Journal of Education for Pure Science- University of Thi-Qar
Vol.15, No. 2 (2025)
DOI: https://doi.org/10.32792/jeps.v15i2.687

Website: jceps.utq.edu.iq Email: jceps@eps.utq.edu.iq

Study The General Boolean Sum (GBS) of New Modification For Baskakov-Bate Operators

Ahmed F. Jabber [1,], Amal K. Hassan[2,]

[1,2] Department of Mathematics , College of Scenic , University of Basra , Basra, Iraq.

[*] Corresponding email: Pgs.ahmed.fadhil@uobasrah.edu.iq

Received 11 / 3 /2025, Accepted 5 / 5 /2025, Published 1/ 6 /2025

This work is licensed under a Creative Commons Attribution 4.0 International License.

Abstract:

In this paper, we construct a new sequence of bivariate Basakov – Bate operators based on parameter S . We discuss the rate of convergence of these operators using the partial and total moduli of continuity. A Voronovskaya-type asymptotic result is also established. Further, we introduce the associated GBS (Generalized Boolean Sum) operators and estimate the degree of approximation using the mixed modulus of continuity and a class of the Lipschitz of Bögel type continuous functions .

Keywords: Bivariate Baskakov operators, Voronovskaya theorem, Modulud of continuity, Degree of approximation.

1-Introduction

Baskakov [3] studied a sequence of positive linear operators, called Baskakov operators, for suitable functions defined on an unbounded interval $[0,\infty)$.Gurdek [5] introduced the bivariate Baskakov operators in weighted spaces as follows :

$$U_{n,m}(f;x,y)=\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\mathrm{V}_{n,k}(x)\mathrm{V}_{m,j}(y)\ \ g\left(\frac{k}{n},\frac{j}{m}\right), \tag{1.1}$$

where $\mathrm{V}_{n,k}(x)=\binom{n+k-1}{k}x^{k}(1+x)^{-n-k}$,$\mathrm{V}_{m,j}(y)=\binom{m+j-1}{j}y^{j}(1+y)^{-m-j}$.

A non-negative real parametric generalization of the above sequences was proposed by Aral and Erbay [1] and is know as the $\alpha-$ Baskakov operator . Sonker and Priyanka [8] proposed the stance variant of a bivariate parametrically generalized Baskakov operators operator given by

$$Q_{n_1,n_2,\alpha_1,\alpha_2}^{\alpha_1,\alpha_2,\beta_1,\beta_2}(f;x,y)=\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}V_{n_1,n_2,k_1,k_2}^{(\alpha_1,\alpha_2)}(x,y)\ \ f\left(\frac{k_1+\alpha_1}{n_1+\beta_1},\frac{k_2+\alpha_2}{n_2+\beta_2}\right), \tag{1.2}$$

where $V^{(\alpha_1,\alpha_2)}_{n_1,n_2,k_1,k_2}(x,y) = V^{\alpha_1}_{n_1,k_1}(x)V^{\alpha_2}_{n_2,k_2}(y)$, such that $V^{\alpha_1}_{n_1,k_1}(x)$ defined in [1]. A new sequence of Generalized Bivariate Baskakov Durrmeyer Operators and Associated GBS Operators was designed by Rani and Rao [7]. A new modified form of Baskakov-Bate operators based on parameters S was presented by jabbar and Hassan [6] in a recent study.

$$L_n(f;x) = \sum_{k=0}^{\infty} \mathrm{P}_{n,k}(x) \int_0^{\infty} B_{n,k}(t)\, f\left(x + \frac{(t-x)}{n^S}\right) dt\,, \tag{1.3}$$

where
$$\mathrm{P}_{n,k}(x) = A(x,n)\mathrm{V}_{n+1,k}(x) + A(-1-x,n)\mathrm{V}_{n+1,k-1}(x)\,,$$
$$A(x,n) = \mu(n) + \varrho(n)x \text{ and } B_{n,k}(t) = \frac{(n+k)!\ t^k}{k!(n-1)!\ (1+t)^{n+k+1}}.$$

2. Construction of bivariate operators $H_{n,m}(f;x,y)$

For $u,v > 0$ and $f \in C_{u,v}(I^2) = \{f \in C(I^2): |f(x,y)| \le M(1+x^u)(1+y^v), \forall (x,y) \in I^2\}$, where $I^2 = [0,\infty) \times [0,\infty)$ and M is positive constant drpendent on f, we introduce a new bivariate Baskakov – Bate operators as

$$H_{n,m}(f;x,y) = \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} \mathrm{P}_{n,k}(x)\mathrm{P}_{m,j}(y) \times \int_0^{\infty}\int_0^{\infty} B_{n,k}(t)\, B_{m,j}(z)\, f\left(x + \frac{(t-x)}{n^S}, y + \frac{(z-y)}{m^S}\right) dt\, dz\,, \tag{2.1}$$

where
$$\mathrm{P}_{n,k}(x) = A(x,n)\mathrm{V}_{n+1,k}(x) + A(-1-x,n)\mathrm{V}_{n+1,k-1}(x)\,,$$
$$\mathrm{P}_{m,j}(y) = T(y,m)\mathrm{V}_{m+1,j}(y) + T(-1-y,m)\mathrm{V}_{m+1,j-1}(y)\,,$$
$$B_{n,k}(t) = \frac{(n+k)!\ t^k}{k!(n-1)!\ (1+t)^{n+k+1}}\,, B_{m,j}(z) = \frac{(m+j)!\ z^j}{j!(m-1)!\ (1+z)^{m+j+1}}\,,$$

and $A(x,n) = \mu(n) + \varrho(n)x$, $T(y,m) = \mu(m) + \varrho(m)y$. Hear $\mu(n), \varrho(n)$, $\mu(m)$ and $\varrho(m)$, are unknow sequences which are to be determined by imposing suitable convergence conditions . For $\mu(n) = \varrho(n) = \mu(m) = \varrho(m) = 1$ and $S = 0$ the operator (2.1) reduces to the ordinary Baskakov – Bate bivariate operator .

Lemma 2.1 : Let $e_{r_1 r_2} = t^{r_1} z^{r_2}$, $r_1, r_2 \in N^0$, then the operator $H_{n,m}(f;x,y)$ satisfy the following identities

(1) $H_{n,m}(e_{00};x,y) = \big(2\,\mu(n) - \varrho(n)\big)\big(2\mu(m) - \varrho(m)\big)$,

(2) $H_{n,m}(e_{10};x,y) = \left[\big(2\,\mu(n) - \varrho(n)\big)x + \frac{\big(4\,\mu(n) - 3\varrho(n)\big)x + \big(3\,\mu(n) - 2\varrho(n)\big)}{n^S(n-1)}\right][2\mu(m) - \varrho(m)]$,

(3) $H_{n,m}(e_{01};x,y) = \left[\big(2\,\mu(m) - \varrho(m)\big)y + \frac{\big(4\,\mu(m) - 3\varrho(m)\big)y + \big(3\,\mu(m) - 2\varrho(m)\big)}{m^S(m-1)}\right][2\mu(n) - \varrho(n)]$,

(4) $H_{n,m}(e_{11};x,y)$
$$= \left[\big(2\,\mu(n) - \varrho(n)\big)x + \frac{\big(4\,\mu(n) - 3\varrho(n)\big)x + \big(3\,\mu(n) - 2\varrho(n)\big)}{n^S(n-1)}\right]\left[\big(2\,\mu(m) - \varrho(m)\big)y + \frac{\big(4\,\mu(m) - 3\varrho(m)\big)y + \big(3\,\mu(m) - 2\varrho(m)\big)}{m^S(m-1)}\right],$$

(5) $H_{n,m}(e_{20};x,y)$

$$= \Big[\big(2\mu(n) - \varrho(n)\big)x^2 + \frac{[(8x^2+6x)\mu(n) - (6x^2+4x)\varrho(n)]}{n^{S-1}(n-1)(n-2)} + \frac{[(4x^2+4x)\mu(n) - (2x^2+2x)\varrho(n)]}{n^{2S-1}(n-1)(n-2)} - \frac{[(12x+16x^2)\mu(n) - (8x+12x^2)\varrho(n)]}{n^{S}(n-1)(n-2)} + \frac{[(8+22x+16x^2)\mu(n) - (6+18x+14x^2)\varrho(n)]}{n^{2S}(n-1)(n-2)} \Big] [2\mu(m) - \varrho(m)],$$

(6) $H_{n,m}(e_{02};x,y)$

$$= \Big[\big(2\mu(m) - \varrho(m)\big)y^2 + \frac{[(8y^2+6y)\mu(m) - (6y^2+4y)\varrho(m)]}{m^{S-1}(m-1)(m-2)} + \frac{[(4y^2+4y)\mu(m) - (2y^2+2y)\varrho(m)]}{m^{2S-1}(m-1)(m-2)} - \frac{[(12y+16y^2)\mu(m) - (8y+12y^2)\varrho(m)]}{m^{S}(m-1)(m-2)} + \frac{[(8+22y+16y^2)\mu(m) - (6+18y+14y^2)\varrho(m)]}{m^{2S}(m-1)(m-2)} \Big] [2\mu(n) - \varrho(n)].$$

Proof: It is simple to get the desired result from (2.1). ■

In order to study the uniformly convergence , we consider that sequences $\mu(n)$, $\varrho(n)$, $\mu(m)$ and $\varrho(m)$ Verify the conditions

$$2\mu(n) - \varrho(n) = 1 \quad \text{and} \quad 2\mu(n) - \varrho(n) = 1 \ . \tag{2.2}$$

For the sequences, we will examine two cases.

Case 1 : Let

$$\mu(n) - \varrho(n) \geq 0 \ , \ \mu(n) \geq 0 \quad \text{and} \quad \mu(m) - \varrho(m) \geq 0 \ , \ \mu(m) \geq 0 \tag{2.3}$$

Using (2.2) ,we get $0 \leq \mu(n) \leq 1$, $-1 \leq \varrho(n) \leq 1$ and $0 \leq \mu(m) \leq 1$, $-1 \leq \varrho(m) \leq 1$. The operator (2.2) is positive in the present case.

Case 2 : Let

$$\mu(n) - \varrho(n) < 0 \ \text{or} \ \mu(n) < 0 \ , \mu(m) - \varrho(m) < 0 \ \text{or} \ \mu(m) < 0 \ . \tag{2.4}$$

If $\mu(n) - \varrho(n) < 0$, $\mu(m) - \varrho(m) < 0$ then $\mu(n) > 1$, $\mu(m) > 1$, if $\mu(n) < 0$, $\mu(m) < 0$ then $\mu(n) - \varrho(n) > 1$, $\mu(n) - \varrho(n) > 1$. In this case , the operator (2.1) is non positive .

In the next theorem, we prove the $H_{n,m}(f;x,y)$ operator's convergence uniformly to the function $f(x,y)$ using the Korovkin–type theorem.

Theorem 2.2: Let $\mu(n)$, $\varrho(n)$ and $\mu(m)$, $\varrho(m)$ be convergence sequences which verify the conditions (2.2) and (2.3) . If $f \in C_{u,v}(I^2)$, then $H_{n,m}(f;x,y)$ converges uniformly to $f(x,y)$ as $n, m \to \infty$ on $I_{hk} = [0,h] \times [0,k]$ be compact sub set of I^2.

Proof : From Lemma (2.1) , we get

$$\lim_{n,m\to\infty} H_{n,m}\big(e_{r_1 r_2};x,y\big) = e_{r_1 r_2}(x,y) \quad \text{for } r_1, r_2 \in \{(0,0), (0,1), (1,0)\}$$

and

$$\lim_{n,m\to\infty} H_{n,m}(e_{20}+e_{02};x,y)=(e_{20}+e_{02})(x,y) \text{ uniformly on } I_{hk}.$$

By applying Theorem (2.1) in [2] , we get the required . ■

Definition 2.3 . For $(\lambda,\ \gamma)\in N^0\times N^0$, we define the $(\lambda,\gamma)-$ order moment for the sequence $H_{n,m}(f;x,y)$ by

$$\mathcal{F}_{n,m,\lambda,\gamma}(x,y)=H_{n,m}\big((t-x)^{\lambda}(z-y)^{\gamma};x,y\big). \tag{2.5}$$

Lemma 2.4 . For $(\lambda,\gamma)\in N^0\times N^0$, the following properties satisfied

(**1**) $\mathcal{F}_{n,m,1,0}(x,y)=\dfrac{\mu(n)-(1+x)\varrho(n)}{n^{S}(n-1)}$,

(**2**) $\mathcal{F}_{n,m,0,1}(x,y)=\dfrac{\mu(m)-(1+y)\varrho(m)}{m^{S}(m-1)}$,

(**3**) $\mathcal{F}_{n,m,1,1}(x,y)=\left[\dfrac{\mu(n)-(1+x)\varrho(n)}{n^{S}(n-1)}\right]\left[\dfrac{\mu(m)-(1+y)\varrho(m)}{m^{S}(m-1)}\right]$,

(**4**) $$\mathcal{F}_{n,m,0,2}(x,y)=\left[\frac{[(4x^2+4x)\mu(n)-(2x^2+2x)\varrho(n)]}{n^{2S-1}(n-1)(n-2)}+\frac{[(8+22x+16x^2)\mu(n)-(6+18x+14x^2)\varrho(n)]}{n^{2S}(n-1)(n-2)}\right],$$

(**5**) $$\mathcal{F}_{n,m,0,2}(x,y)=\left[\frac{[(4y^2+4y)\mu(m)-(2y^2+2y)\varrho(m)]}{m^{2S-1}(m-1)(m-2)}+\frac{[(8+22y+16y^2)\mu(m)-(6+18y+14y^2)\varrho(m)]}{m^{2S}(m-1)(m-2)}\right],$$

(**6**) $\mathcal{F}_{n,m,4,0}(x,y)=O\left(\dfrac{1}{n^{2S+2}}\right)$,

(**7**) $\mathcal{F}_{n,m,0,4}(x,y)=O\left(\frac{1}{m^{2S+2}}\right)$.

Lemma 2.5 : The operator $H_{n,n}(f;x,y)$ verifies following identies

(**1**) $\lim\limits_{n\to\infty} n^{S+1}\mathcal{F}_{n,n,1,0}(x,y)=L_1-(1+x)L_2$,

(**2**) $\lim\limits_{n\to\infty} n^{S+1}\mathcal{F}_{n,n,0,1}(x,y)=L_3-(1+y)L_4$,

(**3**) $\lim\limits_{n\to\infty} n^{S+1}\mathcal{F}_{n,n,1,1}(x,y)=[L_1-(1+x)L_2]\,[L_3-(1+y)L_4]$,

(4) $\lim_{n\to\infty} n^{S+1}\mathcal{F}_{n,n,2,0}(x,y) = n^{s+1}\lim_{n\to\infty}\mathcal{F}_{n,n,0,2}(x,y) = 0$,

where $\lim_{n\to\infty}\mu(n) = L_1$, $\lim_{n\to\infty}\varrho(n) = L_2$, $\lim_{n\to\infty}\mu(m) = L_3$ and $\lim_{n\to\infty}\varrho(m) = L_4$.

In the following result, we establish a voronovskaya type asymptotic theorem .

Theorem 2.6: Let $f'' \in C_{u,v}(I^2)$ and the conditions (2.2) , (2.3) are satisfies . Then , we have

$$\begin{aligned}\lim_{n\to\infty} n^{S+1}\left(H_{n,n}(f;x,y) - f(x)\right) &= (L_1-(1+x)L_2)\, f_x(x,y) + (L_3-(1+y)L_4)\, f_y(x,y)\\ &+ ([L_1-(1+x)L_2]\,[L_3-(1+y)L_4])\, f_{xy}(x,y)\end{aligned}$$

Proof :For $f'' \in C_{u,v}(I^2)$, by the Taylor formula , we get

$$f(t,z) = f(x,y) + (t-x)f_x(x,y) + (z-y)f_y(x,y) + \frac{1}{2}\Big((t-x)^2 f_{xx}(x,y) + 2(t-x)(z-y)f_{xy}(x,y) + (z-y)^2 f_{yy}(x,y)\Big) + \eta(t,z;x,y)\sqrt{(t-x)^4+(z-y)^4}\ .$$

where $\eta(t,z;x,y) \longrightarrow 0$ as $(t,z) \longrightarrow (x,y)$.

Using $H_{n,n}(.\,;x,y)$ on both sides , we obtain

$$\begin{aligned}H_{n,n}(f;x,y) - f(x,y) &= f_x(x,y)\, H_{n,n}\big((t-x);x,y\big) + f_y(x,y)H_{n,n}\big((z-y);x,y\big)\\ &+ \frac{1}{2}f_{xx}(x,y)H_{n,n}\,((t-x)^2;x,y) + f_{xy}(x,y)H_{n,n}\big((t-x)(z-y);x,y\big)\\ &+\frac{1}{2}f_{yy}(x,y)H_{n,n}\,((s-y)^2;x,y) + H_{n,n}\left(\eta(t,z;x,y)\sqrt{(t-x)^4+(z-y)^4};x,y\right).\end{aligned}$$

Hence, using Lemma (2.5) , we get

$$\begin{aligned}\lim_{n\to\infty} n^{S+1}\left(H_{n,n}(f;x,y) - f(x)\right) &= (L_1-(1+x)L_2)\, f_x(x,y) + (L_3-(1+y)L_4)\, f_y(x,y)\\ &+ ([L_1-(1+x)L_2]\,[L_3-(1+y)L_4]) f_{xy}(x,y)\\ &+ \lim_{n\to\infty} n^{S+1}H_{n,n}\left(\eta(t,z;x,y)\sqrt{(t-x)^4+(z-y)^4};x,y\right)\end{aligned} \quad (2.6)$$

Now , by applying Cauchy – Schwartz inequality to the least term of (2.6), we have

$$\begin{aligned}\left|H_{n,n}\left(\eta(t,z;x,y)\sqrt{(t-x)^4+(z-y)^4};x,y\right)\right| &\le \left(H_{n,n}(\eta^2(t,z;x,y);x,y)\right)^{\frac{1}{2}}\left(H_{n,n}\,((t-x)^4+(z-y)^4;x,y)\right)^{\frac{1}{2}}\\ &= \left(H_{n,n}(\eta^2(t,z;x,y);x,y)\right)^{\frac{1}{2}}\left(H_{n,n}((t-x)^4;x,y) + H_{n,n}((z-y)^4;x,y)\right)^{\frac{1}{2}}.\end{aligned}$$

From Theorem (2.2) $H_{n,n}(\eta^2(t,z;x,y);x,y) \longrightarrow 0$ as $n \longrightarrow \infty$ and from Lemma (2.4) , we have

$$H_{n,n}((t-x)^4;x,y)=O\left(\frac{1}{n^{2S+2}}\right),\ H_{n,n}((z-y)^4;x,y)=O\left(\frac{1}{n^{2S+2}}\right).$$

Therefore ,

$$\lim_{n\to\infty} n^{S+1}H_{n,n}\left(\eta(t,z;x,y)\sqrt{(t-x)^4+(z-y)^4};x,y\right)=0\,. \tag{2.7}$$

Combining (2.6) and (2.7) yields the desired result. ■

3.Degree of approximation

Let $C_W(I^2)$ be the space of all uniformly continuous functions on I^2 equipped with the norm

$$\|f\|=\sup\ |f(x,y)|\ ,\ \ (x,y)\in I^2.$$

The complete modulus of continuity for the bivariate situation is defined as follows for $f\in C_W(I^2)$ and $\delta_1,\delta_2>0$:

$$\omega(f;\delta_1,\delta_2)=\sup\{|f(t,z)-f(x,y)|:(t,z),(x,y)\in I^2 \text{ and } |t-x|\le\delta_1\,,|z-y|\le\delta_2\,\} \tag{3.1}$$

Further, the partial moduli of continuity with respect to x and y is defined by

$$\omega_1(f;\delta_1)=\sup\{|f(t,z)-f(x,z)|:\ z\in I \text{ and } |t-x|\le\delta_1\,\} \tag{3.2}$$

$$\omega_2(f;\delta_2)=\sup\{|f(t,z)-f(t,y)|:\ t\in I \text{ and } |z-y|\le\delta_2\,\} \tag{3.3}$$

The following inequalities hold for the aforementioned smoothness moduli .

(i) $|f(t,z)-f(x,y)|\le\omega\left(f;\ \sqrt{(t-x)^2+(z-y)^2}\right)$,
(ii) $|f(t,z)-f(x,z)|\le\omega_1(f;|t-x|)$,
(iii) $|f(t,z)-f(t,y)|\le\omega_2(f;|z-y|)$.

Theorem 3.1 : For $f\in C_W(I^2)$ and $(x,y)\in I^2$, we arrive

$$\left|H_{n,m}(f;x,y)-f(x,y)\right|\le 2\big(\omega(f;\delta\,)\big) \tag{3.4}$$

Proof . From the definition of the complete modulus of continuity for $f\in C_W(I^2)$, we may write

$$\left|H_{n,m}(f;x,y)-f(x,y)\right|\le H_{n,m}(|f(t,z)-f(x,y)|;x,y)$$

$$\le H_{n,m}\left(\omega\left(f;\ \sqrt{(t-x)^2+(z-y)^2}\right);x,y\right)$$

$$\le\omega(f;\delta\,)\left[1+\frac{1}{\delta}H_{n,m}\left(\sqrt{(t-x)^2+(z-y)^2};x,y\right)\right].$$

For any $\delta>0$, we have

$$|H_{n,m}(f;x,y)-f(x,y)|\le\omega(f;\delta)\left[1+\frac{1}{\delta}\left(H_{n,m}((t-x)^2+(z-y)^2;x,y)\right)^{\frac{1}{2}}\right]$$

Now , choosing $\delta=\sqrt{\mathcal{F}_{n,m,2,0}(x,y)+\mathcal{F}_{n,m,0,2}(x,y)}$, we get the desired result . ■

Theorem 3.2 : For $f\in C_W(I^2)$, then for all $(x,y)\in I^2$, we have

$$|H_{n,m}(f;x,y)-f(x,y)|\le 2\big(\omega_1(f;\delta_1)+\omega_2(f;\delta_2)\big). \tag{3.5}$$

Proof . For $f\in C_W(I^2)$, we get

$$\begin{aligned}|H_{n,m}(f;x,y)-f(x,y)|&\le H_{n,m}(|f(t,z)-f(x,y)|;x,y)\\&\le H_{n,m}(|f(t,z)-f(x,z)|;x,y)+H_{n,m}(|f(x,z)-f(x,y)|;x,y)\\&\le\omega_1(f;\delta_1)[1+\frac{1}{\delta_1}H_{n,m}(|t-x|;x,y)]\;+\omega_2(f;\delta_2)[1+\frac{1}{\delta_2}H_{n,m}(|z-y|;x,y)].\end{aligned}$$

Utilizing the Cauchy – Schwartz inequality for any $\delta_1,\delta_2>0$, we get

$$\begin{aligned}&|H_{n,m}(f;x,y)-f(x,y)|\\&\le\omega_1(f;\delta_1)\left[1+\frac{1}{\delta_1}\sqrt{H_{n,m}((t-x)^2;x,y)}\right]+\omega_2(f;\delta_2)\left[1+\frac{1}{\delta_2}\sqrt{H_{n,m}((z-y)^2;x,y)}\right].\end{aligned}$$

Now , taking $\delta_1=\sqrt{\mathcal{F}_{n,m,2,0}(x,y)}$ and $\delta_2=\sqrt{\mathcal{F}_{n,m,0,2}(x,y)}$, we obtain the proof of this theorem .
■

Theorem 3.3 : Let $f\in C_W^1(I^2)$. Then ,we have

$$|H_{n,m}(f;x,y)-f(x,y)|\le\left\|\frac{\partial f(x,y)}{\partial x}\right\|\sqrt{\mathcal{F}_{n,m,2,0}(x,y)}+\left\|\frac{\partial f(x,y)}{\partial y}\right\|\sqrt{\mathcal{F}_{n,m,0,2}(x,y)}.$$

Proof . By Taylors formula we can write

$$f(t,z)-f(x,y)=\int_x^t\frac{\partial f(\tilde{a},y)}{\partial\tilde{a}}\,d\tilde{a}+\int_y^z\frac{\partial f(x,\tilde{u})}{\partial\tilde{u}}\,d\tilde{u}. \tag{3.6}$$

Now applying $H_{n,m}(.;x,y)$ on both sides (3.6) ,we are lead to

$$|H_{n,m}(f;x,y)-f(x,y)|\le H_{n,m}\left(\left|\int_x^t\frac{\partial f(\tilde{a},y)}{\partial\tilde{a}}\,d\tilde{a}\right|;x,y\right)+H_{n,m}\left(\left|\int_y^z\frac{\partial f(x,\tilde{u})}{\partial\tilde{u}}\,d\tilde{u}\right|;x,y\right).$$

Since

$$\left|\int_x^t\frac{\partial f(\tilde{a},y)}{\partial\tilde{a}}\,d\tilde{a}\right|\le\int_x^t\left|\frac{\partial f(\tilde{a},y)}{\partial\tilde{a}}\,d\tilde{a}\right|\le\left\|\frac{\partial f(x,y)}{\partial x}\right\||t-x|$$

and

$$\left|\int_{y}^{z} \frac{\partial f\ (x,\tilde{u})}{\partial \tilde{u}}\ d\tilde{u}\right| \leq \int_{y}^{z} \left|\frac{\partial f\ (x,\tilde{u})}{\partial \tilde{u}}\ d\tilde{u}\right| \leq \left\|\frac{\partial\ f(x,y)}{\partial y}\right\| \ |z-y|\ .$$

We obtain

$$\left|H_{n,m}(f;x,y) - f(x,y)\right| \leq \left\|\frac{\partial\ f(x,y)}{\partial x}\right\| H_{n,m}(|t-x|;x,y) + \left\|\frac{\partial\ f(x,y)}{\partial y}\right\| H_{n,m}(|z-y|;x,y)\ .$$

Now, applying the Cauchy - Schwarz inequality, we get

$$\left|H_{n,m}(f;x,y) - f(x,y)\right| \leq \left\|\frac{\partial\ f(x,y)}{\partial x}\right\| \sqrt{H_{n,m}((t-x)^2;x,y)} + \left\|\frac{\partial\ f(x,y)}{\partial y}\right\| \sqrt{H_{n,m}((z-y)^2;x,y)}$$

This completes the proof . ■

4.Some approximation results in Bögel space

Using Bögel continuous functions, we present the GBS case of the operators $H_{n,m}(f;x,y)$ in this section. Bögel was the first to use the concept of the B-continuous and B-bounded functions [4].Let X and Y be a compact real intervals and $\Delta_{(t,z)}f(x,y)$ be mixed difference operators of f defined by $\Delta_{(t,z)}f(x,y) = f(t,z) - f(t,y) - f(x,z) + f(x,y)$ for $(t,z),(x,y) \in X \times Y$.

A function $f: X \times Y \to \mathbb{R}$ is called B - continuous (Bögel continuous) at $(x,y) \in X \times Y$, if

$$\lim_{(t,z)\to(x,y)} \Delta_{(t,z)}f(x,y) = 0 \tag{4.1}$$

A function $f: X \times Y \to \mathbb{R}$ is called B-differentiable (Bögel differentiable) at $(x,y) \in X \times Y$, if the following limit exists and is finite

$$\lim_{(t,z)\to(x,y)} \frac{\Delta_{(t,z)}f(x,y)}{(t-x)(z-y)} \tag{4.2}$$

Note that by $C_b(X \times Y)$ and $D_b(X \times Y)$,we denote the spaces of all B-continuous and B-differentiable functions on $X \times Y$, respectively .

A function $f: X \times Y \to \mathbb{R}$ is called B-bounded (Bögel bounded) in $X \times Y$, if there exists $M > 0$ such that $\left|\Delta_{(t,z)}f(x,y)\right| < M$ for all $(t,z),(x,y) \in X \times Y$.
Let $B_b(X \times Y)$ be the space of all B-bounded on $X \times Y$, with the norm

$$\|f\|_B = \sup_{(t,z),(x,y)\ \in X \times Y} \left|\Delta_{(t,z)}f(x,y)\right|$$

For $f \in B_b(I_{hk})$,the mixed modulus of smoothness is defined as

$$\omega_{mixed}(f;\delta_1,\delta_2) = \sup\{|\Delta_{(t,z)}f(x,y)| : \quad |t-x| \le \delta_1, |z-y| \le \delta_2\}. \tag{4.3}$$

For all $(t,z),(x,y) \in X \times Y$ and $\delta_1,\delta_2 > 0$.Also, for any $\varphi_1,\varphi_2 > 0$,the following inequality holds

$$\omega_{mixed}(f;\ \varphi_1\delta_1,\ \varphi_2\delta_2) \le (1+\ \varphi_1)(1+\ \varphi_2)\omega_{mixed}(f;\delta_1,\delta_2) \tag{4.4}$$

Let $C_b(I_{hk})$ denote the set of all B - continuous functions on I_{hk}.

Now, we construct GBS operators for $f \in C_b(I_{hk})$ associated with the operators given by equation (2.1) as follows

$$\mathbb{G}_{n,m}(f;x,y) = H_{n,m}(f(t,y)+f(x,z)-f(t,z);x,y)$$

$$= \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} \mathrm{P}_{n,k}(x)\mathrm{P}_{m,j}(y)$$
$$\times \int_0^{\infty}\int_0^{\infty} B_{n,k}(t)B_{m,j}(z)\left[f\left(x+\frac{(t-x)}{n^S},y\right)+f\left(x,y+\frac{(z-y)}{m^S}\right)\right.$$
$$\left.-f\left(x+\frac{(t-x)}{n^S},y+\frac{(z-y)}{m^S}\right)\right]\ dt\,dz. \tag{4.5}$$

Theorem 4.1 . For every $f \in C_b(I_{hk})$ and for each $(x,y) \in I_{hk}$,there holds the following inequality

$$|\mathbb{G}_{n,m}(f;x,y)-f(x,y)| \le 4\omega_{mixed}\left(f;\sqrt{H_{n,m}((t-x)^2;x,y)},\sqrt{H_{n,m}((z-y)^2;x,y)}\right)$$

Proof. Using the definition of $\omega_{mixed}(f;\delta_1,\delta_2)$ and the property (4.4), we have

$$|\Delta_{(t,z)}f(x,y)| \le \omega_{mixed}(f;|t-x|,|z-y|)$$
$$\le \left(1+\frac{|t-x|}{\delta_1}\right)\left(1+\frac{|z-y|}{\delta_2}\right)\omega_{mixed}(f;\delta_1,\delta_2). \tag{4.6}$$

For all $(t,z),(x,y) \in I_{hk}$ and $\delta_1,\delta_2 > 0$. From the definition of $\Delta_{(t,s)}f(x,y)$, we have

$$f(t,y)+f(x,z)-f(t,z) = f(x,y)-\Delta_{(t,z)}f(x,y). \tag{4.7}$$

Applying the operator $H_{n,m}(f;x,y)$ on both sides equation (4.7) and using Cauchy-Schwarz inequality , we obtain

$$|\mathbb{G}_{n,m}(f;x,y)-f(x,y)|$$
$$\le \omega_{mixed}(f;\delta_1,\delta_2)\left\{1+\frac{1}{\delta_1}\sqrt{H_{n,m}((t-x)^2;x,y)}+\frac{1}{\delta_2}\sqrt{H_{n,m}((z-y)^2;x,y)}\right.$$
$$\left.+\frac{1}{\delta_1\delta_2}\sqrt{H_{n,m}((t-x)^2;x,y)}\sqrt{H_{n,m}((z-y)^2;x,y)}\right\}.$$

Now, taking $\delta_1 = \sqrt{\mathcal{F}_{n,m,2,0}\ (x,y)}$ and $\delta_2 = \sqrt{\mathcal{F}_{n,m,0,2}\ (x,y)}$, leads us to the required result . ■

The Lipschitz class of B-continuous function : For $c_1, c_2 \in (0,1]$ and $f \in C_b(I_{hk})$, the Lipschitz class $Lip_M(c_1, c_2)$ of f is defined as follows

$$Lip_M(c_1, c_2) = [\, f \in C_b(I_{hk}) : \left|\Delta_{(t,z)} f(x,y)\right| \le M|t-x|^{c_1}|z-y|^{c_2}]$$

For all $(t,z),(x,y) \in I_{hk}$ and M is positive constant.

Theorem 4.2 . Let $f \in Lip_M(c_1, c_2)$ and $c_1, c_2 \in (0,1]$.Then for any $(x,y) \in I_{hk}$, we have

$$\left|\, \mathbb{G}_{n,m}(f;x,y) - f(x,y)\right| \le M \left(H_{n,m}((t-x)^2;x,y)\right)^{\frac{c_1}{2}} \left(H_{n,m}((z-y)^2;x,y)\right)^{\frac{c_2}{2}}$$

Proof . For $f \in Lip_M(c_1, c_2)$, we may write

$$\begin{aligned}\left|\, \mathbb{G}_{n,m}(f;x,y) - f(x,y)\right| &\le H_{n,m}(\left|\Delta_{(t,z)} f(x,y)\right|;x,y)\\ &\le M\, H_{n,m}(|t-x|^{c_1}|z-y|^{c_2};x,y)\\ &\le M\, H_{n,m}(|t-x|^{c_1};x,y)\, H_{n,m}(|z-y|^{c_2};x,y)\end{aligned}$$

In the view of Lemma (2.1) and using Holder's inequality for $(p_1, q_1) = \left(\frac{2}{c_1}, \frac{2}{2-c_1}\right)$ and $(p_2, q_2) = \left(\frac{2}{c_2}, \frac{2}{2-c_2}\right)$,we get

$$\begin{aligned}\left|\, \mathbb{G}_{n,m}(f;x,y) - f(x,y)\right| &\le M \left(H_{n,m}((t-x)^2;x,y)\right)^{\frac{c_1}{2}} \left(H_{n,m}(\, e_{00};x,y)\right)^{\frac{2-c_1}{2}}\\ &\quad \times \left(H_{n,m}((z-y)^2;x,y)\right)^{\frac{c_2}{2}} \left(H_{n,m}(e_{00};x,y)\right)^{\frac{2-c_2}{2}}\\ &= M \left(H_{n,m}((t-x)^2;x,y)\right)^{\frac{c_1}{2}} \left(H_{n,m}((z-y)^2;x,y)\right)^{\frac{c_2}{2}}\end{aligned}$$

This completes the result . ∎

5. Erroe estimation tables

In this secation , with the help of Maple software , we compare the error convergence of operators (2.1) and (4.6) .

Let $R_n^S = \left|H_{n,m}(f;x,y) - f(x,y)\right|$ and $D_n^S = \left|\mathbb{G}_{n,m}(f;x,y) - f(x,y)\right|$.

Table 1 : $f(x,y) = \sin 2x + \sin 2y$, $0 \le x, y \le 2$

(x,y)	R_{30}^0	R_{30}^2	D_{30}^2
(1.1,1.1)	0.513223748	0.000203277	0.000013069
(0.9,0.9)	0.460413109	0.000066988	0.000000075
(1.7,1.7)	0.2465443197	0.0116939153	0.0110236246
(0.6,0.6)	0.200323323	0.000088441	0.000000001
(1.5,1.5)	0.1448846430	0.0014629709	0.0010966631
(0.3,0.3)	0.057546796	0.000164324	0

Table 2 : $f(x,y) = y^2 e^{-\frac{x}{3}}$, $0 \le x, y \le 2$

(x,y)	R_{20}^0	R_{20}^3	D_{30}^3
(1.7,1.7)	0.560299071	0.000212499	0.0002036992
(1.5,1.5)	0.489527117	0.000006371	0.0000016050
(1.1,1.1)	0. 3472298217	0.0000170284	0.0000000039
(0.9,0.9)	0. 2774285549	0.0000141636	0.0000000001
(0.6,0.6)	0.1765908599	0.0000093067	0
(0.3,0.3)	0.08400748318	0.00000441155	0.00000000004

6. Conclusion

In this manuscript, we noticed that the new Baskakov-Bate operator is better for a function than the classical Baskakov-Bate operator. We discussed operators' convergence rates using partial moduli of continuity. Finally,we constructed the GBS operator of these operators and studied approximation in Bögel continuous functions using a mixed modulus of continuity.

ORCID
Ahmed F. Jabbar https://orcid.org/0009-0009-3122-0747

Amal K. Hassan https://orcid.org/0000-0002-7175-9976

References

[1] **A. Aral and H.Erbay** , "Parametric generalization of Baskakov operators , " *Mathematical Communications*, vol. 24, pp. 119-131 , 2019.

[2] **D. Barbosu and C. Muraru ,** "Approximating B-continuous functions using GBS operators of Bernstein–Schurer–Stancu type based on q-integers ," *Applied Mathematics and Computation*, vol. 259 , pp. 80-87 , 2015 , http://dx.doi.org/10.1016/j.amc.2015.02.030 .

[3] **V. Baskakov,** "An instance of a sequence of linear positive operators in the space of continuous functions , " *Doklady Akademii Nayk SSSR*, vol. 113 , no.2 , pp. 249-251 , 1957.

[4] **K. Bogel,** " Uber die mehrdimensionale differentiation," *Jahresber. Dtsch. Math. Ver.*, vol.65 , pp. 45-71,1962.

[5] **M. Gurdek, L. Rempulska and M. Skorupka ,** " The Baskakov operators for functions of two variables , " *Collectanea Mathematica* , vol. 50 , no.3 , pp. 289- 302, 1999.

[6] **A. Jabbar and A. Hassan ,** " Improving the approximation order of Baskakov-Beta operators based on a parameter ," *International Journal of Mathematics and Computer Science* , vol.20, no.2 , pp. 543-548, 2025 , DOI: https://doi.org/10.69793/ijmcs/02.2025/jabber .

[7] **M. Rani , N.Rao and P.Mailk ,** " Generalized Bivariate Baskakov Durrmeyer Operators and Associated GBS Operators ," *Filomat* , vol.36, no.5 , pp. 1539-1555, 2022 , https://doi.org/10.2298/FIL2205539R .

[8] **S. Sonker and Priyanka ,** " Rate of Convergence of parametrically generalized bivariate Baskakov-Stancu operators," *Filomat* , vol.37,no.27 , pp. 9197-9214, 2023 , https://doi.org/10.2298/FIL2327197S .