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Advanced Techniques in Natural Transform and Homotopy Analysis for Fractional Differential Equations

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Abstract:

This study employs the natural transformation approach combined with the homotopy perturbation technique, a new and effective hybrid method that integrates the homotopy approach with the natural transformation, using a simplified iterative procedure that reduces computational demands. This method provides fast, convergent, and sequential solutions. The reliability of the method is verified through its application to two case studies of ST-TE within the context of the Caputo derivative, which involves the definition of non-singular kernel functions. The study also includes comprehensive comparisons between approximate and exact solutions, utilizing relevant scientific literature to assess the accuracy and effectiveness of the method. Graphical representations highlight the impact of incorrect temporal and spatial parameters on the solution's behavior. The findings suggest that this method is easy to implement and well-suited for studying complex physical models governed by nonlinear partial differential equations with fractional time components.

Keywords: Fractional Differential Equations; Natural Transform; Atangana-Baleanu Fractional Derivative; Homotopy Method

1-Introduction

Linear and nonlinear fractional differential equations are crucial tools for modeling complex phenomena in various fields, such as fluid dynamics, acoustics, electromagnetism, signal processing, analytical chemistry, biology, and other areas of physical sciences and engineering. In recent years, numerous analytical and computational methods have been developed to solve these equations, including the Adomian Decomposition Method (ADM), the Homotopy Analysis Method (HAM), the Reduced Transform Method (RTM), and the Iterative Differential Transform Method (IDTM), among others [2][3]. These methods offer powerful tools for obtaining approximate solutions to fractional-order models that are encountered in diverse scientific and engineering problems.

Fractional differential equations involve derivatives of non-integer order, providing a more general and flexible framework compared to traditional integer-order differential equations. This flexibility allows for more accurate modeling of real-world phenomena that exhibit memory and hereditary properties, such as viscoelastic materials, anomalous diffusion, and systems with long-range dependence. However, solving these equations analytically remains a challenging task, particularly when they include nonlinear terms or complex boundary conditions.

Given the complexity of these problems, there has been significant interest in the development of efficient and robust solution techniques. Among these, the Natural Transform Homotopy Method (NTHM) has gained attention for its ability to effectively address both linear and nonlinear fractional differential equations. This research aims to explore and develop advanced solution techniques for fractional differential equations using the Natural Transform Homotopy method. The paper will discuss the theoretical foundations, implementation strategies, and real-world applications of this method across various disciplines.

In addition to NTHM, other notable methods for solving fractional differential equations include:

- **The Reduced Transform Method (RTM)**: This method has shown considerable success in solving both linear and nonlinear fractional differential equations, particularly in engineering applications.
- **The Iterative Differential Transform Method (IDTM)**: An effective numerical method that iteratively solves fractional differential equations, proving useful in obtaining accurate approximate solutions for complex problems.
- **The Laplace Transform Method**: Often used for linear fractional differential equations, providing a solid analytical approach for solving certain types of fractional models.

These methods, along with the Natural Transform Homotopy Method, represent a growing toolkit of analytical and numerical techniques for solving fractional differential equations.

This paper also aims to examine the strengths and limitations of these methods and to determine the most effective approach for obtaining numerical solutions. Through a thorough theoretical and numerical analysis, we aim to highlight the best methods for achieving accurate and efficient solutions to fractional differential equations.

2- Natural Transform

This section introduces the N-T transform, covering its basic definition and properties. This transform has been used in various applications for solving differential equations, especially in the context of fractional calculus and other advanced mathematical problems. the **N-T**, which was later renamed the Natural Transform by Belgacem and Silambarasan [4].

A list of special Natural Transforms for common functions is provided below:

$$1 \longrightarrow \frac{1}{s}$$

$$\mathrm{t} \longrightarrow \frac{\mathfrak{U}}{s^2}$$

$$\mathrm{e}^{at} \longrightarrow \frac{1}{s - a\mathfrak{U}}$$

$$\frac{\mathrm{t}^{n-1}}{(n-1)!}, n = 1,2, \dots \longrightarrow \frac{\mathfrak{U}^{n-1}}{s^n}$$

$$\sin(\mathrm{t}) \longrightarrow \frac{\mathfrak{U}}{s^2 + \mathfrak{U}^2}$$

Definition 1[5] The N-T of the function $\mathscr{b}(\mathrm{t}) > 0$ and $\mathscr{b}(\mathrm{t}) = 0$ for $\mathrm{t} < 0$ is specified for the collection of functions:

$$\beta = \left\{ \mathscr{b}(\mathrm{t}): \exists M, \tau_1, \tau_2 > 0, |\mathscr{b}(\mathrm{t})| < M e^{\frac{|\mathrm{t}|}{\tau_j}} \mathrm{If} \mathrm{t} \in (-1)^j [0, \infty), \mathrm{j} = 1,2, \dots \right\}$$

By the following integral:

$$N^{=}[\mathscr{b}(\mathrm{t})] = V(s,\mathfrak{U}) = \frac{1}{\mathfrak{U}}\int_0^{\infty} e^{-\frac{st}{\mathfrak{U}}}dt, s > 0, \mathfrak{U} > 0$$

Definition 2[1] The inverse N-T of a function is defined as:

$$N^{-1}[V(s,\mathfrak{U})] = \mathscr{b}(\mathrm{t}) = \frac{1}{2i\pi}\int_{p-\infty}^{p+\infty} e^{\frac{st}{\mathfrak{U}}}V(s,\mathfrak{U})ds$$

Where s and $\mathfrak{U}$ are N-T variables and $\mathfrak{J}$ is real constant.

Definition 3. [30], Suppose that the function $\varphi \in H^1(a,b), a > b$, The Atangana-Baleanu Fractional Derivative (ABFD) of φ is:

$$_a^{AB}D_t^{\gamma}\varphi(t) = \frac{M(\gamma)}{1-\gamma}\int_a^t E_{\gamma}\left(\frac{-\gamma(t-\xi)^{\gamma}}{\gamma-1}\right)\varphi'(\xi)d\xi \qquad t \geq 0$$

where $0 < \gamma < 1$ and $M(\gamma)$ is a normalization function, such that $M(0) = M(1) = 1$. and $\varphi'(\xi)$ is the derivative of φ.

In view of (2.9), the ABFD of a constant and Natural transform of this derivative defined as follows:

, where c is a constant.:

$1 - {}_a^{AB}D_t^{\gamma}c = 0$

$.2 - L\{{}_a^{AB}D_t^{\gamma}\varphi(\xi,t)\} = \frac{s^{\gamma}L\varphi(\xi,t)}{s^{\gamma}(1-\gamma)+\gamma} - \frac{s^{\gamma-1}L\{\varphi(\xi,0)\}}{s^{\gamma}(1-\gamma)+\gamma}$

$$3 - {}_a^{AB}D_t^{\gamma}\ \mathcal{T}^k = \frac{M(\gamma)}{1-\gamma}.\Gamma(k+1)\mathcal{T}^k.E_{\gamma,k+1}\left(\frac{-\gamma\ x^{\gamma}}{1-\gamma}\right).$$

3- Analysis of the proposed Method

Take into account the fractional partial differential equation involving the Atangana-Baleanu-Caputo operator in the subsequent form:

$$^{ABC}D_{\mathrm{t}}^{\mathfrak{J}}\mathfrak{U}(\mathfrak{x},\mathrm{t}) + \zeta[\mathfrak{U}(\mathfrak{x},\mathrm{t})] + \mu[\mathfrak{U}(\mathfrak{x},\mathrm{t})] = g(\mathfrak{x},\mathrm{t}) \tag{1}$$

With the starting condition $\mathfrak{U}(\mathfrak{x},0) = \mathfrak{U}_0(\mathfrak{x})$, where $^{ABC}D_{\mathrm{t}}^{\mathfrak{J}}$ is the Atangana-Balenu-Caputo operator, ζ is a linear operator, μ is a nonlinear operator, and g represents the source term. By applying the Natural Transform, we obtain initial condition.

$$\frac{B(\mathfrak{J})}{1-\mathfrak{J}+\mathfrak{J}\left(\frac{\mathscr{b}}{\delta}\right)^{\mathfrak{J}}}\left(N(\mathscr{b},\mathfrak{J}) - \frac{1}{\mathfrak{J}}\mathfrak{U}(0)\right) = N(g(\mathfrak{x},\mathrm{t}) - \zeta[\mathfrak{U}(\mathfrak{x},\mathrm{t})] - \mu[\mathfrak{U}(\mathfrak{x},\mathrm{t})]) \tag{2}$$

By substituting initial condition in (1)

$$\bar{\mathfrak{U}} = \frac{1}{\mathfrak{J}}\mathfrak{U}_0(\mathfrak{x}) - \frac{1-\delta+\delta\left(\frac{\mathscr{b}}{\mathfrak{J}}\right)^{\delta}}{B(\delta)}N(\zeta[\mathfrak{U}] + \mu[\mathfrak{U}] - g). \tag{3}$$

By applying $(N^{-1}T)$ to both sides of equation (3), the result is:

$$\mathfrak{U} = \mathfrak{U}_0(\mathfrak{x}) + N^{-1}\left(\frac{1-\delta+\delta(\mathscr{b})}{B(\mathfrak{J})}N(g)\right) - N^{-1}\left(\frac{1-\delta+\delta\left(\frac{\mathscr{b}}{\mathfrak{J}}\right)^{\delta}}{B(\delta)}N(\zeta[\mathfrak{U}]+\mu[\mathfrak{U}])\right). \tag{4}$$

Applying homotopy Permutation method,

$$\mathfrak{U}(\mathfrak{x},\mathrm{t}) = \sum_{n=0}^{\infty}\rho^{n}\,(\mathfrak{x},\mathrm{t}), N[\mathscr{b}(\mathfrak{x},\mathrm{t})] = \sum_{n=0}^{\infty}\rho^{n}\,H_n(\mathfrak{U}) \tag{5}$$

Where

$$H_n(\mathfrak{U}_1,\mathfrak{U}_2,\mathfrak{U}_3,\dots,\mathfrak{U}_n) = \frac{1}{n!}\frac{\partial^n}{\partial\rho^n}\left[N\left(\sum_{n=0}^{\infty}\rho^{i}\,\mathfrak{U}_i(\mathfrak{x},\mathrm{t})\right)\right]\rho = 0, \mathrm{n} = 0,1,2,\cdots \tag{6}$$

By replacing equation (5) into equation (4), yields the following result:

$$\sum_{n=0}^{\infty}\rho^{n}\,\mathfrak{U}_n(\mathfrak{x},\mathrm{t}) = \vartheta(\mathfrak{x},\mathrm{t}) - \rho\left(N^{-1}\left(\frac{1-\delta+\delta\left(\frac{\mathscr{b}}{\mathfrak{J}}\right)^{\delta}}{B(\delta)}N\left(\sum_{n=0}^{\infty}\rho^{n}\,\zeta[\mathfrak{U}_n] + \sum_{n=0}^{\infty}\rho^{n}\,H_n(\mathfrak{U})\right)\right)\right) \tag{7}$$

By equating both sides of the equation, the following conclusion is obtained:

$$\rho^0: \mathfrak{U}_0(\mathfrak{x},\mathrm{t}) = \vartheta(\mathfrak{x},\mathrm{t}),$$

$$\rho^1: \mathfrak{U}_1(\mathfrak{x},\mathrm{t}) = -N^{-1}\left(\frac{1-\delta+\delta\left(\frac{\mathscr{b}}{\mathfrak{J}}\right)^{\delta}}{B(\delta)}N(\zeta[\mathfrak{U}_0] + H_0[\mathfrak{U}])\right)$$

$$\rho^n: \mathfrak{U}_n(\mathfrak{x},\mathrm{t}) = -N^{-1}\left(\frac{1-\delta+\delta\left(\frac{\mathscr{b}}{\mathfrak{J}}\right)^{\delta}}{B(\delta)}N(\zeta[\mathfrak{U}_{n-1}] + H_{n-1}[\mathfrak{U}])\right)$$

Utilizing the parameter ρ, we express the solution in the subsequent form:

$$\mathfrak{U}(\mathfrak{x},\mathrm{t}) = \sum_{n=0}^{\infty}\rho^{n}\mathfrak{U}_n(\mathfrak{x},\mathrm{t}). \tag{8}$$

Setting $\rho = 1$ results in the soluting of Eq.(8)

$$\mathfrak{U}(\mathfrak{x},\mathrm{t}) = \lim_{\rho\to 1}\sum_{n=0}^{\infty}\rho^{n}\,\mathfrak{U}_n(\mathfrak{x},\mathrm{t}) = \sum_{n=0}^{\infty}\mathfrak{U}_n\,(\mathfrak{x},\mathrm{t})$$

4. Illustrative Examples

Example 1 Consider the following fractional diffusion equation expressed as:

$${}^{AB}D_{\mathrm{t}}^{\alpha}\mathscr{b} + \mathscr{b}_{\mathfrak{x}\mathfrak{x}} + \mathscr{b}_{yy} + \mathscr{b}_{ZZ} = 0 \qquad ,-\infty < \mathfrak{x}, y, Z < \infty, \mathrm{t} > 0. \tag{11}$$

$$\mathscr{b}(\mathfrak{x}, y, Z, 0) = e^{\mathfrak{x}+y+Z}, \alpha \in (0,1). \tag{12}$$

By applying the N- T to equation (11) and incorporating the specified initial condition, we obtain:

$$\mathscr{b}(\mathfrak{x}, y, Z, \mathfrak{U}) = \frac{e^{\mathfrak{x}+y+Z}}{s} + \frac{\mathfrak{U}^{\alpha}}{s^{\alpha}}N^{+}\left(\mathscr{b}_{\mathfrak{x}\mathfrak{x}} + \mathscr{b}_{yy} + \mathscr{b}_{ZZ}\right). \tag{13}$$

Seizing the inverse N-T of equation (13), we obtain:

$$\mathscr{b}(\mathfrak{x},\mathscr{y},\mathcal{Z},\mathfrak{t}) = e^{\mathfrak{x}+\mathscr{y}+\mathcal{Z}} + N^{-1}\left(\frac{\mathfrak{U}^{\alpha}}{s^{\alpha}} N^{+}\left[\mathscr{b}_{\mathfrak{x}\mathfrak{x}} + \mathscr{b}_{\mathscr{y}\mathscr{y}} + \mathscr{b}_{\mathcal{Z}\mathcal{Z}}\right]\right) \quad (14)$$

At this point, we employ the (HPM) to address the issue:

$$\mathscr{b}(\mathfrak{x},\mathscr{y},\mathcal{Z},\mathfrak{t}) = \sum_{n=0}^{\infty} \rho^{n}\, \mathscr{b}_{n}(\mathfrak{x},\mathscr{y},\mathcal{Z},\mathfrak{t}). \quad (15)$$

Subsequently, equation (15) will transform into:

$$\sum_{n=0}^{\infty} \rho^{n}\, \mathscr{b}_{n}(\mathfrak{x},\mathscr{y},\mathcal{Z},\mathfrak{t}) = e^{\mathfrak{x}+\mathscr{y}+\mathcal{Z}} - \rho\left(N^{-1}\left[\frac{\mathfrak{U}^{\alpha}}{s^{\alpha}} \cdot N^{+}\left[\sum_{n=0}^{\infty} \rho^{n}\, \mathscr{b}_{n\mathfrak{x}\mathfrak{x}} + \sum_{n=0}^{\infty} \rho^{n}\, \mathscr{b}_{n\mathscr{y}\mathscr{y}} + \sum_{n=0}^{\infty} \rho^{n}\, \mathscr{b}_{n\mathcal{Z}\mathcal{Z}}\right]\right]\right) \quad (16)$$

By equating the factors corresponding to similar powers of ρ in equation (16),
we derive the following approximations:

$$\rho^{0}: \mathscr{b}_{0}(\mathfrak{x},\mathscr{y},\mathcal{Z},\mathfrak{t}) = e^{\mathfrak{x}+\mathscr{y}+\mathcal{Z}},$$

$$\rho^{1}: \mathscr{b}_{1}(\mathfrak{x},\mathscr{y},\mathcal{Z},\mathfrak{t}) = -N^{-1}\left[\frac{\mathfrak{U}^{\alpha}}{s^{\alpha}} N^{+}\left[\mathscr{b}_{0\mathfrak{x}\mathfrak{x}} + \mathscr{b}_{0\mathscr{y}\mathscr{y}} + \mathscr{b}_{0\mathcal{Z}\mathcal{Z}}\right]\right.$$

$$= -\frac{e^{\mathfrak{x}+\mathscr{y}+\mathcal{Z}}\mathfrak{t}^{\alpha}}{\Gamma(\alpha+1)}$$

$$\rho^{2}: \mathscr{b}_{2}(\mathfrak{x},\mathscr{y},\mathcal{Z},\mathfrak{t})$$

$$\rho^{2}: \mathscr{b}_{2}(\mathfrak{x},\mathscr{y},\mathcal{Z},\mathfrak{t}) = -N^{-1}\left[\frac{\mathfrak{U}^{\alpha}}{s^{\alpha}} N^{+}\left[\mathscr{b}_{1\mathfrak{x}\mathfrak{x}} + \mathscr{b}_{1\mathscr{y}\mathscr{y}} + \mathscr{b}_{1\mathcal{Z}\mathcal{Z}}\right]\right.$$

$$= -\frac{e^{\mathfrak{x}+\mathscr{y}+\mathcal{Z}}\mathfrak{t}^{2}\alpha}{\Gamma(2\alpha+1)}$$

$$\rho^{3}: \mathscr{b}_{3}(\mathfrak{x},\mathscr{y},\mathcal{Z},\mathfrak{t})$$

$$\rho^{3}: \mathscr{b}_{3}(\mathfrak{x},\mathscr{y},\mathcal{Z},\mathfrak{t}) = -N^{-1}\left[\frac{\mathfrak{U}^{\alpha}}{s^{\alpha}} N^{+}\left[\mathscr{b}_{2\mathfrak{x}\mathfrak{x}} + \mathscr{b}_{2\mathscr{y}\mathscr{y}} + \mathscr{b}_{2\mathcal{Z}\mathcal{Z}}\right]\right.$$

$$= -\frac{e^{\mathfrak{x}+\mathscr{y}+\mathcal{Z}}\mathfrak{t}^{3}\alpha}{\Gamma(3\alpha+1)}$$

Then, the series solution for equations (11) and (12) is expressed as:

$$\mathscr{b}(\mathfrak{x},\mathscr{y},\mathcal{Z},\mathfrak{t}) = \lim_{N\to\infty}\sum_{n=0}^{N} \mathscr{b}_{n}(\mathfrak{x},\mathscr{y},\mathcal{Z},\mathfrak{t}) = \mathscr{b}_{0}(\mathfrak{x},\mathscr{y},\mathcal{Z},\mathfrak{t}) + \mathscr{b}_{1}(\mathfrak{x},\mathscr{y},\mathcal{Z},\mathfrak{t}) + \mathscr{b}_{2}(\mathfrak{x},\mathscr{y},\mathcal{Z},\mathfrak{t}) + \mathscr{b}_{3}(\mathfrak{x},\mathscr{y},\mathcal{Z},\mathfrak{t}) + \cdots$$

$$= e^{\mathfrak{x},\mathscr{y},\mathcal{Z}}\left(1 - \frac{\mathfrak{t}^{\alpha}}{\Gamma(+1)} + \mathfrak{t}^{2\alpha}\Gamma(2\alpha+1) - \frac{\mathfrak{t}^{3\alpha}}{\Gamma(3\alpha+1)} + \cdots\right)$$

$$= e^{\mathfrak{x}+\mathcal{y}+\mathcal{Z}}\left(1+\sum_{m=1}^{\infty}\frac{(-\mathrm{t}^{\alpha})^{m}}{\Gamma(m\alpha+1)}\right)=e^{\mathfrak{x}+\mathcal{y}+\mathcal{Z}}E_{\alpha}(-\mathrm{t}^{\alpha})$$

When $\alpha = 1$, the following result is derived:

$$\mathscr{b}(\mathfrak{x},\mathcal{y},\mathcal{Z},\mathrm{t}) = \lim \sum \mathscr{b}_n(\mathfrak{x},\mathcal{y},\mathcal{Z},\mathrm{t})$$

$$= \mathscr{b}_0(\mathfrak{x},\mathcal{y},\mathcal{Z},\mathrm{t}) + \mathscr{b}_1(\mathfrak{x},\mathcal{y},\mathcal{Z},\mathrm{t}) + \mathscr{b}_2(\mathfrak{x},\mathcal{y},\mathcal{Z},\mathrm{t}) + \mathscr{b}_3(\mathfrak{x},\mathcal{y},\mathcal{Z},\mathrm{t})$$

$$= e^{\mathfrak{x}+\mathcal{y}+\mathcal{Z}}\left(1-\frac{\mathrm{t}}{1!}+\frac{\mathrm{t}^2}{2!}-\frac{\mathrm{t}^3}{3!}\cdots\right)$$

$$= e^{\mathfrak{x}+\mathcal{y}+\mathcal{Z}-\mathrm{t}}$$

Example 2 The fractional-order differential equation is given by:

$${}^{AB}D_{\mathrm{t}}^{\alpha w} = \rho + \frac{\partial^2 w}{\partial \mathcal{Z}^2} + \frac{1}{\mathcal{Z}}\frac{\partial w}{\partial \mathcal{Z}} \ldots \tag{17}$$

Based on the initial conditions.

$$\mathfrak{U}(\mathscr{b},0) = 1-\mathcal{Z}^2. \tag{18}$$

The Laplace transform is:

$$\iota[w(\mathcal{Z},\mathrm{t})] - \frac{1}{s}(1-\mathcal{Z}^2) - \frac{1}{s^{\alpha}}\iota\left[\frac{\partial^2 w}{\partial \mathcal{Z}^2} + \frac{1}{\mathcal{Z}}\frac{\partial w}{\partial \mathcal{Z}} = 0. \tag{19}$$

The nonlinear term is:

$$N[\Phi(\mathcal{Z},\mathrm{t},q)] = \iota[\Phi(\mathcal{Z},\mathrm{t},q)] - \frac{1}{s}(1-\mathcal{Z}^2) - \frac{1}{s^{\alpha}}\iota[\rho] - \frac{1}{s^{\alpha}}\iota\left[\frac{\partial^2}{\partial \mathcal{Z}^2}\Phi(\mathcal{Z},\mathrm{t},q) + \frac{1}{\mathcal{Z}}\frac{\partial}{\partial \mathscr{b}}\Phi(\mathcal{Z},q,\mathrm{t})\right] \tag{20}$$

Thus

$$R_r\left(w_{r-1}(\mathcal{Z},\mathrm{t})\right) = \iota[w_{r-1}(\mathcal{Z},\mathrm{t})] - \frac{1}{s^{\alpha}}\iota\left[\frac{\partial^2}{\partial \mathcal{Z}^2}w_{r-1} + \frac{1}{\mathcal{Z}}\frac{\partial}{\partial \mathcal{Z}}\right] - (1-\mathfrak{x}_r)\frac{1}{s}(1-\mathcal{Z}^2) - \frac{1}{s^{\alpha+1}}\iota[\rho](1-\mathfrak{x}_r) \tag{21}$$

The r^{th} order deformation equation is

$$\iota[w_r(\mathcal{Z},\mathrm{t}) - \mathfrak{x}_r w_{r-1}(\mathcal{Z},\mathrm{t})] = hR_r(w_{r-1}(\mathcal{Z},\mathrm{t})] \tag{22}$$

The next step is to apply the inverse Laplace transform to obtain the solution in the original domain.

$$\mathrm{w}_m(\mathcal{Z},\mathrm{t}) - \mathfrak{x}_m w_{m-1}(\mathcal{Z},\mathrm{t}) = h\iota^{-1}\left[R_m\left(w_{m-1}(\mathcal{Z},\mathrm{t})\right)\right]. \tag{23}$$

By solving the above Eq(23), for $r = 1,2,3 \ldots$, then

$$\mathrm{w}_0(\mathcal{Z},\mathrm{t}) = 1 - \mathcal{Z}^2$$

$$w_1(\mathcal{Z},\mathrm{t}) = -\frac{h(-4+P)}{\Gamma(\alpha+1}\mathrm{t}^{\alpha}$$

$$w_2(\mathcal{Z},\mathrm{t}) = -\frac{h(-4+P)(1+h)}{\Gamma(\alpha+1}\mathrm{t}^{\alpha}$$

$$w_3(\mathcal{Z},\mathrm{t}) = -\frac{h(-4+P)(1+h)^2}{\Gamma(\alpha+1}\mathrm{t}^{\alpha}$$

$$w_4(\mathcal{Z},\mathrm{t}) = -\frac{h(-4+P)(1+h)^3}{\Gamma(\alpha+1}\mathrm{t}^{\alpha}$$

And so on. Concluding that

$$w(\mathscr{b},\mathrm{t}) = w_0(\mathcal{Z},\mathrm{t}) + w_1(\mathcal{Z},\mathrm{t}) + w_2(\mathcal{Z},\mathrm{t}) + w_3(\mathcal{Z},\mathrm{t}) + w_4(\mathcal{Z},\mathrm{t}) + \cdots$$

Then

$$w(\mathcal{Z},\mathrm{t}) = 1 - \mathcal{Z}^2 - \frac{h(-4+P)}{\Gamma(\alpha+1}\mathrm{t}^{\alpha} - \frac{h(-4+P)(1+h)}{\Gamma(\alpha+1}\mathrm{t}^{\alpha}$$

$$-\frac{h(-4+P)(1+h)^2}{\Gamma(\alpha+1)}\mathrm{t}^{\alpha} - \frac{h(-4+P)(1+h)^3}{\Gamma(\alpha+1)}\mathrm{t}^{\alpha} + \cdots$$

$$w(\mathcal{Z},\mathrm{t}) = 1 - \mathcal{Z}^2 - \frac{h(-4+P)}{\Gamma(\alpha+1)}\mathrm{t}^{\alpha}[1 + (1+h) + (1+h)^2 + (1+h)^3 + \cdots]$$

If h<0, the geometric series will converge as it approaches infinity, and the solution takes the following form:

$$w(\mathcal{Z},\mathrm{t}) = 1 - \mathcal{Z}^2 + \frac{-4+p}{\Gamma(\alpha+1)}\mathrm{t}^{\alpha}$$

If $\alpha = 1$, then the solution $w(\mathcal{Z},\mathrm{t}) = 1 - \mathcal{Z}^2 + (-4+P)$ (24)

Example 3 Let us examine the subsequent nonlinear equation within the Atangana-Baleanu-Caputo framewor:

$${}^{AB}D_{\tau}^{\delta}\psi(\mu,\tau) = -\frac{\partial}{\partial\mu}\left(\frac{12}{\mu}\psi - \mu\right)\psi + \frac{\partial^2}{\partial\mu^2}\psi^2, 0 < \delta \leq 1 \tag{25}$$

Subject to the initial condition $\psi(\mu, 0) = \mu^2$ (26)

By applying the natural transform to both sides of the equation, we obtain:

$$N\left[{}^{ABC}D_\tau^\delta \psi(\mu,\tau) = \frac{12}{\mu^2}\psi^2 - \frac{12}{\mu}\psi_\mu\psi + \psi + (\psi^2)_{\mu\mu}\right], \tag{27}$$

By applying the inverse natural transform to both sides of Eq. (27) and utilizing the initial condition, we obtain:,

$$\psi(\mu,\tau) = \mu^2 + N^{-1}\left[\left(1-\delta+\delta\left(\frac{\mathfrak{U}}{\mathfrak{I}}\right)^{\left[\frac{12}{\mu^2}\right.}\psi^2 - \frac{12}{\mu}\psi_\mu\psi + (\psi^2)_{\mu\mu} + \psi\right]\right].$$

.

By applying homotopy permutation method,

$$\sum_{n=0}^{\infty}\rho^n\psi_n = \mu^2 - \rho N^{-1}\left[\left(1-\delta+\delta\left(\frac{\mu}{\mathfrak{I}}\right)^\delta\right)N\left[\sum_{n=0}^{\infty}\rho^n A_n - \sum_{n=0}^{\infty}\rho^n B_n + \sum_{n=0}^{\infty}\rho^n C_n + \sum_{n=0}^{\infty}\rho^n\psi_n\right.\right.$$

By The following result is obtined,

$$\rho^0:\psi_0 = \mu^2$$
$$\rho^1:\psi_1 = N^{-1}\left[\left(1-\delta+\delta\left(\frac{\mu}{\mathfrak{I}}\right)^\delta\right)N(\rho^0A_0 - \rho^0B_0 + \rho^0c_0 + \rho^0\psi_0)\right] \quad ,$$
$$\rho^2:\psi_2 = N^{-1}\left[\left(1-\delta+\delta\left(\frac{\mu}{\mathfrak{I}}\right)^\delta\right)N(\rho^1A_1 - \rho^1B_1 + \rho^1c_1 + \rho^1\psi_1)\right],$$

By the above algorithms,

$$\psi_0 = \mu^2$$
$$\psi_1 = \mu^2\left(1-\delta+\delta\frac{\tau^\delta}{\Gamma(\delta+1)}\right) \quad ,$$
$$\psi_2 = \mu^2(1-2\delta+\delta^2) + (2\delta-2\delta^2)\frac{\tau^\delta}{\Gamma(\delta+1)} + \delta^2\frac{\tau^{2\delta}}{\Gamma(2\delta+1)}),$$

Therefore, the series solution $\psi(\mu,\tau)$ is given by

$$\psi(\mu,\tau) = \mu^2\left[(3-3\delta+\delta^2) + (3\delta-2\delta^2)\frac{\tau^\delta}{\Gamma(\delta+1)} + \delta^2\frac{\tau^{2\delta}}{\Gamma(2\delta+1)} + \cdots\right] \tag{29}$$

Ifweput $\delta = 1$ in (), we get the approximate and exast solution

$$\psi(\mu,\tau) = \mu^2\left(1+\frac{\tau}{1!}+\frac{\tau^2}{2!}+\cdots\right) = \mu^2\exp(\tau). \tag{30}$$

Example 4: Let us examine the non-homogeneous fractional equation in the following form:

$${}^{AB}D_\tau^{\mathfrak{I}}(\rho,\tau) = \mathfrak{x}(\rho,\tau) - \frac{1}{2}\frac{\partial\mathfrak{x}^2(\rho,\tau)}{\partial\rho} - \mathfrak{x}^2(\rho,\tau) - e^{\tau-\rho}. \tag{31}$$

Subject to initial condition $\mathfrak{x}(\rho,0) = 1 - e^{-\rho}$.

Applying the (NT) to both sides of equation (31), we obtain:

$$N\left(\mathfrak{x}(\rho,\tau) = \frac{1-e^{-\rho}}{s^{\mathfrak{I}}} + \frac{w^{\mathfrak{I}}e^{-\rho}}{s^{\mathfrak{I}}(s^{\mathfrak{I}}-w^{\mathfrak{I}})} + \frac{w^{\mathfrak{I}}}{s^{\mathfrak{I}}}\left(N\left(\mathfrak{x}(\rho,\tau) - \frac{1}{2}\frac{\partial\mathfrak{x}^2(\rho,\tau)}{\partial\rho} - \mathfrak{x}^2(\rho,\tau)\right)\right)\right). \tag{32}$$

By taking the inverse Natural transform of equation (32), we get

$$\mathfrak{x}(\rho,\tau)=1-e^{-\rho}E_{\mathfrak{J}}(\tau)+N^{-1}\left(\frac{w^{\mathfrak{J}}}{s^{\mathfrak{J}}}\left(N\left(\mathfrak{x}(\rho,\tau)-\frac{1}{2}\frac{\partial\mathfrak{x}^2(\rho,\tau)}{\partial\rho}-\mathfrak{x}^2(\rho,\tau)\right)\right)\right). \tag{33}$$

We now suppose that

$$\mathfrak{x}(\rho,\tau)=\sum_{n=0}^{\infty}\rho^n\,\mathfrak{x}_n(\rho,\tau) \tag{35}$$

Subsequently, by applying equation (34), we can express equation (33) as:

$$\sum_{n=0}^{\infty}\rho^n\,\mathfrak{x}_n=1-e^{-\rho}E_{\mathfrak{J}}(\tau)+N^{-1}\left(\frac{w^{\mathfrak{J}}}{s^{\mathfrak{J}}}\left(N\sum_{n=0}^{\infty}\rho^n\,\mathfrak{x}_n-\frac{1}{2}\sum_{n=0}^{\infty}\rho^n H_n-\sum_{n=0}^{\infty}\rho^n G_n\right)\right)\Bigg) \tag{36}$$

Where H_n and G_n these are the Adomian polynomials that represent the $\mathfrak{x}^2(\rho,\tau)$ respectively.

By equating both sides of equation (36), we can straightforwardly derive the recursive relation as:

$$\rho^0:\mathfrak{x}_0(\rho,\tau)=1-e^{-\rho}E_{\mathfrak{J}}(\tau)$$

$$\rho^1:\mathfrak{x}_1(\rho,\tau)=N^{-1}\left(\frac{W^{\mathfrak{J}}}{s^{\mathfrak{J}}}\left(N\left(\mathfrak{x}_0-\frac{1}{2}H_0-G_0\right)\right)\right)$$

$$=N^{-1}\left(\frac{w^{\mathfrak{J}}}{s^{\mathfrak{J}}}\left(N\left(\mathfrak{x}_0(\rho,\tau)\frac{1}{2}\frac{\partial\mathfrak{x}_0^2(\rho,\tau)}{\partial\rho}-\mathfrak{x}_0^2(\rho,\tau)\right)\right)\right)=0$$

$$\rho^2:\mathfrak{x}_2(\rho,\tau)=N^{-1}\left(\frac{W^{\mathfrak{J}}}{s^{\mathfrak{J}}}\left(N\left(\mathfrak{x}_1-\frac{1}{2}H_1-G_1\right)\right)\right)$$

$$:\mathfrak{x}(\rho,\tau)=1-e^{-\rho}E_{\mathfrak{J}}(\tau). \tag{37}$$

For $\mathfrak{J}=1$ The given solution converges to the precise solution $.\ \mathfrak{x}(\rho,\tau)=1-e^{\tau-\rho}$. (38)

Example 5: Let us consider the nonlinear system of time-fractional differential equations governed by the Atangana-Baleanu-Caputo operator:

$$\begin{aligned}&{}^{AB}D_{\mathrm{t}}^{\alpha}\mathscr{b}(\mathfrak{x},\mathrm{t})-\mathscr{b}_{\mathfrak{x}\mathfrak{x}}-2\mathscr{b}\mathscr{b}_{\mathfrak{x}}+(\mathscr{b}w)_{\mathfrak{x}}=0\\&{}^{AB}D_{\mathrm{t}}^{\mathfrak{J}}w(\mathfrak{x},\mathrm{t})-w_{\mathfrak{x}\mathfrak{x}}-2ww_{\mathfrak{x}}+(\mathscr{b}w)_{\mathfrak{x}}=0.\end{aligned} \tag{39}$$

where $0<\alpha,\mathfrak{J}\leq 1$ and the initial condition are $\mathscr{b}(\mathfrak{x},0)=\sin(\mathfrak{x}),w(\mathfrak{x},0)\}=\sin(\mathfrak{x})$

$$\mathrm{N}\left\{{}^{ABC}D_{\mathrm{t}}^{\alpha}\mathscr{b}(\mathfrak{x},\mathrm{t})=\mathscr{b}(\mathfrak{x},0)+\left(1-\alpha+\alpha\left(\frac{\mathfrak{U}}{s}\right)^{\alpha}\right)N\left(\frac{\partial^2\mathscr{b}}{\partial\mathfrak{x}^2}+2\mathscr{b}\frac{\partial\mathscr{b}}{\partial\mathfrak{x}}-\frac{\partial}{\partial\mathfrak{x}}(\mathscr{b}w)\right)$$

$$\mathrm{N}\{{}^{ABC}D_{\mathfrak{t}}^{\mathfrak{I}}\mathscr{b}(\mathfrak{x},\mathfrak{t})\} = w(\mathfrak{x},0) + \left(1-\mathfrak{I}+\mathfrak{I}\left(\frac{\mathfrak{u}}{s}\right)^{\mathfrak{I}}\right) N\left(\frac{\partial^2 w}{\partial \mathfrak{x}^2} + 2w\frac{\partial w}{\partial \mathfrak{x}} - \frac{\partial}{\partial \mathfrak{x}}(\mathscr{b}w)\right) \tag{40}$$

By applying the (NT) to both sides of eq. (39), we obtain:

$$\mathscr{b}(\mathfrak{x},\mathfrak{t}) = \sin(\mathfrak{x}) + N^{-1}\left(\left(1-a+a\left(\frac{\mathfrak{u}}{s}\right)^{a}\right) N\left(\frac{\partial^2 \mathscr{b}}{\partial \mathfrak{x}^2} + 2\mathscr{b}\frac{\partial \mathscr{b}}{\partial \mathfrak{x}} - \frac{\partial}{\partial \mathfrak{x}}(\mathscr{b}w)\right)\right.$$

$$w(\mathfrak{x},\mathfrak{t}) = \sin(\mathfrak{x}) + N^{-1}\left(\left(1-\mathfrak{I}+\mathfrak{I}\left(\frac{\mathfrak{u}}{s}\right)^{\mathfrak{I}}\right) N\left(\frac{\partial^2 w}{\partial \mathfrak{x}^2} + 2w\frac{\partial w}{\partial \mathfrak{x}} - \frac{\partial}{\partial \mathfrak{x}}(\mathscr{b}w)\right)\right. \tag{41}$$

Now, we express the solution as an infinite series given by:

$\mathscr{b}(\mathfrak{x},\mathfrak{t}) = \sum_{n=0}^{\infty}\rho^n \mathscr{b}_n(\mathfrak{x},\mathfrak{t}), w(\mathfrak{x},\mathfrak{t}) = \sum_{n=0}^{\infty}\rho^n w_n(\mathfrak{x},\mathfrak{t})$.

and the nonlinear terms can be expanded as

$$\mathscr{b}\mathscr{b}_{\mathfrak{x}} - (\mathscr{b}w)_{\mathfrak{x}} = \sum_{n=0}^{\infty}\rho^n A_n(\mathfrak{x},\mathfrak{t}), ww_{\mathfrak{x}} - (\mathscr{b}w)_{\mathfrak{x}}\sum_{n=0}^{\infty}\rho^n B_n(\mathfrak{x},\mathfrak{t}). \tag{42}$$

. .

Substituting(41) and (40) in(39) and by applying (HPM) on Eq.(39)

$$\sum_{n=0}^{\infty}\rho^n\mathscr{b}_n = \sin(\mathfrak{x}) - \rho N^{-1}\left(1-a+a\left(\frac{\mathfrak{u}}{s}\right)^{a}\right) N\left(\sum_{n=0}^{\infty}\rho^n\frac{\partial^2 \mathscr{b}_n}{\partial \mathfrak{x}^2} + \sum_{n=0}^{\infty}\rho^n A_n\right)\Bigg) \tag{43}$$

By equating both sides of equation (43),

$$\rho^0: \mathscr{b}_0 = \mathscr{b}(\mathfrak{x},0), \rho^0: w_0 = w(\mathfrak{x},0)$$

$$\rho^1: \mathscr{b}_1 = -N^{-1}\left(\left(1-a+a\left(\frac{\mathfrak{U}}{s}\right)^{a}\right) N\left(\frac{\partial^2 \mathfrak{U}_0}{\partial \mathfrak{x}^2} + A_0\right)\right)$$

$$\rho^1: w_1 = -N^{-1}\left(\left(1-\mathfrak{I}+\mathfrak{I}\left(\frac{\mathfrak{U}}{s}\right)^{\mathfrak{I}}\right) N\left(\frac{\partial^2 \mathscr{b}_0}{\partial \mathfrak{x}^2} + B_0\right)\right)$$

$$\rho^2: \mathscr{b}_2 = -N^{-1}\left(\left(1-a+a\left(\frac{\mathfrak{U}}{s}\right)^{a}\right) N\left(\frac{\partial^2 \mathfrak{U}_0}{\partial \mathfrak{x}^2} + A_1\right)\right)$$

$$\rho^2: w_2 = -N^{-1}\left(\left(1-\mathfrak{I}+\mathfrak{I}\left(\frac{\mathfrak{U}}{s}\right)^{\mathfrak{I}}\right) N\left(\frac{\partial^2 \mathscr{b}_0}{\partial \mathfrak{x}^2} + B_1 - C_1\right)\right)$$

By the above algorithms, $\mathscr{b}_0 = \sin(\mathfrak{x}), w_0 = \sin(\mathfrak{x})$

$$\mathscr{b}_1 = -\sin(\mathfrak{x})\left(1-a+a\frac{\mathfrak{t}^{a}}{\Gamma(a+1)}\right)$$

$$w_1 = -\sin(\mathfrak{x})\left(1-\mathfrak{I}+\mathfrak{I}\frac{\mathfrak{t}^{\mathfrak{I}}}{\Gamma(\mathfrak{I}+1)}\right)$$

$$v_2 = \sin(\mathfrak{x})\left((1-a)^2 + (2a-2a^2)\frac{t^a}{\Gamma(a+1)} + a^2\frac{t^{2a}}{\Gamma(2a+1)} + 2(a-B)\cos(\mathfrak{x})\left((1-a) + (2a-1)\frac{t^a}{\Gamma(a+1)} - a^2\frac{t^{2a}}{\Gamma(2a+1)}\right)\right.$$

$$w_2 = \sin(\mathfrak{x})\left((1-\mathfrak{J})^2 + (2\mathfrak{J}-2\mathfrak{J}^2)\frac{t^{\mathfrak{J}}}{\Gamma(\mathfrak{J}+1)} + \mathfrak{J}^2\frac{t^{2\mathfrak{J}}}{\Gamma(2\mathfrak{J}+1)} + 2(\mathfrak{J}-B)\cos(\mathfrak{x})((1-\mathfrak{J}) + (2\mathfrak{J}-1)\frac{t^{\mathfrak{J}}}{\Gamma(\mathfrak{J}+1)} - \mathfrak{J}^2\frac{t^{2\mathfrak{J}}}{\Gamma(2\mathfrak{J}+1)}\right)$$

$$\begin{aligned}
\mathscr{b}(\mathfrak{x},t) &= \sin(\mathfrak{x})\left((1-a+a^2) + (a-2a^2)\frac{t^a}{\Gamma(a+1)} + a^2\frac{t^{2a}}{\Gamma(2a+1)} + \right. \\
&\quad 2(a-\mathfrak{J})\cos(\mathfrak{x})\left((1-a)(2a-1)\frac{t^a}{\Gamma(a+1)} - a^2\frac{t^{2a}}{\Gamma(2a+1)} + \cdots\right) \\
w(\mathfrak{x},t) &= \sin(\mathfrak{x})\left((1-\mathfrak{J}+\mathfrak{J}^2) + (\mathfrak{J}-2\mathfrak{J}^2)\frac{t^{\mathfrak{J}}}{\Gamma(\mathfrak{J}+1)} + \mathfrak{J}^2\frac{t^{2\mathfrak{J}}}{\Gamma(2\mathfrak{J}+1)} + \right. \\
&\quad 2(\mathfrak{J}-\mathfrak{J})\cos(\mathfrak{x})\left((1-\mathfrak{J})(2\mathfrak{J}-1)\frac{t^{\mathfrak{J}}}{\Gamma(\mathfrak{J}+1)} - \mathfrak{J}^2\frac{t^2\mathfrak{J}}{\Gamma(2\mathfrak{J}+1)}\right] + \cdots
\end{aligned} \tag{44}$$

If we substitute $\alpha \to 1$ in (44)

$$\mathfrak{U}(\mathfrak{x},t) = \sin(\mathfrak{x})\left(1 - t + \frac{t^2}{2!} - \cdots\right), \mathscr{b}(\mathfrak{x},t) = \sin(\mathfrak{x})\left(1 - t + \frac{t^2}{2!} - \cdots\right)$$

$$\mathfrak{U}(\mathfrak{x},t) = \sin(\mathfrak{x})e^{-t}, \mathscr{b}(\mathfrak{x},t) = \sin(\mathfrak{x})e^{-t}$$

5- Conclusion

In this research, a hybrid method combining the Homotopy approach with the Natural Transformation has been presented as an effective tool for solving fractional differential equations, whether linear or nonlinear. The results demonstrated that this method provides fast and accurate approximate solutions, enabling efficient handling of complex mathematical problems involving fractional derivatives. Furthermore, this method reduces computational demands, making it a powerful and reliable tool for studying and analyzing systems governed by fractional differential equations. These findings represent an important step toward improving numerical solution methods in various fields such as mathematical physics and engineering, making this approach a promising tool for future applications in solving fractional differential equations..

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