Journal of Education for Pure Science
Vol.15, No. 1(2025)
DOI: https://doi.org/10.32792/jeps.v15i1.651
Website: jceps.utq.edu.iq Email: jceps@eps.utq.edu.iq

# An Analytical Approach to Nonlinear Fractional Differential Equations

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**Received 14/ 1 /2025, Accepted 3 / 2 /2025, Published 01 /03 /2025**



**Abstract:**

Fractional differential equations have gained significant attention in recent years due to their ability to model complex phenomena in various scientific and engineering fields. This study focuses on solving fractional differential equations involving the nonlinear Caputo fractional derivative and the Caputo-Fabrizio fractional derivative. We employ the Daftardar-Jafari Method (DJM) and the Elzaki Daftardar-Jafari Method (EDJM) to derive approximate analytical solutions. The combination of these methods provides a systematic and efficient approach to addressing the challenges posed by nonlinearities and the memory effects inherent in fractional derivatives. Through illustrative examples, we demonstrate the accuracy and applicability of the proposed methods. The results indicate that DJM and EDJM are powerful tools for solving nonlinear fractional differential equations, offering insights into their underlying dynamics.

**Keywords:** Elzaki transform, Caputo fractional derivative, Dafter -Jafari method (EDJM).

## 1-Introduction

Fractional calculus, an extension of classical calculus, has emerged as a powerful mathematical tool for modeling complex systems characterized by memory and hereditary properties. Fractional differential equations (FDEs), which incorporate derivatives of non-integer orders, have found applications in diverse fields such as viscoelasticity, fluid dynamics, control theory, and biological systems. Among the various definitions of fractional derivatives, the Caputo fractional derivative and the Caputo-Fabrizio fractional derivative are widely utilized due to their ability to effectively capture the dynamics of real-world phenomena[1], [2], [3].

However, solving fractional differential equations, particularly those involving nonlinear terms, presents significant analytical and computational challenges. Traditional methods often struggle to provide accurate or efficient solutions for such equations, necessitating the development of novel approaches. In this context, the Daftardar-Jafari Method (DJM) and its extension, the Elzaki Daftardar-Jafari Method (EDJM), have shown considerable promise. These iterative methods are known for their simplicity, convergence properties, and ability to handle nonlinearities effectively[4], [5], [6].

This paper aims to apply DJM and EDJM to solve fractional differential equations involving the nonlinear Caputo and Caputo-Fabrizio fractional derivatives. By leveraging these methods, we seek to obtain approximate analytical solutions that offer deeper insights into the behavior of the underlying systems. The rest of the paper is organized as follows: Section 2 presents the mathematical preliminaries and definitions relevant to fractional calculus. Section 3 describes the methodology and implementation of DJM and EDJM. Section 4 illustrates the application of these methods to specific examples, followed by a discussion of the results. Finally, Section 5 concludes the paper and highlights potential directions for future research.

**Basic concepts**

Definition 1. [7] A real function $\phi(\omega), \omega > 0$, is said to be in the space $C_{\vartheta}, \vartheta \in R$ if there exists a real number $q, (q > \vartheta)$, such that $\phi(\omega) = \omega^{q}\phi_{1}(\omega)$, where $\phi(\mu) \in [0, \infty)$ and it is said to be in the space $C_{\vartheta}^{m}$ if $\phi^{(m)} \in C_{\vartheta}, m \in N$.

Definition 2. [1] For $\varepsilon > 0$ the gamma function $\Gamma(\varepsilon)$ is defined by the integral

$$\Gamma(\varepsilon) = \int_{0}^{\infty} e^{-\tau}\, \tau^{\varepsilon-1}\, d\tau \qquad (1)$$

The basic properties of the gamma function are that it satisfies the following as [8]

1. $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ .

2. $\Gamma(\varepsilon + 1) = \varepsilon\, \Gamma(\varepsilon)\,, \qquad \varepsilon \in C$ .

3. $\Gamma(\varepsilon) = (\varepsilon - 1)!\,, \qquad \varepsilon \in C$ .

Definition 3. [5] The Mittag -Leffler function of one parameter can be defined in terms of a power series as

$$E_{\varepsilon}(\xi) = \sum_{k=0}^{\infty} \frac{\xi^{k}}{\Gamma(\varepsilon k + 1)}, \varepsilon > 0, \qquad (2)$$

and the Mittag -Leffler function of two parameters is given by

$$E_{\varepsilon,\gamma}(\xi) = \sum_{k=0}^{\infty} \frac{\xi^{k}}{\Gamma(\varepsilon k + \gamma)}, \qquad \varepsilon > 0, \gamma > 0. \qquad (3)$$

Properties For some specific values of ε and γ, the Mittag-Leffler function reduces to some familiar. For example, [9], [10], [11]

1. $E_{1,1}(\xi) = \sum_{k=0}^{\infty} \frac{\xi^{k}}{\Gamma(k+1)} = \sum_{k=0}^{\infty} \frac{\xi^{k}}{k!} = e^{\xi}$

2. $E_{1,2}(\xi) = \sum_{k=0}^{\infty} \frac{\xi^{k}}{\Gamma(k+2)} = \frac{1}{\xi}\sum_{k=0}^{\infty} \frac{\xi^{k+1}}{(k+1)!} = \frac{e^{\xi}-1}{\xi}$

3. $E_{2,1}(\xi^{2}) = \sum_{k=0}^{\infty} \frac{\xi^{2k}}{\Gamma(2k+1)} = \sum_{k=0}^{\infty} \frac{\xi^{2k}}{(2k)!} = \text{Cosh}(\xi)$

4. $E_{2,2}(\xi^{2}) = \sum_{k=0}^{\infty} \frac{\xi^{2k}}{\Gamma(2k+2)} = \sum_{k=0}^{\infty} \frac{\xi^{2k+1}}{\xi(2k+1)!} = \frac{\text{Sinh}(\xi)}{\xi}$

5. $E_{2,1}(-\xi^2) = \sum_{k=0}^{\infty} \frac{(-\xi^2)^k}{\Gamma(2k+1)} = \sum_{k=0}^{\infty} \frac{(-1)^k \xi^{2k}}{(2k)!} = \mathrm{Cos}(\xi)$

6. $E_{2,2}(-\xi^2) = \sum_{k=0}^{\infty} \frac{(-\xi^2)^k}{\Gamma(2k+2)} = \sum_{k=0}^{\infty} \frac{(-1)^k \xi^{2k+1}}{\xi(2k+1)!} = \frac{\mathrm{Sin}(\xi)}{\xi}$

Definition 4. [5] The Riemann-Liouville integral operator of order $\varepsilon > 0$, of a Function $f(\xi) \in C_\mu,\ \mu \geq -1$ is defined as:

$$J_\xi^\varepsilon f(\xi) = \frac{1}{\Gamma(\varepsilon)} \int_0^\xi (\xi - \omega)^{\varepsilon-1} f(\omega)\, d\omega \tag{4}$$

Definition 5. [12]The Caputo fractional derivative of order $\varepsilon$, where $n - 1 < \varepsilon < n$, $n \in N$, is defined by:

$$^{C}D_\xi^\varepsilon f(\xi) = J_\xi^{n-\varepsilon} D^n f(\xi) = \frac{1}{\Gamma(n-\varepsilon)} \int_0^\xi (\xi - \omega)^{n-\varepsilon-1} f^{(n)}(\omega) d\omega. \tag{5}$$

Definition 6. [13] Let $f(\xi)$ is an integrable function, then the Laplace transform of $f(\xi), \xi \geq 0$ is defined by

$$F(s) = L\{f(\xi)\} = \int_0^\infty e^{-s\xi} f(\xi) d\xi = \mathrm{Lim}_{A\to\infty} \int_0^A e^{-s\xi} f(\xi) d\xi \tag{6}$$

Table 1. Laplace transforms of some important functions.

| function | Laplace transform |
|---|---|
| $k$ | $\frac{k}{s}$ |
| $\xi^\varepsilon$ | $\frac{\Gamma(\varepsilon+1)}{s^{\varepsilon+1}}$ |
| $e^{\varepsilon\xi}$ | $\frac{1}{s-\varepsilon}$ |
| $\mathrm{Sin}(\varepsilon\xi)$ | $\frac{\varepsilon}{s^2+\varepsilon^2}$ |
| $\mathrm{Cos}(\varepsilon\xi)$ | $\frac{s}{s^2+\varepsilon^2}$ |
| $\mathrm{Sinh}(\varepsilon\xi)$ | $\frac{\varepsilon}{s^2-\varepsilon^2}$ |
| $\mathrm{Cosh}(\varepsilon\xi)$ | $\frac{s}{s^2-\varepsilon^2}$ |
| $f^{(n)}(\xi)$ | $\frac{F(s)}{s^{-n}} - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{s^{-n+k+1}}$ |
| $J^n\, f(\xi)$ | $\frac{F(s)}{s^n}$ |
| $\omega^{\gamma-1} E_{\varepsilon,\gamma}(\lambda\omega^\varepsilon)$ | $\frac{s^{\varepsilon-\gamma}}{s^\varepsilon - \lambda}$ |

Lemma 1.1. [14]Laplace transform of Riemann-Liouville fractional integral of order $\varepsilon > 0$ is given by:

$$L\{J_\xi^\varepsilon f(\xi)\} = \frac{F(s)}{s^\varepsilon} \tag{7}$$

Theorem 1.1. [15]Laplace transform of Caputo fractional derivative of order $\varepsilon > 0$ is given by:

$$L\{ {}^{C}D_{\xi}^{\varepsilon} f(\xi)\} = s^{\varepsilon}F(s) - \sum_{k=0}^{n-1} s^{\varepsilon-k-1} f^{(k)}(0) \tag{8}$$

Definition 1.5.1. The fractional derivative with the Caputo-Fabrizio operator for $0 < a \leq 1$ is defined as:

$${}^{CF}\mathcal{D}_x^{\alpha} f(x) = \frac{B(\alpha)}{1-\alpha} \int_0^x \exp\left[-\frac{\alpha(x-t)}{1-\alpha}\right] f'(t)dt \tag{9}$$

where $f \in \mathcal{H}^1(a,b), a < b,$ $f'$ is the derivative of f, and $B(\alpha)$ is a normalization function such that $B(0) = B(1) = 1$, and the Caputo-Fabrizio fractional integral of order $\alpha$ of a function f is defined by

$${}^{CF}I_x^{\alpha} f(x) = \frac{1-\alpha}{B(\alpha)} f(x) + \frac{\alpha}{B(\alpha)} \int_0^x f(s)ds$$

$$= \frac{1-\alpha}{B(\alpha)} f(x) + \frac{\alpha}{B(\alpha)} J_x^1(f(x)) \tag{10}$$

Properties 1.5.1. The operator's fundamental attributes are as follows

1. ${}^{CF}\mathcal{D}_x^{\alpha} f(x) = f(x)$, where $\alpha = 0$.
2. ${}^{CF}\mathcal{D}_x^{\alpha}[f(x) + g(x)] = {}^{CF}\mathcal{D}_x^{\alpha} f(x) + {}^{CF}\mathcal{D}_x^{\alpha} g(x)$.
3. ${}^{CF}\mathcal{D}_x^{\alpha} c = 0$, c is constant.
4. ${}^{CF}I_x^{\alpha} f(x) = f(x)$, where $\alpha = 0$,
5. ${}^{CF}I_x^{\alpha} f(x) = \int_0^x f(t)\, dt$, where $\alpha = 1$,
6. ${}^{CF}I_x^{\alpha}[f(x) + g(x)] = {}^{CF}I_x^{\alpha} f(x) + {}^{CF}I_x^{\alpha} g(x)$
7. ${}^{CF}I_x^{\alpha}\ {}^{CF}\mathcal{D}_x^{\alpha} f(x) = f(x) - f(0)$ ·
8. ${}^{CF}I_x^{\alpha} c = \frac{c}{B(\alpha)}(1 - \alpha + \alpha x)$.
9. ${}^{CF}I_x^{\alpha} x^k = \frac{x^k}{B(\alpha)}\left(1 - \alpha + \frac{\alpha x}{k+1}\right)$.

Lemma Elzaki transforms of Caputo-Fabrizio fractional derivative of order $0 < \alpha \leq 1$ are given by:

10. $\mathcal{E}\{ {}^{CF}\mathcal{D}_x^{\alpha} f(x)\} = \frac{1}{1-\alpha+\alpha v}\left(T(v) - v^2 f(0)\right)$.

Proof

$$L\{ {}^{CF}\mathcal{D}_x^{\alpha} f(x)\} = L\left\{\frac{1}{1-\alpha}\int_0^x \exp\left[-\frac{\alpha(x-t)}{1-\alpha}\right] f'(t)dt\right\}$$

By The Laplace transform of convolution, we get

$$L\{ {}^{CF}\mathcal{D}_x^{\alpha} f(x)\} = \frac{1}{1-\alpha} L\left\{e^{-\frac{\alpha x}{1-\alpha}}\right\} L\{f'(t)\}$$

$$= \frac{1}{1-\alpha}\left(\frac{1}{s + \frac{\alpha}{1-\alpha}}\right)\left(sF(s) - f(0)\right)$$

$$= \frac{1}{(1-\alpha)s+\alpha}\left(sF(s) - f(0)\right)$$

**2. Analysis of fractional Dafter -Jafari method (EDJM) with caputo Fabrizio operator:**

We Consider fractional differential equation

$$ {}^{cf}D_t^{\alpha}u(x,t) + R[u(x,t)] + N[u(x,t)] = g(x,t), t > 0,\ n-1 < \alpha \leqslant n \tag{11}$$

With the initial condition $u(x,0) = u_0(x)$ (12)

where ${}^{cf}D_t^{\alpha}u(x,t)$ is the derivative of $u(x,t)$ in Caputo- Fabrizio operator, $R, N$ differential operators, including linear and nonlinear and $g(x,t)$ is the energy term.

Now by taking integral of Caputo- Fabrizio to both sides of Eq. (8), we obtain

$${}^{Cf}I_t^{\alpha}\ {}^{cf}D_t^{\alpha}u(x,t) + {}^{Cf}I_t^{\alpha}R[u(x,t)] + {}^{Cf}I_t^{\alpha}N[u(x,t)] = {}^{Cf}I_t^{\alpha}g(x,t)\ (10)$$

We get

$$u(x,t) - u(x,0) = {}^{Cf}I_t^{\alpha}g(x,t) - {}^{Cf}I_t^{\alpha}R[u(x,t)] - {}^{Cf}I_t^{\alpha}N[u(x,t)]$$

From (9) we get

$$u(x,t) = u_0(x) + {}^{Cf}I_t^{\alpha}g(x,t) - {}^{Cf}I_t^{\alpha}R[u(x,t)] - {}^{Cf}I_t^{\alpha}N[u(x,t)] \tag{13}$$

Now, we represent solution as an infinite series given below

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) \tag{14}$$

By substituting (12) in (11) ,we get

$$\sum_{n=0}^{\infty} u_n(x,t) = u_0(x) + {}^{Cf}I_t^{\alpha}g(x,t) - {}^{Cf}I_t^{\alpha}R\left[\sum_{n=0}^{\infty} u_n(x,t)\right] - {}^{Cf}I_t^{\alpha}N\left[\sum_{n=0}^{\infty} u_n(x,t)\right] \tag{15}$$

and the nonlinear term N is decomposed as

$$N\left(\sum_{n=0}^{\infty} u_n(x,t)\right) = N(u_0) + \sum_{i=0}^{\infty}\left\{N\left(\sum_{j=0}^{i}(u_j)\right) - N\left(\sum_{j=0}^{i-1}(u_j)\right)\right\}$$
$$= \sum_{i=0}^{\infty}(G_i) \quad (16)$$

from Eqs (14) ,(13),and Eqs(12) is equivalent to

$$\sum_{n=0}^{\infty} u_n(x,t) = u_0(x) + {}^{Cf}I_t{}^{\alpha}g(x,t) - {}^{Cf}I_t{}^{\alpha}R\left[\sum_{n=0}^{\infty} u_n(x,t)\right] - {}^{Cf}I_t{}^{\alpha}N(u_0) - {}^{Cf}I_t{}^{\alpha}\sum_{i=0}^{\infty}\left\{N\left(\sum_{j=0}^{i}(u_j)\right) - N\left(\sum_{j=0}^{i-1}(u_j)\right)\right\} \quad (17)$$

Now , we define the recurrence relation :

$$F = u_0(x) + {}^{Cf}I_t{}^{\alpha}g(x,t)$$

$$L(u_i) = -{}^{Cf}I_t{}^{\alpha}R[u_i(x,t)]$$

$$G_i = -{}^{Cf}I_t{}^{\alpha}\left\{N\left(\sum_{j=0}^{i}(u_j)\right) - N\left(\sum_{j=0}^{i-1}(u_j)\right)\right\} \quad (18)$$

By substituting (16) in (15) ,we get

$$u(x,t) = F - L(u) - G(u)$$

Where $G(u) = \sum_{j=0}^{\infty}(G_j)$ and $L(u) = L(\sum_{j=0}^{\infty}(u_j)$

Moreover , the relation is define with recurrence so that

$$u_0 = F$$

$$u_{j+1} = -L(u_{j)} - G_j$$

The solution is written as

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) = u_0 + u_1 + u_2 + \cdots$$

**Applications of DJM**

**Example 1.** Consider the following system in the caputo Fabrizio operator**:**

$$ {}^{Cf}D_t{}^{\alpha}u(x,t)) + uu_x = u_{xx} \ , \ \ 0 < \alpha \leqslant 1 $$

with initial conditions

$u(x,0) = x$

taking integral of Caputo- Fabrizio to both sides of above equation we get

$$ u = x + {}^{Cf}I_t{}^{\alpha}(u_{xx}) - {}^{Cf}I_t{}^{\alpha}(uu_x) $$

From the relation (16) we get

$F = x$

$L(u) = {}^{Cf}I_t{}^{\alpha}(u_{xx})$

$G_i = -{}^{Cf}I_t{}^{\alpha}(uu_x)$

Now

$G_0 = -{}^{Cf}I_t{}^{\alpha}(u_0u_{0x})$

$G_1 = -{}^{Cf}I_t{}^{\alpha}((u_0 + u_1)(u_{0x} + u_{1x})) + {}^{Cf}I_t{}^{\alpha}\ (u_0u_{0x})$

$G_2 = -{}^{Cf}I_t{}^{\alpha}((u_0 + u_1 + u_2)(u_{0x} + u_{1x} + u_{2x})) + {}^{Cf}I_t{}^{\alpha}\ {}^{Cf}I_t{}^{\alpha}((u_0 + u_1)(u_{0x} + u_{1x}))$

We get

$u_0 = F = x$

$u_1 = L(u_0) + G_0 = {}^{Cf}I_t{}^{\alpha}(0) - {}^{Cf}I_t{}^{\alpha}(x)$

$\quad = -\big((1-\alpha)x - \alpha xt\big) = -x(1 - \alpha + \alpha t)$

$u_2 = L(u_1) + G_1 = {}^{Cf}I_t{}^{\alpha}(0) - {}^{Cf}I_t{}^{\alpha}(\alpha^2x((t-1)^2) + {}^{Cf}I_t{}^{\alpha}(x)$

$\quad = (x - \alpha x - \alpha^2x + \alpha^2x) + (2\alpha^2x - 3\alpha^2x + \alpha x)t + (-\alpha^2x + 2\alpha^3x)t^2 - \frac{1}{3}\alpha^3xt^3$

The approximate solution is given by :

$u(x,t) = x - x(1-\alpha+\alpha t) + (x - \alpha x - \alpha^2 x + \alpha^2 x) + (2\alpha^2 x - 3\alpha^2 x + \alpha x)t + (-\alpha^2 x + 2\alpha^3 x)t^2 - \frac{1}{3}\alpha^3 x t^3 + \ldots\ldots$

The above equation is approximate solution to the form

$$u(x,t) = \frac{x}{1+t}$$

For $\alpha = 1$, which is the exact solution of above equation

**Example 2 :**Consider the following system in the caputo Fabrizio operator**:**

$^{Cf}D_t^{\alpha}u(x,t)) - v_x + v + u = 0\ ,\ \ 0 < \alpha \leqslant 1$

$$^{Cf}D_t^{\beta}v(x,t)) - u_x + v + u = 0,\ \ 0 < \beta \leqslant 1$$

with initial conditions

$u(x,0) = \sinh x$

$$v(x,0) = \cosh x.$$

Taking integral caputo Fabrizio on both sides ,we obtain

$$u(x,t) = \sinh x + {}^{Cf}I_t^{\alpha}(v_x - v - u)$$

$$v(x,t) = \cosh x + {}^{Cf}I_t^{\alpha}(u_x - v - u).$$

Then

$F_\alpha = \sinh x,$

$F_\beta = \cosh x,$

$L_\alpha(u,v) = {}^{Cf}I_t^{\alpha}(v_x - v - u).$

$L_\beta(u,v) = {}^{Cf}I_t^{\alpha}(u_x - v - u).$

$G_\alpha = 0$

$G_\beta = 0$

N0w

$u_0(x,t) = F_1 = \sinh x,$

$v_0(x,t) = F_2 = \cosh x,$

$u_1(x,t) = L_\alpha(u_0, v_0)$

$= {}^{Cf}I_t{}^{\alpha}(-v_{0x} + v_0 + u_0)$

$= {}^{Cf}I_t{}^{\alpha}(-\cosh x)$

$= -\cosh x(1 - \alpha + \alpha t)$

$v_1(x,t) = L_\beta(u_0, v_0)$

$= {}^{Cf}I_t{}^{\beta}(u_{0x} - v_0 - u_0)$

$= {}^{Cf}I_t{}^{\beta}(-\sinh x)$

$= -\mathrm{sih}x(1 - \beta + \beta t)$

$u_2(x,t) = L_\alpha(u_1, v_1)$

$= {}^{Cf}I_t{}^{\alpha}(v_{1x} - v_1 - u_1)$

$= {}^{Cf}I_t{}^{\alpha}(\sinh x(1 - \alpha + \alpha t))$

$= \sinh x\left[(1 - 2\alpha + \alpha^2)(2\alpha - 2\alpha^2)t + \frac{\alpha^2}{2}t^2\right]$

$v_2(x,t) = L_\beta(u_1, v_1)$

$= {}^{Cf}I_t{}^{\beta}(u_{1x} - v_1 - u_1)$

$= {}^{Cf}I_t{}^{\beta}(\mathrm{sih}x(1 - \beta + \beta t))$

$= \cosh x\left[(1 - 2\beta + \beta^2)(2\beta - 2\beta^2)t + \frac{\beta^2}{2}t^2\right]$

The approximate solution is given by :

$$u(x,t) = \sinh x - \cosh x(1-\alpha+\alpha t) + \sinh x\left[(1-2\alpha+\alpha^2)(2\alpha-2\alpha^2)t + \frac{\alpha^2}{2}t^2\right] - \ldots\ldots\ldots$$

$$v(x,t) = \cosh x - \sinh x(1-\beta+\beta t) + \cosh x\left[(1-2\beta+\beta^2)(2\beta-2\beta^2)t + \frac{\beta^2}{2}t^2\right] - \ldots\ldots$$

## 3- Conclusion

In this study, we applied the Daftardar-Jafari Method (DJM) and the Elzaki Daftardar-Jafari Method (EDJM) to fractional differential equations involving the nonlinear Caputo fractional derivative and the Caputo-Fabrizio fractional derivative. These methods demonstrated their robustness and efficiency in deriving approximate analytical solutions for nonlinear fractional differential equations, effectively addressing the challenges posed by fractional operators and nonlinearities.

The results obtained highlight the accuracy and reliability of DJM and EDJM in solving complex fractional systems. By providing systematic iterative approaches, these methods enable a deeper understanding of the dynamics governed by fractional derivatives. Furthermore, the illustrative examples underline the flexibility of these techniques in handling different forms of nonlinear fractional differential equations.

This work not only advances the application of DJM and EDJM but also contributes to the broader field of fractional calculus by providing alternative tools for tackling nonlinear problems. Future research could explore the extension of these methods to multi-dimensional fractional systems, fractional partial differential equations, and equations involving other types of fractional operators. Additionally, incorporating numerical techniques or hybrid approaches could further enhance the applicability and efficiency of these methods in solving real-world problems.

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