Journal of Education for Pure Science- University of Thi-Qar
Vol.15, No. 2 (2025)
DOI: https://doi.org/10.32792/jeps.v15i2.673
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# Semi-Analytical Solutions of Nonlinear Time-Fractional Fornberg–Whitham Equation

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**Received 15/ 2 /2025, Accepted 22/ 4 /202X, Published 1 / 6 /2025**



## Abstract:

In this paper, we present an approximate solution for the Fornberg–Whitham equation using fractional operators, specifically the Atangana–Baleanu derivative, and apply the Yang Adomian Decomposition Method (YADM). The results indicate high efficiency and show a significant agreement with the exact solution.

**Keywords:** Adomian Decomposition Method; Yang transform; Atangana-Baleanu derivative.

## 1-Introduction

In recent decades, fractional calculus (FC) has been applied to numerous phenomena across natural sciences, engineering, fluid dynamics, life sciences, and other practical sciences. FC provides a robust framework for effectively describing these phenomena using fractional mathematical tools. Fractional derivatives (FDs) are especially valuable in capturing the hereditary and memory characteristics of various processes and materials. Applications of FDs appear in multiple engineering and scientific challenges, including diffusion and reaction processes, frequency-dependent signal processing, system modeling, damping behaviors in materials, and the relaxation and creep of viscoelastic substances. [1-3].

The examination of nonlinear wave equations and their answers holds vital significance across many academic disciplines. One of the most compelling solutions for nonlinear fractional partial differential equations (FPDEs) is the concept of traveling waves. Nonlinear FPDEs are frequently associated with complex physical and mechanical processes, making the exact solutions of these equations, particularly traveling wave solutions, highly desirable. Certain FPDEs, such as the Korteweg-de Vries and Camassa-Holm equations, exhibit various traveling wave solutions and are instrumental in modeling nonlinear, Omnidirectional dispersing waves in shallow depths. [4] Additionally, the Fornberg–Whitham equation (FWE) is notably significant in mathematical physics, with applications spanning multiple scientific disciplines.

The Fornberg–Whitham equation (FWE) [5],[6] is formulated as follows:

$$D_t^{\vartheta} u - u_{xxt} + u_x + u u_x = 3 u_x u_{xx} + u u_{xxx}$$

This equation describes the qualitative behavior of wave breaking in nonlinear dispersive waves. The FWE is known to allow peakon solutions, providing a numerical framework for understanding limiting wave heights and the occurrence of wave breaks. In 1978, Fornberg and Whitham introduced a peaked solution given by:

$$u(x,t) = Ce^{\frac{x}{2} - \frac{2t}{3}}$$

where $C$ is a constant. The analysis of FWEs has been approached through various numerical and analytical methods, including the Laplace decomposition technique [7], Lie symmetry [8], variational iteration technique [9], differential transformation technique [10], new iterative technique [11], homotopy-perturbation technique [12], and homotopy analysis transform technique [13], In addition to the numerous scientific studies in which solutions and methods for solving differential equations of both fractional and classical orders have been proposed [14-17], [19-32]

The Adomian Decomposition Method (ADM), pioneered by George Adomian, has emerged as a robust analytical framework for solving nonlinear differential equations. Unlike iterative or perturbation-based techniques, ADM systematically decomposes nonlinear terms into computable Adomian polynomials, enabling precise handling of complex nonlinearities without restrictive linearization assumptions [29–31]. This method has been extensively applied to fractional-order systems, particularly under operators like Caputo, Riemann-Liouville, and the modern Atangana-Baleanu derivative, which preserves memory effects critical for modeling hereditary phenomena [33,34].

A key advancement in ADM's application to fractional equations lies in its integration with integral transforms. In this study, the Yang transform is combined with ADM to streamline the resolution of the time-fractional Fornberg–Whitham equation (FWE). While traditional mesh-based methods [18] provide discrete approximations, the Yang-ADM synergy generates continuous, rapidly convergent series solutions, avoiding the numerical overhead of spatial discretization. Recent innovations in ADM, such as optimized algorithms for Adomian polynomial generation [34], have further enhanced its computational efficiency, addressing historical critiques about polynomial complexity.

This manuscript advances the analytical resolution of the fractional FWE using the Atangana-Baleanu fractional **operator**, a non-singular kernel derivative that accurately models memory-dependent wave-breaking dynamics. While prior studies employed methods like differential transformation [10] or homotopy techniques [12,13], their reliance on linearization or small parameters limits generality. In contrast, ADM's direct decomposition framework circumvents these constraints, yielding semi-analytical solutions in series form with minimal computational effort. The method's convergence, rigorously established for fractional PDEs [35,36], ensures reliability, while its avoidance of discretization errors aligns with the need for precise physical insights into wave behavior.

Study Contributions:

1. **Yang-ADM Hybridization**: Unifies the Yang transform's operational simplicity with ADM's nonlinear decomposition, enabling efficient handling of Atangana-Baleanu fractional terms.
2. **Streamlined Polynomial Computation**: Leverages recent algorithmic improvements [38] to accelerate Adomian polynomial generation, enhancing practicality.

3. **Memory-Effect Preservation**: Utilizes the Atangana-Baleanu operator to maintain the FWE's intrinsic memory properties, critical for modeling wave-breaking.

This work underscores ADM's adaptability to modern fractional calculus challenges, demonstrating its superiority over iterative and mesh-dependent methods in balancing accuracy, efficiency, and analytical transparency. By achieving semi-analytical solutions with clear convergence, the study provides a template for solving broader classes of FPDEs prevalent in engineering and physics.

## 2-Preliminaries

**Definition.1**. The Caputo fractional derivative of function $f(x)$ , $x > 0$ is defined by [32]

$$D_x^{\propto} f(t) = \begin{cases} \frac{1}{\Gamma(n-\propto)} \int_0^x (x-t)^{n-\propto-1} f^{(n)}(t)dt, & n-1 < \propto \leq n \in \mathbb{N} \\ \frac{d^n}{dx^n} f(x), & \propto = n \in \mathbb{N} \end{cases} \quad (1)$$

**Note. 1**.From **Definition 1**, the following **outcome** is **derived**:

$$D_t^{\propto} t^{\beta} = \begin{cases} \frac{\Gamma(\beta+1)}{\Gamma(\beta-\propto+1)} t^{\beta-\propto}, & n-1 < \propto \leq n, \quad \beta > n-1, \ \beta \in \mathcal{R} \\ 0 & n-1 < \propto \leq n, \quad n > \beta, \ \beta \in \mathbb{N} \end{cases} \quad (2)$$

**Definition.2.** The Atangana-Baleanu fractional derivative is expressed as[33][34]:

$$ {}^{AB}_{a}\mathcal{D}_t^{\vartheta} u(t) = \frac{\mathcal{M}(\vartheta)}{1-\vartheta} \int_a^t \mathbb{E}_{\vartheta} \left( \frac{-\vartheta (t-x)^{\vartheta}}{1-\vartheta} \right) u'(x)dx, \quad (3)$$

Where $0 < \vartheta < 1$ *and* $\mathcal{M}(\vartheta)$ *is* a normalization function, such that $\mathcal{M}(0) = \mathcal{M}(1) = 1$.

**Definition.3.** The Atangana-Baleanu Fractional Integral (ABFI) of order $\vartheta$ is defined as follows[34]:

$$ {}^{AB}_{a}I_t^{\vartheta} u(t) = \frac{1-\vartheta}{\mathcal{M}(\vartheta)} u(t) + \frac{\vartheta}{\mathcal{M}(\vartheta)\Gamma(\vartheta)} \int_a^t (t-x)^{\vartheta-1} u(x)dx, \quad (4)$$

Where $0 < \vartheta < 1$ *and* $\mathcal{M}(\vartheta)$ *is* a normalization function, such that $\mathcal{M}(0) = \mathcal{M}(1) = 1$.

**Definition.4**. The Yang transform YT is stated as[35]

$$Y\{u(t)\} = \int_0^{\infty} e^{-\frac{t}{v}} u(t) \, dt \, , t > 0, \quad (5)$$

with $v$ representing the transform variable.

**Some of properties of it:**

1. $Y\{1\} = v$
2. $Y\{t\} = v^2$
3. $Y\{t^n\} = v^{n+1} n!$
4. $Y\{t^{\vartheta}\} = v^{\vartheta+1}\Gamma(\vartheta+1), \vartheta \in R$

**Theorem**. The Yang transform of fractional order derivative is defined by

a. $$Y\left\{ {}^{C}D_x^{\vartheta} u(t) \right\} = \frac{Y\{u(t)\}}{v^{\vartheta}} - \sum_{k=0}^{n-1} \frac{u^{(k.)}(0)}{v^{\propto-k-1}}, \quad n-1 < \vartheta \leq n \quad (6)$$

b. $$Y\left\{ {}^{AB}_{a}\mathcal{D}_t^{\vartheta} u(t) \right\} = \frac{\mathcal{M}(\vartheta)}{1-\vartheta+\vartheta \, v^{\vartheta}} \left( \mathrm{Y}\{u(\mathrm{t})\} - v u(0) \right), \ 0 < \vartheta \leq 1, \quad (7)$$

**Proof.** The proof of (a) as in [36]. To prove (b) We take the transformation into Eq(3) and then utilize the convolution property to obtain the desired result after simplification. Thus, the proof is completed

**Definition 5**. The Mittag-Leffler function with two Parameters is defined by [37], [38]

$$\mathbb{E}_{\vartheta,\wp}(\mathcal{Z}) = \sum_{n=0}^{\infty} \frac{\mathcal{Z}^n}{\Gamma(n\vartheta+\wp)} \quad , \quad \propto, \wp, \mathcal{Z} \in \mathbb{C} \quad , \quad Re(\vartheta) > 0 \ , \quad Re(\wp) > 0$$

**Note.2**. From definition5. The following result is obtained:

(1) $\mathbb{E}_{2,1}(x^2) = \cos h(x)$.

(2) $\mathbb{E}_{2,2}(x^2) = \frac{\sin h(x)}{x}$.

(3) $\mathbb{E}_{2,3}(x^2) = \frac{1}{x^2}[-1 + \cos h(x)]$.

## 3-Fornberg–Whitham equation with Atangana-Baleanu fractional derivative

Consider the fractional Fornberg–Whitham equation

$$^{AB}\mathfrak{D}_t^{\vartheta} u - u_{xxt} + u_x + uu_x = 3u_x u_{xx} + uu_{xxx} \ , \ 0 < \vartheta \le 1, \tag{8}$$

With initial condition

$$u(x,0) = g(x), \tag{9}$$

Through the application of the Yang transform to Equation (8), we get:

$$Y\{^{AB}\mathfrak{D}_t^{\vartheta} u\} = \mathrm{Y}\{u_{xxt} - u_x - uu_x + 3u_x u_{xx} + uu_{xxx}\},$$

Or equivalently,

$$\frac{[\mathrm{Y}\{u\} - \mathrm{v}u(x,0)]}{1 - \vartheta + \vartheta v^{\vartheta}} = \mathrm{Y}\{u_{xxt} - u_x - uu_x + 3u_x u_{xx} + uu_{xxx}\},$$

Rearrange,

$$\mathrm{Y}\{u\} = \mathrm{v}u(x,0) + (1 - \vartheta + \vartheta v^{\vartheta})\mathrm{Y}\{u_{xxt} - u_x - uu_x + 3u_x u_{xx} + uu_{xxx}\},$$

Taking inverse of Yang Transform, we obtain

$$u(x,t) = u(x,0) + Y^{-1}\left[(1 - \vartheta + \vartheta v^{\vartheta})\mathrm{Y}\{u_{xxt} - u_x - uu_x + 3u_x u_{xx} + uu_{xxx}\}\right], \tag{10}$$

Let

$$u(x,t) = \sum_{n=0}^{\infty} u_n, \tag{11}$$

$$uu_x = \sum_{n=0}^{\infty} \mathrm{A}_n, \tag{12}$$

$$u_x u_{xx} = \sum_{n=0}^{\infty} \mathrm{B}_n, \tag{13}$$

$$uu_{xxx} = \sum_{n=0}^{\infty} \mathrm{C}_n, \tag{14}$$

Where

$$A_0 = u_0 u_{0x} \ , \quad B_0 = u_{0x} u_{0xx}, \qquad C_0 = u_0 u_{0xxx},$$

$$A_1 = u_0 u_{1x} + u_1 u_{0x}\,, \quad B_0 = u_{0x} u_{1xx} + u_{1x} u_{0xx}, \qquad C_1 = u_0 u_{1xxx} + u_1 u_{0xxx},$$

$$A_2 = u_0 u_{2x} + u_1 u_{1x} + u_2 u_{0x}\,, \quad B_2 = u_{0x} u_{2xx} + u_{1x} u_{1xx} + u_{2x} u_{0xx},$$

$$C_2 = u_0 u_{2xxx} + u_1 u_{1xxx} + u_2 u_{0xxx},$$

$$\vdots$$

By combining Equation (11), (12) , (13) and (14) into Equation (10) yields the following result

$$\sum_{n=0}^{\infty} u_n = u(x,0) + Y^{-1}\left[(1-\vartheta+\vartheta v^{\vartheta})\mathrm{Y}\left\{\sum_{n=0}^{\infty} u_{nxxt} - \sum_{n=0}^{\infty} u_{nx} - \sum_{n=0}^{\infty} \mathrm{A}_n + 3\sum_{n=0}^{\infty} \mathrm{B}_n + \sum_{n=0}^{\infty} \mathrm{C}_n\right\}\right], \tag{15}$$

We present recursive relations as a continuation of the FADM,

$$u_0 = (x,0),$$

$$u_{n+1} = Y^{-1}\left[(1-\vartheta+\vartheta v^{\vartheta})\mathrm{Y}\{u_{\mathrm{n}xxt} - u_{\mathrm{n}x} - \mathrm{A}_n + 3\mathrm{B}_n + \mathrm{C}_n\}\right], \qquad n \geq 0, \tag{16}$$

Consequently, the solution for Equation (8) is

$$u(x,t) = u_0 + u_1 + u_2 + \cdots \tag{17}$$

## 4-Application

**Example.1**. Consider the fractional Fornberg–Whitham equation

$$^{AB}\mathfrak{D}_t^{\vartheta} u - u_{xxt} + u_x + u u_x = 3 u_x u_{xx} + u u_{xxx}, \tag{18}$$

With initial condition

$$u(x,0) = e^{\frac{x}{2}}.$$

using eq.(16), we obtain

$$u_0 = e^{\frac{x}{2}},$$

$$u_1 = -\left[1-\vartheta+\frac{\vartheta t^{\vartheta}}{\Gamma(\vartheta+1)}\right]\frac{1}{2}e^{\frac{x}{2}},$$

$$u_2 = -\left[(1-\vartheta)\frac{\vartheta t^{\vartheta-1}}{\Gamma(\vartheta)} + \frac{\vartheta^2 t^{2\vartheta-1}}{\Gamma(2\vartheta)}\right]\frac{1}{8}e^{\frac{x}{2}} + \left[(1-\vartheta)^2 + 2(1-\vartheta)\frac{\vartheta t^{\vartheta}}{\Gamma(\vartheta+1)} + \frac{\vartheta^2 t^{2\vartheta}}{\Gamma(2\vartheta+1)}\right]\frac{1}{4}e^{\frac{x}{2}}.$$

$$u_3 = -\left[(1-\vartheta)^2\vartheta\frac{t^{\vartheta-2}}{\Gamma(\vartheta-1)} + 2(1-\vartheta)\frac{\vartheta^2 t^{2\vartheta-2}}{\Gamma(2\vartheta-1)} + \frac{\vartheta^3 t^{3\vartheta-2}}{\Gamma(3\vartheta-1)}\right]\frac{1}{32}e^{\frac{x}{2}}$$
$$+\left[3(1-\vartheta)^2\vartheta\frac{t^{\vartheta-1}}{\Gamma(\vartheta)} + 5(1-\vartheta)\frac{\vartheta^2 t^{2\vartheta-1}}{\Gamma(2\vartheta)} + 2\frac{\vartheta^3 t^{3\vartheta-1}}{\Gamma(3\vartheta)}\right]\frac{1}{16}e^{\frac{x}{2}}$$
$$-\left[(1-\vartheta)^3 + 3(1-\vartheta)^2\vartheta\frac{t^{\vartheta}}{\Gamma(\vartheta+1)} + 3(1-\vartheta)\frac{\vartheta^2 t^{2\vartheta}}{\Gamma(2\vartheta+1)} + \frac{\vartheta^3 t^{3\vartheta}}{\Gamma(3\vartheta+1)}\right]\frac{1}{8}e^{\frac{x}{2}}.$$

Then ,

$$u(x,t) = e^{\frac{x}{2}} - \left[1-\vartheta+\frac{\vartheta t^{\vartheta}}{\Gamma(\vartheta+1)}\right]\frac{1}{2}e^{\frac{x}{2}} - \left[(1-\vartheta)\frac{\vartheta t^{\vartheta-1}}{\Gamma(\vartheta)} + \frac{\vartheta^2 t^{2\vartheta-1}}{\Gamma(2\vartheta)}\right]\frac{1}{8}e^{\frac{x}{2}}$$
$$+\left[(1-\vartheta)^2 + 2(1-\vartheta)\frac{\vartheta t^{\vartheta}}{\Gamma(\vartheta+1)} + \frac{\vartheta^2 t^{2\vartheta}}{\Gamma(2\vartheta+1)}\right]\frac{1}{4}e^{\frac{x}{2}}$$
$$-\left[(1-\vartheta)^2\vartheta\frac{t^{\vartheta-2}}{\Gamma(\vartheta-1)} + 2(1-\vartheta)\frac{\vartheta^2 t^{2\vartheta-2}}{\Gamma(2\vartheta-1)} + \frac{\vartheta^3 t^{3\vartheta-2}}{\Gamma(3\vartheta-1)}\right]\frac{1}{32}e^{\frac{x}{2}}$$
$$+\left[3(1-\vartheta)^2\vartheta\frac{t^{\vartheta-1}}{\Gamma(\vartheta)} + 5(1-\vartheta)\frac{\vartheta^2 t^{2\vartheta-1}}{\Gamma(2\vartheta)} + 2\frac{\vartheta^3 t^{3\vartheta-1}}{\Gamma(3\vartheta)}\right]\frac{1}{16}e^{\frac{x}{2}}$$
$$-\left[(1-\vartheta)^3 + 3(1-\vartheta)^2\vartheta\frac{t^{\vartheta}}{\Gamma(\vartheta+1)} + 3(1-\vartheta)\frac{\vartheta^2 t^{2\vartheta}}{\Gamma(2\vartheta+1)} + \frac{\vartheta^3 t^{3\vartheta}}{\Gamma(3\vartheta+1)}\right]\frac{1}{8}e^{\frac{x}{2}}\ldots$$

The exact solution at $\vartheta = 1$, $u(x,t) = e^{\frac{x}{2}-\frac{2}{3}t}$.

**Example.2**. Consider the fractional Fornberg–Whitham equation

$$^{AB}\mathfrak{D}_t^{\vartheta}u - u_{xxt} + u_x + uu_x = 3u_x u_{xx} + uu_{xxx}, \tag{19}$$

With initial condition

$$u(x,0) = (\cosh\frac{x}{4})^2,$$

Apply eq.(16), We achieve

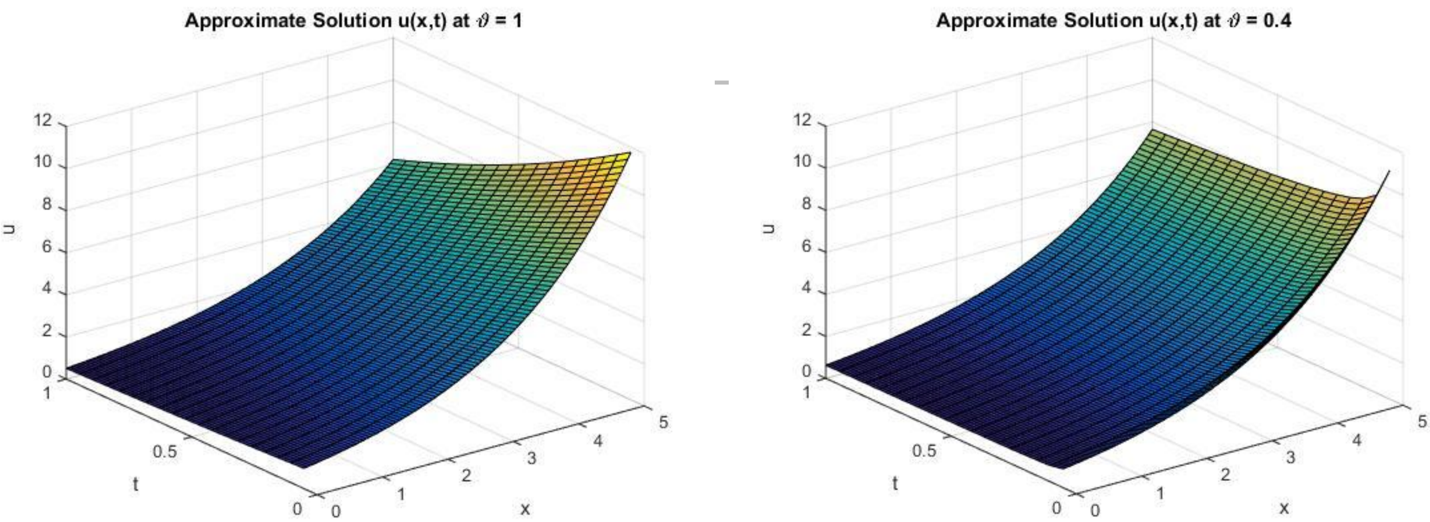
$$u_0 = (\cosh\frac{x}{4})^2.$$

$$u_1 = -\left(1-\vartheta+\vartheta\frac{t^{\vartheta}}{\Gamma(\vartheta+1)}\right)\left[\frac{11}{32}\sinh\frac{x}{2}\right].$$

$$u_2 = \frac{121}{512}\cosh\frac{x}{2}\left[(1-\vartheta)^2 + 2(1-\vartheta)\vartheta\frac{t^{\vartheta}}{\Gamma(\vartheta+1)} + \frac{t^{2\vartheta}}{\Gamma(2\vartheta+1)}\right] - \frac{11}{128}\sinh\frac{x}{2}\left[\vartheta(1-\vartheta)\frac{t^{\vartheta-1}}{\Gamma(\vartheta)} + \vartheta^2\frac{t^{2\vartheta-1}}{\Gamma(2\vartheta)}\right].$$

$$u_3 = -\left[(1-\vartheta)^3 + 3(1-\vartheta)^2\vartheta\frac{t^{\vartheta}}{\Gamma(\vartheta+1)} + 3(1-\vartheta)\frac{\vartheta^2 t^{2\vartheta}}{\Gamma(2\vartheta+1)} + \frac{\vartheta^3 t^{3\vartheta}}{\Gamma(3\vartheta+1)}\right]\frac{1331}{8192}\sinh\frac{x}{2}$$
$$+\frac{121}{2048}\cosh\frac{x}{2}\left[3(1-\vartheta)^2\vartheta\frac{t^{\vartheta-1}}{\Gamma(\vartheta)} + 5(1-\vartheta)\frac{\vartheta^2 t^{2\vartheta-1}}{\Gamma(2\vartheta)} + 2\frac{\vartheta^3 t^{3\vartheta-1}}{\Gamma(3\vartheta)}\right]$$
$$-\frac{11}{512}\sinh\frac{x}{2}\left[(1-\vartheta)^2\vartheta\frac{t^{\vartheta-2}}{\Gamma(\vartheta-1)} + 2(1-\vartheta)\frac{\vartheta^2 t^{2\vartheta-2}}{\Gamma(2\vartheta-1)} + \frac{\vartheta^3 t^{3\vartheta-2}}{\Gamma(3\vartheta-1)}\right].$$

The approximate solution is




$$u(x,t)=\frac{1}{2}+\frac{1}{2}\cosh\frac{x}{2}-\frac{11}{32}\sinh\frac{x}{2}\left[1-\vartheta+\vartheta\frac{t^{\vartheta}}{\Gamma(\vartheta+1)}\right]+\frac{121}{512}\cosh\frac{x}{2}\left[(1-\vartheta)^2+2(1-\vartheta)\vartheta\frac{t^{\vartheta}}{\Gamma(\vartheta+1)}+\frac{t^{2\vartheta}}{\Gamma(2\vartheta+1)}\right]$$
$$-\frac{11}{128}\sinh\frac{x}{2}\left[\vartheta(1-\vartheta)\frac{t^{\vartheta-1}}{\Gamma(\vartheta)}+\vartheta^2\frac{t^{2\vartheta-1}}{\Gamma(2\vartheta)}\right]$$
$$-\left[(1-\vartheta)^3+3(1-\vartheta)^2\vartheta\frac{t^{\vartheta}}{\Gamma(\vartheta+1)}+3(1-\vartheta)\frac{\vartheta^2t^{2\vartheta}}{\Gamma(2\vartheta+1)}+\frac{\vartheta^3t^{3\vartheta}}{\Gamma(3\vartheta+1)}\right]\frac{1331}{8192}\sinh\frac{x}{2}$$
$$+\frac{121}{2048}\cosh\frac{x}{2}\left[3(1-\vartheta)^2\vartheta\frac{t^{\vartheta-1}}{\Gamma(\vartheta)}+5(1-\vartheta)\frac{\vartheta^2t^{2\vartheta-1}}{\Gamma(2\vartheta)}+2\frac{\vartheta^3t^{3\vartheta-1}}{\Gamma(3\vartheta)}\right]$$
$$-\frac{11}{512}\sinh\frac{x}{2}\left[(1-\vartheta)^2\vartheta\frac{t^{\vartheta-2}}{\Gamma(\vartheta-1)}+2(1-\vartheta)\frac{\vartheta^2t^{2\vartheta-2}}{\Gamma(2\vartheta-1)}+\frac{\vartheta^3t^{3\vartheta-2}}{\Gamma(3\vartheta-1)}\right]\dots$$

When selcting $\vartheta=1$ , we get exact solution $u(x,t)=\cosh^2\left(\frac{x}{4}-\frac{11}{24}t\right)$.

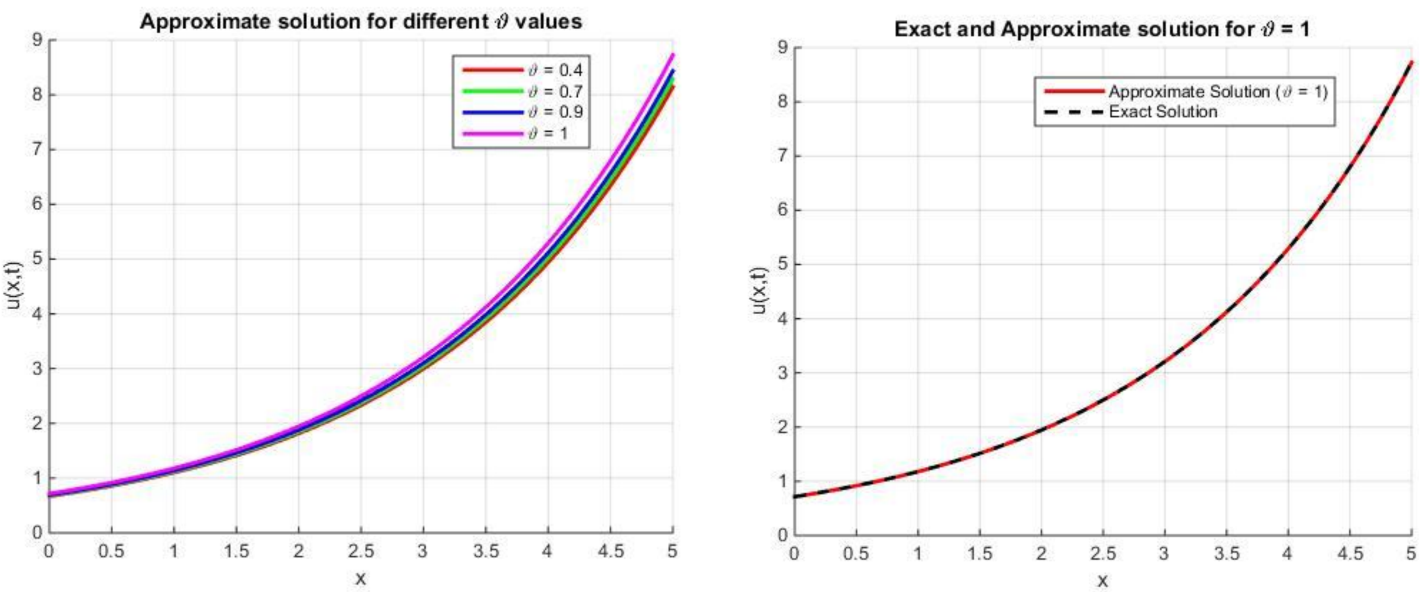
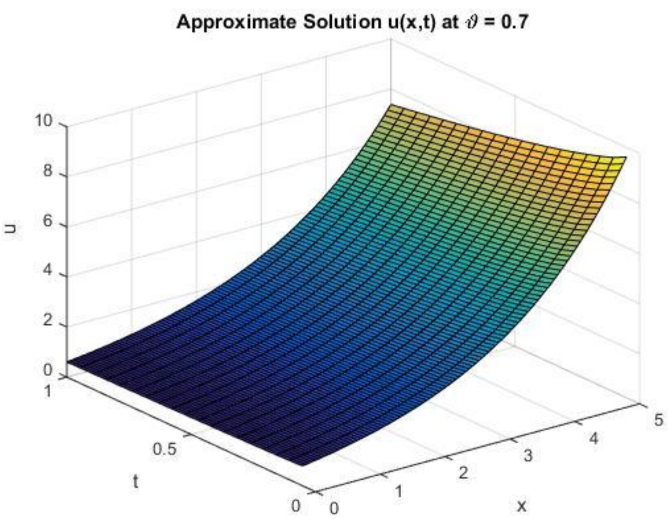
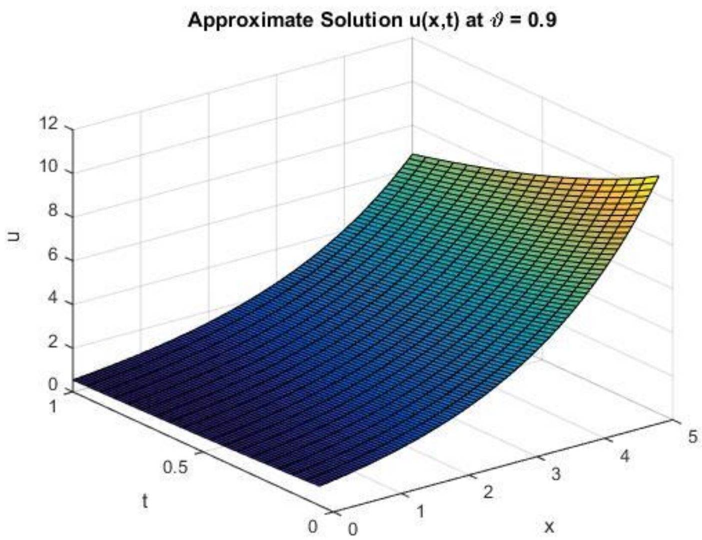



Figure 1 Plot showing the solutions for various values of $\vartheta$, alongside the exact solution at $\vartheta = 1$, for **Example 1**.



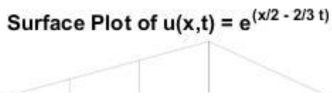




Figure 2. Graph for different values of ϑ and the exact solution at ϑ = 1 for Example 1

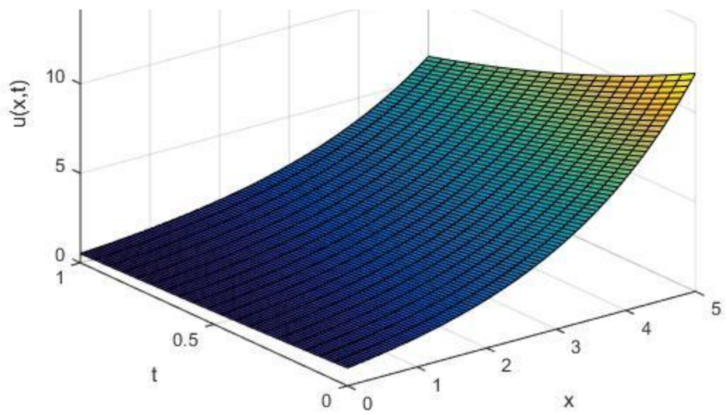


Table 1.absolute error for Example 1

| $x$ | $\vartheta = 1$ | $\vartheta = 0.9$ | $\vartheta = 0.7$ | $\vartheta = 0.4$ |
|---|---|---|---|---|
| **0.5** | 0.00049496 | 0.031087 | 0.04746 | 0.061608 |
| **1** | 0.00063554 | 0.039917 | 0.060537 | 0.079106 |
| **1.5** | 0.00081606 | 0.051254 | 0.077731 | 0.10157 |
| **2** | 0.0010478 | 0.065811 | 0.099809 | 0.13042 |
| **2.5** | 0.0013454 | 0.084503 | 0.12816 | 0.16747 |
| **3** | 0.0017276 | 0.1085 | 0.16456 | 0.21503 |
| **3.5** | 0.0022183 | 0.13932 | 0.2113 | 0.27611 |
| **4** | 0.0028483 | 0.17889 | 0.27131 | 0.35453 |
| **4.5** | 0.0036573 | 0.2297 | 0.34837 | 0.45523 |
| **5** | 0.0046961 | 0.29495 | 0.44731 | 0.58452 |

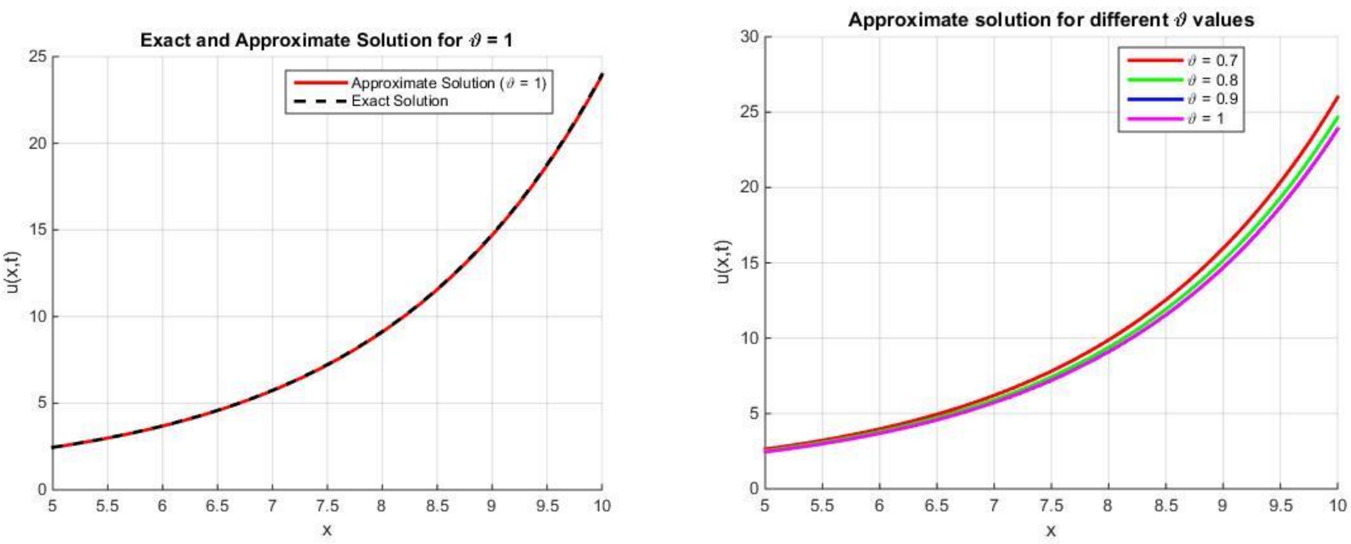



Figure 3. Graph for different values of ϑ and the exact solution at ϑ = 1 **for Example 2.**

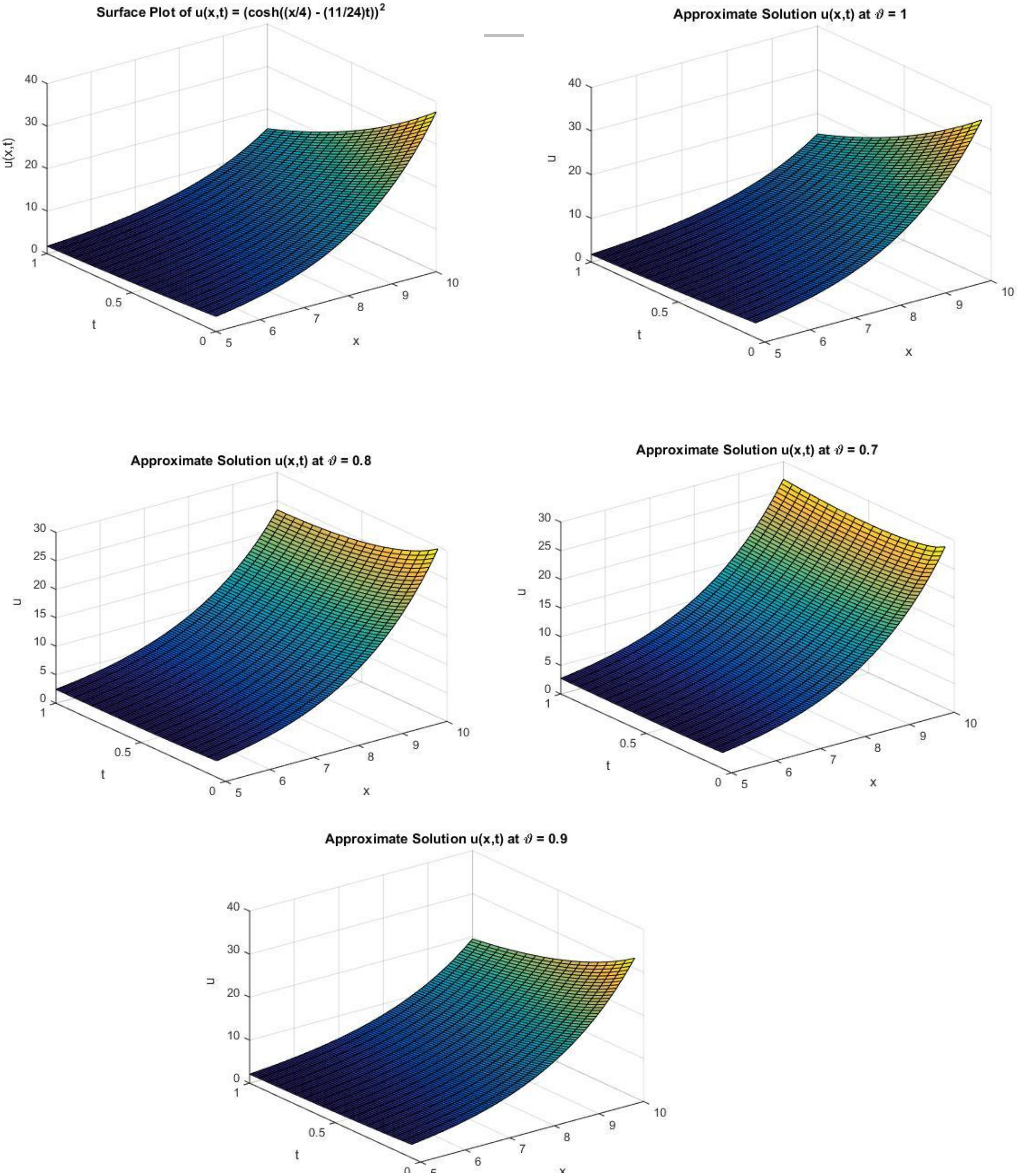



Figure 4. Visualization of the solutions for varies values of ϑ, including the exact solution at ϑ = 1, in **Example 2**.

Table 2. absolute error for **Example 2.**

| $x$ | $\vartheta = 1$ | $\vartheta = 0.9$ | $\vartheta = 0.8$ | $\vartheta = 0.7$ |
|---|---|---|---|---|
| **5** | 0.005757 | 0.0039514 | 0.065314 | 0.17893 |
| **5.5** | 0.0070306 | 0.0070339 | 0.079532 | 0.22314 |
| **6** | 0.008746 | 0.010558 | 0.098746 | 0.28137 |
| **6.5** | 0.011011 | 0.014746 | 0.12416 | 0.35728 |
| **7** | 0.013967 | 0.01986 | 0.15738 | 0.45564 |
| **7.5** | 0.017801 | 0.026222 | 0.20049 | 0.58262 |
| **8** | 0.022754 | 0.034232 | 0.25619 | 0.74621 |
| **8.5** | 0.029136 | 0.044392 | 0.32799 | 0.95667 |
| **9** | 0.037348 | 0.057341 | 0.42039 | 1.2272 |
| **9.5** | 0.047907 | 0.073892 | 0.53921 | 1.5749 |

## Conclusion

We investigated the Fornberg–Whitham equation using the fractional Atangana–Baleanu operator and employed the Yang Adomian Decomposition Method (YADM) to obtain an approximate solution. The results demonstrated the high accuracy and efficiency of the proposed method, as they closely aligned with the exact solution. This confirms the applicability of fractional operators in modeling complex nonlinear phenomena and highlights the effectiveness of YADM in solving fractional differential equations. Future work may explore the extension of this approach to other nonlinear fractional models and compare its performance with alternative numerical and analytical techniques.

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