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Analytical Solution of Fractional Differential Equations Using Natural Variation Iteration Method

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Abstract:

The approach for resolving a set of linear and nonlinear equations using fractional natural variation iterations is proposed in this article. According to the findings, this strategy is far more successful and promising than previous numerical approaches.

Keywords: Variation iteration method, Natural transform, Fractional differential Equations.

1-Introduction

Modern technology has radically changed the world and our way of life. Numerous engineering fields, such as fluid dynamics, aerodynamics, the sciences of the body, and finance, utilise technology. The modeling of mathematical objects has a profound impact on and shapes the design of technology. Mathematical computations can be used to model many diseases, and data collection and careful analysis can be used to control them [9]. Fractional order differential equations (FDEs) are the name given to the non-integer order differential equations [1,6]. Fractional calculus is the area of mathematics concerned with FDEs [12].The operators for fractional derivatives have been provided by numerous academics in great number. The fractional derivative operator by Caputo [10] is the most well-known. The fractional order integral operator was developed by Li et al. to handle differential equations [11].

In this paper, we will deal with the natural transform iterative method (NTIM), a combination of the natural transform and the new iterative method which is variation iteration method (VIM) [4,5].

2-Preliminaries

This section goes over several fractional calculus principles and symbols that will come in handy during this inquiry [2, 3].

Definition 2.1. Suppose $\nu(\zeta) \in R, \zeta > 0$, which is in the space $C_m, m \in R$ if there exists

$$\{ \rho, (\rho > m), s.t. \nu(\zeta) = \zeta^{\rho} \nu_1(\zeta), where\ \nu_1(\zeta) \in C[0,8) \}$$

and $\nu(\zeta)$ is known as in the space C_m^n $when\ \nu^{\,n} \in C_m, m \in N$.

Definition 2.2. The fractional integral operator of order $\gamma \geq 0$ for Riemann Liouville of $\nu(\zeta) \in C_m, m \geq -1$ is given by the form

$$I^{\gamma}\, \nu(\zeta) = \begin{cases} \frac{1}{\Gamma(\gamma)} \int_0^{\zeta} (\zeta - \xi)^{\gamma-1}\, \nu(\xi)\, d\xi, & \gamma > 0, \zeta > 0 \\ I^{\,0} \nu(\zeta) = \nu(\zeta), & \gamma = 0 \end{cases} \tag{2.1}$$

where $\Gamma(\cdot)$ is the recognizable Gamma function. The following are the characteristics of the operator I^{γ} : $For\ \nu \in Cm, m \geq -1, \gamma, \sigma \geq 0, then$

1. $I^{\gamma} I^{\sigma}\, \nu(\zeta) = I^{\gamma+\sigma} \nu(\zeta)$
2. $I^{\gamma} I^{\sigma}\, \nu(\zeta) = I^{\sigma} I^{\gamma}\, \nu(\zeta)$

Definition 2.3. In the understanding of Caputo, $\nu(\zeta)$'s fractional derivative is as follows:

$$D^{\gamma}\, \nu(\zeta) = I^{n-\gamma} D^{n}\, \nu(\zeta) = \frac{1}{\Gamma(n - \gamma)} \int_0^{\zeta} (\zeta - \xi)^{n-\gamma-1}\, \nu^{(n)}(\xi)\, d\xi, \tag{2.2}$$

$such\ that\ n - 1 < \gamma \leq n, n \in N, \zeta > 0\ and\ \nu \in C_{-1}^{n}$

Definition 2.4. The following formula gives the Mittag-Leffler function E_{γ} if it satisfies the following:

$$E_{\gamma}(z) = \sum_{n=0}^{\infty} \frac{z^{\gamma}}{\Gamma(n\gamma + 1)} \quad , \text{for each } \gamma > 0 \tag{2.3}$$

Definition 2.5. [3]The function $\nu(\zeta)$ for $\zeta \in R$ has a natural transform defined by

$$N[\nu(\zeta)] = R(S, U) = \int_{-\infty}^{\infty} e^{-S\zeta}\, \nu(U\zeta) d\zeta,\ S, U \in (-\infty, \infty) \tag{2.4}$$

3-Analysis Fractional Natural Variation Iteration Method (FNVIM)

Suppose that the general fractional nonlinear PDEs with Caputo fractional operator

$$D_{\zeta}^{\gamma}\nu(\ell,\zeta)+R\nu(\ell,\zeta)+F\nu(\ell,\zeta)=\eth(\ell,\zeta)\qquad m-1<\gamma\leq m \tag{3.1}$$

depending on the initial condition

$$\nu(\ell,0)=\eth(\ell), \tag{3.2}$$

the derivative of $\nu(\ell,\zeta)$ is $D_{\zeta}^{\gamma}\nu(\ell,\zeta)$ in Caputo sense, R is linear differential operator, F nonlinear differential operator, and the source phrase is $\eth(\ell,\zeta)$. Now, by taking NT on both sides of (1)

$$N\left[D_{\zeta}^{\gamma}\nu(\ell,\zeta)\right]+N\left[R\nu(\ell,\zeta)+F\nu(\ell,\zeta)\right]=N\left[\eth(\ell,\zeta)\right],$$

or

$$\frac{S^{\gamma}}{U^{\gamma}}\nu(\ell,\zeta)-\sum_{k=0}^{n}\frac{S^{\gamma-(k+1)}}{U^{\gamma-k}}\nu(\ell,0)+N[R\nu(\ell,\zeta)+F\nu(\ell,\zeta)-\eth(\ell,\zeta)]=0.$$

The iteration formula:

$$\nu_{n+1}(\ell,\zeta)=\nu_n+\lambda(\xi)\left[\frac{S^{\gamma}}{U^{\gamma}}\nu_n-\sum_{k=0}^{n}\frac{S^{\gamma-(k+1)}}{U^{\gamma-k}}\nu(\ell,0)+N[R\nu_n(\ell,\zeta)+F\nu_n(\ell,\zeta)-\eth(\ell,\zeta)]\right],$$

where $\lambda(\xi)$ Lagrange multiplier

Taking variation

$$\delta[\nu_{n+1}(\ell,\zeta)]=\delta[\nu_n]+\lambda(\xi)\delta\left[\frac{S^{\gamma}}{U^{\gamma}}\nu_n-\sum_{k=0}^{n}\frac{S^{\gamma-(k+1)}}{U^{\gamma-k}}\nu(\ell,0)+N[R\nu_n(\ell,\zeta)+F\nu_n(\ell,\zeta)-\eth(\ell,\zeta)]\right].$$

By using computation $\delta[\nu_{n+1}]=\delta[\nu_n]+\lambda(\xi)\frac{S^{\gamma}}{U^{\gamma}}\delta[\nu_n]$

We impose the condition $\frac{\delta[\nu_{n+1}]}{\delta[\nu_n]}=0$

$$1+\lambda(\xi)\frac{S^{\gamma}}{U^{\gamma}}=0$$

Hence $\lambda(\xi)=-\frac{U^{\gamma}}{S^{\gamma}}$

$$\nu_{n+1}(\ell,\zeta)=\nu_n-\nu_n+\frac{1}{S}\,\nu(\ell,0)-\frac{U^{\gamma}}{S^{\gamma}}N\left[R\nu_n(\ell,\zeta)+F\nu_n(\ell,\zeta)-\eth(\ell,\zeta)\right].$$

By applying Natural inverse after placing the value of $\lambda(\xi)$, its follow:

$$\nu_{n+1}(\ell,\zeta)=\nu(\ell,0)-N^{-1}\left[\frac{U^{\gamma}}{S^{\gamma}}N\left[R\nu_n(\ell,\zeta)+F\nu_n(\ell,\zeta)-\eth(\ell,\zeta)\right.\right.$$

The solution is provided by

$$v(\ell,\zeta) = \lim_{n\to\infty} v_n$$

4- Applications

Example 4.1 Let us consider the FPDE

$$D_{\zeta}^{\gamma} v(\ell,\zeta) = v_{\ell\ell}(\ell,\zeta) - v(\ell,\zeta), \quad 0 < \gamma \le 1$$

with initial condition

$$v(\ell,0) = e^{-\ell} + \ell$$

Taking the natural transform of (4.1), we get

$$N[D_{\zeta}^{\gamma} v(\ell,\zeta)] = N[v_{\ell\ell}(\ell,\zeta) - v(\ell,\zeta)]$$

$$\frac{S^{\gamma}}{U^{\gamma}} v(\ell,\zeta) - \frac{S^{\gamma-1}}{U^{\gamma}} v(\ell,0) = N[v_{\ell\ell}(\ell,\zeta) - v(\ell,\zeta)].$$

$$\frac{S^{\gamma}}{U^{\gamma}} v(\ell,\zeta) - \frac{S^{\gamma-1}}{U^{\gamma}} v(\ell,0) - N\left[\frac{\partial^2 v_n(\ell,\zeta)}{\partial \ell^2} - v_n(\ell,\zeta)\right] \quad v_{n+1}(\ell,\zeta) = v_n + \lambda(\xi)[$$

$$v_{n+1}(\ell,\zeta) = \frac{1}{S} v(\ell,0) + \frac{U^{\gamma}}{S^{\gamma}} N\left[\frac{\partial^2 v_n(\ell,\zeta)}{\partial \ell^2} - v_n(\ell,\zeta)\right]$$

Taking the inverse Natural transform

$$v_{n+1}(\ell,\zeta) = v(\ell,0) + N^{-1}\left[\frac{U^{\gamma}}{S^{\gamma}} N\left[\frac{\partial^2 v_n(\ell,\zeta)}{\partial \ell^2} - v_n(\ell,\zeta)\right]\right]$$

$$v_0 = v(\ell,0) = e^{-\ell} + \ell$$

$$v_1 = e^{-\ell} + \ell + N^{-1}\left[\frac{U^{\gamma}}{S^{\gamma}} N\left[e^{-\ell} - e^{-\ell} - \ell\right]\right]$$

$$= e^{-\ell} + \ell - \ell\left[\frac{\zeta^{\gamma}}{\Gamma(\gamma+1)}\right]$$

$$v_2 = e^{-\ell} + \ell + N^{-1}\left[\frac{U^{\gamma}}{S^{\gamma}} N\left[e^{-\ell} - e^{-\ell} - \ell + \ell\left(\frac{\zeta^{\gamma}}{\Gamma(\gamma+1)}\right)\right]\right]$$

$$= e^{-\ell} + \ell - \ell\left[\frac{\zeta^{\gamma}}{\Gamma(\gamma+1)}\right] + \ell\left[\frac{\zeta^{2\gamma}}{\Gamma(2\gamma+1)}\right]$$

$$\vdots$$

$$v_n = (-1)^n \frac{\ell\, \zeta^{n\gamma}}{\Gamma(n\gamma+1)}$$

Therefore, we have $v(\ell,\zeta) = \lim_{n\to\infty} v_n$

$$= e^{-\ell} + \ell\,[1 - \frac{\zeta^{\gamma}}{\Gamma(\gamma+1)} + \frac{\zeta^{2\gamma}}{\Gamma(2\gamma+1)}] = e^{-\ell} + \ell E_{\gamma}(-\zeta^{\gamma})$$

If $\gamma = 1$, and by applying Taylor, the approximation yields

$$\nu(\ell,\zeta) = e^{-\ell} + \ell[1 - \zeta + \frac{\zeta^2}{2!} - \cdots]$$

Example 4.2 Consider the nonlinear fractional equation is given as the following:

$$\mathcal{D}_{\zeta}^{\gamma}\nu(\ell,\vartheta,\zeta) - \nu_{\ell\ell}^2 - \nu_{\vartheta\vartheta}^2 - h\nu = 0$$

w.r.t initial condition $\nu(\ell,\vartheta,\zeta) = \sqrt{\ell\vartheta}$

solution: $N\left[\mathcal{D}_{\zeta}^{\gamma}\nu(\ell,\vartheta,\zeta)\right] - N[\nu_{\ell\ell}^2 + \nu_{\vartheta\vartheta}^2 + h\nu] = 0$

$$\frac{S^{\gamma}}{U^{\gamma}}\nu(\ell,\vartheta,\zeta) - \frac{S^{\gamma-1}}{U^{\gamma}}\nu(\ell,\vartheta,0) - N\left[\frac{\partial^2\nu^2}{\partial\ell^2} + \frac{\partial^2\nu^2}{\partial\vartheta^2} + h\nu\right] = 0$$

$$\nu_{n+1}(\ell,\vartheta,\zeta) = \nu_n(\ell,\vartheta,\zeta) + \lambda(\xi)\left[\frac{S^{\gamma}}{U^{\gamma}}\nu_n - \frac{S^{\gamma-1}}{U^{\gamma}}\nu(\ell,\vartheta,0) - N\left[\frac{\partial^2{\nu_n}^2}{\partial\ell^2} + \frac{\partial^2{\nu_n}^2}{\partial\vartheta^2} + h\nu_n\right]\right]$$

$$\lambda\nu_{n+1}(\ell,\vartheta,\zeta) = \nu_n(\ell,\vartheta,\zeta) - \frac{S^{\gamma}}{U^{\gamma}}\left[\frac{S^{\gamma}}{U^{\gamma}}\nu_n - \frac{S^{\gamma-1}}{U^{\gamma}}\nu(\ell,\vartheta,0) - N\left[\frac{\partial^2{\nu_n}^2}{\partial\ell^2} + \frac{\partial^2{\nu_n}^2}{\partial\vartheta^2} + h\nu_n\right]\right]$$

Taking Natural inverse $\nu_{n+1}(\ell,\vartheta,\zeta) = \nu(\ell,\vartheta,0) + N^{-1}\left[\frac{S^{\gamma}}{U^{\gamma}}\,N\left[\frac{\partial^2{\nu_n}^2}{\partial\ell^2} + \frac{\partial^2{\nu_n}^2}{\partial\vartheta^2} + h\nu_n\right]\right]$

$$\nu_0 = \nu(\ell,\vartheta,0) = \sqrt{\ell\vartheta}$$

$$\nu_1 = \sqrt{\ell\vartheta} + N^{-1}\left[\frac{S^{\gamma}}{U^{\gamma}}N\left[0 + 0 + h\sqrt{\ell\vartheta}\,\right]\right] = \sqrt{\ell\vartheta} + \frac{\zeta^{\gamma}}{\Gamma(\gamma+1)}\,h\sqrt{\ell\vartheta}$$

$$\nu_2 = \sqrt{\ell\vartheta} + N^{-1}\left[\frac{U^{\gamma}}{S^{\gamma}}\;N\left[0 + 0 + h\sqrt{\ell\vartheta} + \frac{\zeta^{\gamma}}{\Gamma(\gamma+1)}\,h^2\sqrt{\ell\vartheta}\right]\right]$$

$$= \sqrt{\ell\vartheta} + \frac{\zeta^{\gamma}}{\Gamma(\gamma+1)}\,h\sqrt{\ell\vartheta} + \frac{\zeta^{2\gamma}}{\Gamma(2\gamma+1)}\,h^2\sqrt{\ell\vartheta}$$

$$\vdots$$

$$\nu_n = \sqrt{\ell\vartheta} + \frac{\zeta^{\gamma}}{\Gamma(\gamma+1)}\,h\sqrt{\ell\vartheta} + \frac{\zeta^{2\gamma}}{\Gamma(2\gamma+1)}\,h^2\sqrt{\ell\vartheta} + \cdots + \frac{\zeta^{n\gamma}}{\Gamma(n\gamma+1)}\,h^n\sqrt{\ell\vartheta}$$

$$\nu(\ell,\vartheta,\zeta) = \lim_{n\to\infty}\nu_n = \sqrt{\ell\vartheta}[1 + \frac{\zeta^{\gamma}}{\Gamma(\gamma+1)}\,h + \frac{\zeta^{2\gamma}}{\Gamma(2\gamma+1)}\,h^2 + \cdots + \frac{\zeta^{n\gamma}}{\Gamma(n\gamma+1)}\,h^n$$

$$= E_{\gamma}(h\zeta^{\gamma})\sqrt{\ell\vartheta}$$

Example 4.3 Consider the nonlinear fractional equations is given as the following:

$$\mathrm{D}_{\zeta}^{\gamma}\nu(l,\zeta) - \nu_{ll} - 2\nu\nu_{l} + (\nu\omega)_{l} = 0$$

$$\mathrm{D}_{\zeta}^{\sigma}\omega(l,\zeta) - \omega_{ll} - 2\omega\omega_{l} + (\nu\omega)_{l} = 0 \qquad 0 < \gamma \leq 1,\ 0 < \sigma \leq 1$$

with the initial condition $\nu(l,0) = e^{l}$

$$\omega(l,0) = e^{l}$$

$$\frac{S^{\gamma}}{U^{\gamma}}V(S,U) - \sum_{k=0}^{n}\frac{S^{\gamma-1}}{U^{\gamma}}\nu(l,0) - \mathrm{N}\left[\nu_{ll} + 2\nu\nu_{l} - (\nu\omega)_{l}\right] = 0$$

$$\frac{S^{\sigma}}{U^{\sigma}}W(S,U) - \sum_{k=0}^{n}\frac{S^{\sigma-1}}{U^{\sigma}}\omega(l,0) - \mathrm{N}\left[\omega_{ll} + 2\omega\omega_{l} - (\nu\omega)_{l}\right] = 0$$

$$V_{n+1} = V_{n} + \lambda_{1}(\xi)\left[\frac{S^{\gamma}}{U^{\gamma}}V_{n} - \frac{S^{\gamma-1}}{U^{\gamma}}\nu(l,0) - \mathrm{N}\left[\frac{\partial^{2}\nu_{n}}{\partial l^{2}} + 2(\nu_{n}\frac{\partial\nu_{n}}{\partial l}) - \frac{\partial(\nu_{n}\,\omega_{n})}{\partial l}\right]\right]$$

$$W_{n+1} = W_{n} + \lambda_{2}(\xi)\left[\frac{S^{\sigma}}{U^{\sigma}}W_{n} - \frac{S^{\sigma-1}}{U^{\sigma}}\omega(l,0) - \mathrm{N}\left[\frac{\partial^{2}\omega_{n}}{\partial l^{2}} + 2(\omega_{n}\frac{\partial\omega_{n}}{\partial l}) - \frac{\partial(\nu_{n}\,\omega_{n})}{\partial l}\right]\right]$$

By applying Natural inverse $\nu_{n+1} = \nu(l,0) + N^{-1}\left[\frac{U^{\gamma}}{S^{\gamma}}N\left[\frac{\partial^{2}\nu_{n}}{\partial l^{2}} + 2\left(\nu_{n}\frac{\partial\nu_{n}}{\partial l}\right) - \frac{\partial(\nu_{n}\,\omega_{n})}{\partial l}\right]\right]$

$$\omega_{n+1} = \omega(l,0) + N^{-1}\left[\frac{U^{\sigma}}{S^{\sigma}}N\left[\frac{\partial^{2}\omega_{n}}{\partial l^{2}} + 2\left(\omega_{n}\frac{\partial\omega_{n}}{\partial l}\right) - \frac{\partial(\nu_{n}\,\omega_{n})}{\partial l}\right]\right]$$

$$\nu_{0} = \nu(l,0) = e^{l} \qquad ; \qquad \omega_{0} = \omega(l,0) = e^{l}$$

$$\nu_{1} = e^{l} + \frac{\zeta^{\gamma}}{\Gamma(\gamma+1)}e^{l} \qquad ; \qquad \omega_{1} = e^{l} + \frac{\zeta^{\sigma}}{\Gamma(\sigma+1)}e^{l}$$

$$\nu_{2} = e^{l} + \frac{\zeta^{\gamma}}{\Gamma(\gamma+1)}e^{l} + \frac{\zeta^{2\gamma}}{\Gamma(2\gamma+1)}e^{l} + \frac{2\,\zeta^{2\gamma}}{\Gamma(2\gamma+1)}e^{2l} - \frac{2\,\zeta^{\gamma+\sigma}}{\Gamma(\gamma+\sigma+1)}e^{2l}$$

$$\omega_{2} = e^{l} + \frac{\zeta^{\sigma}}{\Gamma(\sigma+1)}e^{l} + \frac{\zeta^{2\sigma}}{\Gamma(2\sigma+1)}e^{l} + \frac{2\,\zeta^{2\sigma}}{\Gamma(2\sigma+1)}e^{2l} - \frac{2\,\zeta^{\gamma+\sigma}}{\Gamma(\gamma+\sigma+1)}e^{2l}$$

$$\vdots$$

$$\nu(l,\zeta) = e^{l}\left[1 + \frac{\zeta^{\gamma}}{\Gamma(\gamma+1)} + \frac{\zeta^{2\gamma}}{\Gamma(2\gamma+1)}\dots\right] + 2e^{2l}\left[\frac{\zeta^{2\gamma}}{\Gamma(2\gamma+1)} - \frac{\zeta^{\gamma+\sigma}}{\Gamma(\gamma+\sigma+1)}\dots\right]$$

$$\omega(l,\zeta)=e^{l}\left[1+\frac{\zeta^{\sigma}}{\Gamma(\sigma+1)}+\frac{\zeta^{2\sigma}}{\Gamma(2\sigma+1)}\dots\right]+2e^{2l}\left[\frac{\zeta^{2\sigma}}{\Gamma(2\sigma+1)}-\frac{\zeta^{\gamma+\sigma}}{\Gamma(\gamma+\sigma+1)}\dots\right]$$

Setting $\gamma=\beta=1$, we have $\nu(l,\zeta)=e^{l}\left[1+\zeta+\frac{\zeta^2}{2!}+\cdots\right]=e^{l+\zeta}$

$$\omega(l,\zeta)=e^{l}\left[1+\zeta+\frac{\zeta^2}{2!}+\cdots\right]=e^{l+\zeta}$$

5- Conclusion

In this paper, we studied the analysis of homotopy, then we took many examples that illustrate the solutions method and how approximate method were found for the solution by substituting the initial values given in the question. Also we applied the Elzaki transform for Captu fractional equation to find an approximate solution for it.

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