9707	0.9707	0.0000
0.6677	2.2565	1.9480	1.7192	1.5453	1.4095	1.3008	1.3008	0.0000
0.7784	2.9826	2.5739	2.2655	2.0292	1.8437	1.6948	1.6948	0.0000
0.8892	3.8416	3.3193	2.9174	2.6062	2.3608	2.1632	2.1633	0.0001
1.0000	4.8482	4.2001	3.6903	3.2909	2.9740	2.7181	2.7183	0.0002

Example 3. Consider the BPM at $h = 1$ and $r = 0$

$$ {}^{AB}D_t^{\gamma}\varphi(x,y,t) = \frac{\partial^2\varphi^2}{\partial x^2} + \frac{\partial^2\varphi^2}{\partial y^2} + \varphi, \tag{7}$$

where $0 < \gamma \leq 1$ and subject to the initial condition

$$\varphi(x,y,0) = \sqrt{\sin(x)\sinh(y)}\,.$$

By alghorithm of the fractional reduced differential transform method, we get

$$W_0(x,y) = \sqrt{\sin(x)\sinh(y)}.$$
$$W_1(x,y) = \sqrt{\sin(x)\sinh(y)}\frac{1}{\Gamma(1+\gamma)}t^{\gamma}$$
$$W_2(x,y) = \sqrt{\sin(x)\sinh(y)}\frac{1}{\Gamma(1+2\gamma)}t^{2\gamma}$$
$$W_3(x,y) = \sqrt{\sin(x)\sinh(y)}\frac{1}{\Gamma(1+3\gamma)}t^{3\gamma}$$
$$W_4(x,y) = \sqrt{\sin(x)\sinh(y)}\frac{1}{\Gamma(1+4\gamma)}t^{4\gamma}$$
$$\vdots$$

Thus, the approximate solution of Eq.(5) is given by

$$\begin{aligned} w(x,y,t) &= \sqrt{\sin(x)\sinh(y)} + \sqrt{\sin(x)\sinh(y)}\frac{1}{\Gamma(1+\gamma)}t^{\gamma} + \sqrt{\sin(x)\sinh(y)}\frac{1}{\Gamma(1+2\gamma)}t^{2\gamma} \\ &+ \sqrt{\sin(x)\sinh(y)}\frac{1}{\Gamma(1+3\gamma)}t^{3\gamma} + \sqrt{\sin(x)\sinh(y)}\frac{1}{\Gamma(1+4\gamma)}t^{4\gamma} + \cdots \\ &= \sqrt{\sin(x)\sinh(y)}\ E_{\gamma}(t^{\gamma}). \end{aligned}$$

Table 4. Numerical solutions of Eq.(7) at different fractional orders, where $h = 1, r = \frac{1}{2}$.

x,y,t	$\varphi_{\gamma=0.5}$	$\varphi_{\gamma=0.6}$	$\varphi_{\gamma=0.7}$	$\varphi_{\gamma=0.8}$	$\varphi_{\gamma=0.9}$	$\varphi_{\gamma=1}$	φ_E	$\lvert\varphi_{\gamma=1} - \varphi_E\rvert$
0.0030	0.0015	0.0014	0.0014	0.0014	0.0014	0.0014	0.0014	0.0000
0.1138	0.1375	0.1316	0.1271	0.1237	0.1211	0.1190	0.1190	0.0000
0.2246	0.2982	0.2841	0.2728	0.2636	0.2561	0.2499	0.2499	0.0000
0.3353	0.4789	0.4559	0.4366	0.4205	0.4069	0.3953	0.3954	0.0000
0.4461	0.6779	0.6461	0.6186	0.5949	0.5743	0.5564	0.5564	0.0000
0.5569	0.8945	0.8547	0.8190	0.7873	0.7593	0.7344	0.7344	0.0000
0.6677	1.1284	1.0816	1.0381	0.9986	0.9628	0.9306	0.9306	0.0000
0.7784	1.3791	1.3267	1.2764	1.2294	1.1859	1.1460	1.1461	0.0001
0.8892	1.6461	1.5901	1.5342	1.4803	1.4294	1.3819	1.3820	0.0001
1.0000	1.9287	1.8715	1.8116	1.7519	1.6942	1.6393	1.6395	0.0003

Conclusion

This paper presents analytical solutions for the population model using the differential transformation approach. Through the analytical and numerical solutions, we conclude that the technique used to solve the population model is an efficient and good method with solutions close to the exact solution.

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