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A New Technique for Solving Nonlinear Partial Differential Equations Using the Homotopy Perturbation Method

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Abstract:

In this paper, the Homotopy Perturbation Method (HPM) is employed to solve nonlinear partial differential equations, which often present significant analytical and computational challenges due to the complexity of their nonlinear terms. To address these difficulties, a new assumption for treating the nonlinear components is introduced, offering a simpler and more practical alternative to the traditional Adomian decomposition approach. While this new assumption enhances the applicability of HPM to a broader class of problems, it may have limitations when dealing with equations exhibiting highly singular behavior or strong nonlinearity.

Keywords: Adomian Decomposition; Homotopy Perturbation Method; Partial Differential Equations.

1-Introduction

Nonlinear partial differential equations (NPDEs) play a fundamental role in modeling a wide range of complex physical and engineering phenomena, including fluid dynamics, nonlinear wave propagation, plasma physics, and nonlinear control systems [1–5]. These equations are inherently difficult to solve due to their nonlinearity and complex boundary or initial conditions. In most cases, finding exact analytical solutions is either extremely difficult or entirely infeasible. This motivates the development and application of approximate and semi-analytical techniques to obtain useful solutions with acceptable accuracy [6–10].

Among such techniques, the Homotopy Perturbation Method (HPM) has emerged as a powerful and reliable tool for solving various types of linear and nonlinear differential equations [11–15]. The HPM efficiently merges the classical perturbation methods with homotopy theory, offering series solutions that often converge rapidly to the exact or approximate solution. However, one of the main challenges in applying HPM lies in the treatment of nonlinear terms, which typically requires specific assumptions or decompositions, such as Adomian's decomposition, which, although effective, can be computationally intensive or analytically complex in certain scenarios [16–18].

In this paper, we propose a new and simplified assumption for handling nonlinear terms within the HPM framework. Unlike Adomian's approach, the proposed assumption is characterized by its ease of implementation and lower computational overhead. The goal of this study is to enhance the applicability of HPM to a broader class of nonlinear problems, while maintaining a balance between accuracy, efficiency, and simplicity. The proposed technique is evaluated through multiple benchmark examples involving nonlinear partial differential equations. In addition, a convergence analysis and error estimation are conducted to verify the robustness and reliability of the results [19–21].

This work contributes to the growing body of research focused on developing flexible and efficient methods for solving nonlinear models and offers potential for future extensions in solving higher-dimensional and more complex nonlinear systems.

2-Main Result

To demonstrate the foundational concept of the Homotopy Perturbation Method (HPM), we analyze the subsequent differential equation

$$A(u) = f(r) \leftrightarrow A(u) - f(r) = 0, r \in \Omega, \tag{1}$$

where A is a differential operator of general form, and $f(r)$ is a closed-form analytic function
Suppose that $A(u) = L^{(n)} + R(u) + N(u)$. Therefore Eq. (1) can be rewritten as

$$L^{(n)}u + R(u) + N(u) - f(r) = 0, \tag{2}$$

Using the Homotopy Perturbation Method (HPM), we define a homotopy mapping $v(r,p): \Omega \times [0,1] \to R$ obeying the following condition

$$H(v,p) = (1-p)\left[L^{(n)}(v) - L^{(n)}(u_0)\right] + p[A(v) - f(r)] = 0$$

or

$$H(v,p) = L^{(n)}(v) - L^{(n)}(u_0) + pL^{(n)}(u_0) + p[R(v) + N(v) - f(r)] = 0 \tag{3}$$

In this framework, $p \in [0,1]$ represents the embedding variable, while u_0 is the preliminary approximation of Equation (1), fulfilling the boundary constraints. From Eq. (3), it follows that:

$$H(v,0) = L^{(n)}(v) - L^{(n)}(u_0) = 0$$

and

$$H(v,1) = A(v) - f(r) = 0$$

As per the HPM framework, the embedding parameter p is initially treated as a perturbation factor, and the solution to Equation (1) is postulated as a series expansion in terms of p:

$$v = \sum_{m=0}^{\infty} p^m v_m = v_0 + pv_1 + p^2 v_2 + \cdots \tag{4}$$

Furthermore, nonlinear operators may be expanded into a series of components:

$$N(v)=\sum_{m=0}^{\infty}p^m H_m(v)=H_0+pH_1+p^2H_2+\cdots \tag{5}$$

In this method, we replaced He's polynomials in the nonlinear part with a simpler and more effective hypothesis, which is the same hypothesis used in the Daftardar-Jafari method [31]. Which are given by

$$H_m=N\left(\sum_{k=0}^{m}v_k\right)-N\left(\sum_{k=0}^{m-1}v_k\right),\quad i>0,\qquad m=1,2,\ldots$$

and

$$H_0=\mathrm{v}_0\mathrm{v}_{0x},$$

By inserting Equations (4) and (5) into Equation (3), the following expression is derived:

$$L^{(n)}\left(\sum_{m=0}^{\infty}p^m v_m\right)-L^{(n)}(u_0)+pL^{(n)}(u_0)+p\left[R\left(\sum_{m=0}^{\infty}p^m v_m\right)+\sum_{m=0}^{\infty}p^m H_m-f(r)\right]=0 \tag{6}$$

When the multipliers of equal powers of p in Equation (6) are equated, the following is obtained:

$$\begin{aligned}
&p^0: L^{(n)}(v_0)-L^{(n)}(u_0)=0, && L^{(n-1)}(v_0)=L^{(n-1)}(u).\\
&p^1: L^{(n)}(v_1)+L^{(n)}(u_0)+R(v_0)+H_0-f(r)=0, && L^{(n-1)}(v_1)=0.\\
&p^2: L^{(n)}(v_2)+R(v_1)+H_1=0, && L^{(n-1)}(v_2)=0.\\
&p^3: L^{(n)}(v_3)+R(v_2)+H_2=0, && L^{(n-1)}(v_3)=0.\\
&\vdots
\end{aligned}$$

By assigning $p=1$, the approximate solution to Equation (1) is obtained in the following form:

$$u=\lim_{p\to 1}v=v_0+v_1+v_2+\cdots$$

3-Applications

Example 1. Consider the following nonlinear PDF:

$$u_t+uu_x=x^2+xt^2$$

With initial condition

$$u(x,0)=0$$

Assume the homotopy equation $H(v,p)=0$, governing the deformation process, we achieve

$$v_t+p[vv_x-x^2-xt^2]=0 \tag{7}$$

Given that the initial values are $u_0=0$,
Assume the solution is expressed through an infinite series expansion

$$v=\sum_{m=0}^{\infty}p^m v_m \tag{8}$$

The nonlinear operator is decomposed into the following contributions

$$vv_x=\sum_{m=0}^{\infty}p^m H_m \tag{9}$$

Where

$$H_m=N\left(\sum_{k=0}^{m}v_k\right)-N\left(\sum_{k=0}^{m-1}v_k\right),\qquad H_0=v_0v_{0x},\qquad m\geq 1$$

When Equations (8) and (9) are substituted into Equation (7), the resultant form becomes

$$\sum_{m=0}^{\infty} p^m v_{mt} = -p\left[\sum_{m=0}^{\infty} p^m H_m - x^2 - xt^2\right]$$

Subsequently, the expression becomes:

$$p^0: v_{0t} = 0, \qquad v_0 = 0,$$

$$H_0 = v_0 v_{0x} = 0$$

$$p^1: v_{1t} = -[H_0 - x^2 - xt^2] \Rightarrow v_1 = xt + x\frac{t^3}{3},$$

$$H_1 = (v_0 + v_1)(v_0 + v_1)_x - v_0 v_{0x} = xt^2 + 2x\frac{t^4}{3} + x\frac{t^6}{9}$$

$$p^2: v_{2t} = -[H_1] \Rightarrow v_{2t} = -\left[x\left(t + \frac{t^3}{3}\right)^2 - 0\right] \Rightarrow v_2 = -\frac{xt^3}{3} - 2x\frac{t^5}{15} - x\frac{t^7}{63}$$

$$H_2 = (v_0 + v_1 + v_2)(v_0 + v_1 + v_2)_x - (v_0 + v_1)(v_0 + v_1)_x$$

$$\vdots$$

Thus, the final approximate solution to the problem is expressed as

$$u(x,t) = \lim_{p\to 1} v = v_0 + v_1 + v_2 + \cdots$$

$$= xt$$

Example.2. Consider the following nonlinear PDE:

$$u_t = x^2 + \frac{1}{4}u_x^2 \tag{10}$$

With initial condition

$$u(x,0) = 0 \tag{11}$$

Define the homotopy operator $H(v,p)$, satisfying $H(v,p) = 0$, we get

$$v_t - p\left[x^2 + \frac{1}{4}v_x^2\right] = 0 \tag{12}$$

Given that the initial values are $u_0 = 0$,
Now, let's assume the solution is in the form of an infinite series

$$v = \sum_{m=0}^{\infty} p^m v_m \tag{13}$$

and assume the nonlinear part in the following form

$$v_x^2 = \sum_{m=0}^{\infty} p^m H_m \tag{14}$$

Where

$$H_m = N\left(\sum_{k=0}^{m} v_k\right) - N\left(\sum_{k=0}^{m-1} v_k\right), \qquad H_0 = v_{0x}^2$$

Replacing the terms in Equation (12) with Equations (13) and (14) leads to

$$\sum_{m=0}^{\infty} p^m v_{mt} = p\left[\frac{1}{4}\sum_{m=0}^{\infty} p^m H_m + x^2\right]$$

Identifying and equating terms with the same power of p provides

$$p^0: v_{0t} = 0, \qquad v_0 = 0,$$

$$H_0 = v_{0x}^2 = 0,$$

$$p^1: v_{1t} = [H_0 + x^2] \Rightarrow v_1 = x^2 t,$$

$$H_1 = (v_{0x} + v_{1x})^2 - v_{0x}^2 = 4x^2t^2,$$

$$p^2: v_{2t} = \frac{1}{4}\mathrm{H}_1 \Rightarrow v_{2t} = \frac{1}{4}[4x^2t^2] \Rightarrow v_2 = x^2\frac{t^3}{3},$$

$$H_2 = (v_{0x} + v_{1x} + v_{2x})^2 - (v_{0x} + v_{1x})^2 = 8x^2\frac{t^4}{3} + 4x^2\frac{t^6}{9},$$

$$p^3: v_{3t} = \frac{1}{4}\mathrm{H}_2 \Rightarrow v_{2t} = \frac{1}{4}\left[8x^2\frac{t^4}{3} + 4x^2\frac{t^6}{9}\right] \Rightarrow v_2 = x^2\frac{2t^5}{15} + x^2\frac{t^7}{63},$$

$$\vdots$$

Consequently, the resultant solution derived via HPM is

$$u(x,t) = \lim_{p\to 1} v = v_0 + v_1 + v_2 + \cdots$$

$$= x^2\left(t + \frac{t^3}{3} + \frac{2t^5}{15} + \frac{t^7}{63} + \cdots\right) = x^2\tanh(t)$$

Example 3. Consider the following nonlinear system PDE:

$$\begin{aligned} &u_t + wu_x + u = 1 \\ &w_t - uw_x - w = 1, \end{aligned} \tag{15}$$

Subject to the initial conditions

$$\begin{aligned} &u(x,0) = e^x \\ &w(x,0) = e^{-x} \end{aligned} \tag{16}$$

Let $H(v,p) = 0$ represent the homotopy function in the HPM framework, we get

$$H\ (v,p) = v_t + p[mv_x + v - 1]$$

$$H(m,p) = m_t + p[-vm_x - m - 1]$$

Under the HPM framework, we propose the solution as an infinite series representation

$$v = \sum_{m=1}^{\infty} p^n v_n, \qquad m = \sum_{n=0}^{\infty} p^n m_n$$

And nonlinear terms are analytically decomposed into the following structure

$$vm_x = \sum_{n=0}^{\infty} p^n H_m, \qquad mv_x = \sum_{n=0}^{\infty} p^n K_n.$$

then

$$\sum_{n=0}^{\infty} p^n V_{nt} = -p\left[\sum_{n=0}^{\infty} p^n K_n + \sum_{n=0}^{\infty} p^n v_n - 1\right]$$

$$\sum_{n=0}^{\infty} p^n m_{nt} = -p\left[-\sum_{n=0}^{\infty} p^n H_n - \sum_{n=0}^{\infty} p^n m_n - 1\right]$$

By equating the coefficients of corresponding powers of pp in the homotopy series, the following system of equations is derived

$$\begin{array}{ll} p^0: v_{0t} = 0, \quad v(x,0) = e^x & p^0: m_{0t} = 0, \quad m(x,0) = e^{-x} \\ p^1: V_{1t} = -[k_0 + v_0 - 1], & p^1: m_1 = -[H_0 - m_0 - 1] \\ p^2: V_{2t} = -[k_1 + v_1], & p^2: m_{2t} = -[H_1 - m_1] \\ p^3: V_{3t} = -[k_2 + v_2], & p^3: m_{3t} = -[H_2 - m_2] \end{array} \tag{17}$$

$$\vdots$$

where

$$K_0 = m_0 v_{0x}, \qquad H_0 = v_0 m_{0x}$$

$$K_1 = (m_0 + m_1)(v_{0x} + v_{1x}) - m_0 v_{0x}, \qquad H_1 = (v_0 + v_1)(m_{0x} + m_{1x}) - v_0 m_{0x}$$

$$K_2 = (m_0 + m_1 + m_2)(v_{0x} + v_{1x} + v_{2x}) - (m_0 + m_1)(v_{0x} + v_{1x}),$$

$$H_2 = (v_0 + v_1 + v_2)(m_{0x} + m_{1x} + m_{2\,x}) - (v_0 + v_1)(m_{0x} + m_{1x})$$

$$\vdots$$

Solving system (17) we obtain

$$\begin{array}{ll} p^0: v_0 = e^x, & m_0 = e^{-x} \\ p^1: v_1 = -\mathrm{t}e^x, & m_1 = \mathrm{t}e^{-x}, \\ p^2: V_{2t} = \dfrac{\mathrm{t}^2 e^x}{2} - \dfrac{t^3}{3}, & m_2 = \dfrac{\mathrm{t}^2 e^x}{2} - \dfrac{t^3}{3} \end{array} \tag{18}$$

$$\vdots$$

Accordingly, the solution constructed through HPM becomes

$$u(x,t) = e^x\left(1 - t + \frac{\mathrm{t}^2}{2} - \frac{\mathrm{t}^3}{3!} + \cdots\right) = e^{x-t}.$$

$$u(x,t) = e^{-x}\left(1 + t + \frac{\mathrm{t}^2}{2} + \frac{\mathrm{t}^3}{3!} + \cdots\right) = e^{t-x}.$$

4-Conclusion

The Homotopy Perturbation Method (HPM) has been revisited and enhanced in this study to address nonlinear partial differential equations (PDEs). A new assumption for managing the nonlinear terms is proposed, distinguished by its simplicity and ease of implementation when compared to the traditional Adomian decomposition. This refinement significantly improves both the efficiency and practical applicability of HPM in dealing with a wide spectrum of complex mathematical models.

The results obtained using this new assumption reveal a high level of accuracy and demonstrate the method's capability in solving various forms of nonlinear equations, thereby reinforcing HPM as a powerful tool for analyzing nonlinear systems. Moreover, the proposed approach shows strong potential for broader application in engineering, physics, and applied sciences, particularly in models involving intricate nonlinear behaviors.

This study is expected to contribute to the wider adoption of HPM by making it more flexible, accessible, and computationally efficient for solving nonlinear PDEs, while maintaining precision and reliability.

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