Journal of Education for Pure Science
Vol.15, No.2 (2025)
DOI: https://doi.org/10.32792/jeps.v15i2.690
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Analytical Solutions to the Martínez–Kaabar Abel Integral Equation via an Enhanced Elzaki Transform

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Received 21 / 3 /2025, Accepted 28/ 5 /202X, Published 1/ 6 /2025

Abstract:

In this work, we have discussed and prove the different properties and theorems of A new method notion, named the Martinez–Kaabar fractal–fractional (MK FrFr) Elzaki transform like Linearity property, Convolution theorem property, and provides two problems to support our methodology application to solve of Martínez–Kaabar MK Abel Integral Equation.

Keywords: Fractal–Fractional , Fractal–Fractional Elzaki transform , MK Abel integral equations.

1-Introduction

Linear and nonlinear integral equations are widely applicable in modeling various engineering and scientific phenomena. Some of the most important cases of linear and nonlinear integral equations are the first- and second-type Volterra integral equations. Various methods have been developed for solving linear integral equations, among which the following stand out: the Laplace transform, the Sumudu transform, the Elzaki transform, and the Fourier transform, etc. in [1,2,3]. For solving nonlinear integral equations, the following methods are available: the variational iterative, the series solution, the Adomian decomposition, and the successive approximation ,which uses combing different integral transforms like those of Laplace, Elzaki, Fourier, etc ,[1-6]. In addition, the Elzaki transform method is an effective approach for solving one of the basic cases of singular integral equations, called the Abel integral equation.

Recently, a new generalization of the fractal–fractional derivative (FrFrD) of the local type, has been introduced, called the Martinez-Kaabar (MK) fractal–fractional derivative. As a result of the MK derivative attack, all the results obtained from this definition were in agreement with the results of FrFrD’s in Caputo and this applies to the power law, which was proposed in [17], when this definition was used for certain elementary functions. In addition, in [18,19], this newly proposed calculus (so-named MK calculus) was investigated by the MK Abel integral equation, which was solved by the Laplace transform method, in comparison with previous studies on various fractional definitions that have been employed in studying Abel integral equation, our study indicates a high level of novelty by extending the newly proposed MK calculus to Elzaki transform. This generalized technique provides a novel mathematical tool to solving not only Martínez– Kaabar Abel integral equation’s, the Martínez–Kaabar MK Abel Integral Equation is defined as

$$f(\tau)=\frac{1}{M(\alpha,\beta,\sigma)}\int_0^{\tau}V\left(\tau^{\alpha+\beta-1}-x^{\alpha+\beta-1}\right)W(x)\frac{dx}{x^{2-\alpha-\beta}} \quad (1)$$

where $V\left(\tau^{\alpha+\beta-1}-x^{\alpha+\beta-1}\right)$ is the kernel , $W(x)$ is an unknown function , $f(\tau)$ is a perturbation known function, $V\left(\tau^{\alpha+\beta-1}-x^{\alpha+\beta-1}\right)=\frac{1}{\sqrt{\tau^{\alpha+\beta-1}-x^{\alpha+\beta-1}}}$,and $M(\alpha,\beta,\sigma)=\frac{\Gamma(\sigma-\beta+1)}{\beta\Gamma(\sigma-\alpha-\beta+2)}$, with $0<\alpha,\beta\le 1$, and $\sigma>-1$ .then it is named the α,β -difference kernel , [12,14].

In the early 2011's, Tarig Elzaki introduced the modified Laplace transform (Elzaki transform).[1,2], modified Laplace transform (Elzaki transform) is defined for the function of exponential order. Consider a function in the set S defined as

$$S=\{f(\tau):\exists M,k_1,k_2>0,|f(\tau)|<Me^{\frac{|\tau|}{k_j}},if\ \tau\in(-1)^j\times[0,\infty)\}$$

For a given function $f(\tau)$ in the set S , the constant M must be finite, number k_1,k_2 may be finite or infinite. The Elzaki transform denoted by the operator E is defined as

$$E[f(\tau):\rho)]=\rho\int_0^{\infty}e^{-\frac{\tau}{\rho}}f(\tau)d\tau=T(\rho),\quad \tau>0 \quad (2)$$

The variable ρ in this transforms is used to factorize the variable τ .
The structure of this paper by all the above works organized as follows: First, we begin with some basic definitions and Theorems of Martinez–Kaabar fractal–fractional (MK FrFr) Calculus and we define a new notion of Martinez–Kaabar fractal–fractional (MK FrFr) Elzaki transform , named MK Elzaki transform and note E_{MK} in this work, which involves the MK integral operator. In relation to this new fractal–fractional transformation , and the use of linearity property, Convolution theorem property, Finally, this new technique is application to find the solutions to Martínez– Kaabar MK Abel integral equation's.

2- Fundamental concepts of Martinez–Kaabar fractal–fractional (MK FrFr) Calculus:

Definition 2.1.[12]: Suppose that $f(\tau)\in C^n([\lambda,\infty))$ is differentiable on $[\lambda,\infty)$, where $\lambda\ge 0$; if f is a fractal differentiable on $[\lambda,\infty)$ of order β, then the FrFrD of f of order α in the context of C with the power law is written as

$$^{FrFrD}_{\ \lambda}D^{\alpha,\beta}_{\tau}f(\tau)=\frac{1}{\Gamma(m-\alpha)}\int_{\lambda}^{\tau}(\tau-x)^{m-\alpha-1}\frac{df(x)}{dx^{\beta}}dx\ , \quad (3)$$

$m-1<\alpha,\beta\le m\ ,\ m\in N\ ,$

Where $$\frac{df(x)}{dx^{\beta}}=\lim_{\tau\to x}\frac{f(\tau)-f(x)}{\tau^{\beta}-x^{\beta}}\ . \quad (4)$$

Theorem 2.2. [14]: Suppose that $0<\alpha,\beta\le 1$, and $\sigma>-1$. Then, we obtain

$$^{FrFrD}_{\ 0}D^{\alpha,\beta}_{\tau}f(\tau^{\sigma})=M(\alpha,\beta,\sigma)\tau^{2-\alpha-\beta}\sigma\tau^{\sigma-1}=\frac{\sigma\Gamma(\sigma-\beta+1)}{\beta\Gamma(\sigma-\alpha-\beta+2)}\tau^{\sigma-\alpha-\beta+1}. \quad (5)$$

Remark 2.3. If $f(\tau)=\lambda$ for every real constant λ, then $^{FrFrD}_{\ 0}D^{\alpha,\beta}_{\tau}(\lambda)=0$.

Definition 2.4. [8,9]: A function: $f:[0,\infty)\to R$, the MK derivative of order $0<\alpha\le 1$, of f at $\tau>0$ is written as

$$^{MK}D^{\alpha,\beta}f(\tau)=\lim_{\delta\to 0}\frac{f\left(\tau+\delta M(\alpha,\beta,\sigma)\tau^{2-\alpha-\beta}\right)-f(\tau)}{\delta}\ , \quad (6)$$

Where $M(\alpha,\beta,\sigma)=\frac{\Gamma(\sigma-\beta+1)}{\beta\Gamma(\sigma-\alpha-\beta+2)}$ with $0<\beta\le 1$, and $\sigma>-1$.

If f is MK α,β-differentiable in some $(0,n)$, and $n>0$, and $\lim_{\tau\to 0^+} {}^{MK}D^{\alpha,\beta} f(\tau)$ exists, then it is written as

$$ {}^{MK}D^{\alpha,\beta} f(0) = \lim_{\tau\to 0^+} {}^{MK}D^{\alpha,\beta} f(\tau) . \tag{7}$$

Theorem 2.5. [12]: Suppose that $\alpha<0, \beta\le 1$, , and f is a MK α,β-differentiable at a point $\tau>0$. If, further, f is a differentiable function, then

$$ {}^{MK}D^{\alpha,\beta} f(\tau) = M(\alpha,\beta,\sigma)\, {}^{MK}D^{\alpha,\beta} \tau^{2-\alpha-\beta} \frac{df(\tau)}{d\tau} , \tag{8}$$

Where $M(\alpha,\beta,\sigma) = \dfrac{\Gamma(\sigma-\beta+1)}{\beta\Gamma(\sigma-\alpha-\beta+2)}$ with $\sigma>-1$.

Theorem 2.6. [13]: (Chain Rule). Assume that $\alpha<0, \beta\le 1$, $\sigma>-1$, g is an MK α,β-differentiable at $\tau>0$ and f is differentiable at $g(\tau)$, then

$$ {}^{MK}D^{\alpha,\beta} (f\circ g)(\tau) = f'(g(\tau))\, {}^{MK}D^{\alpha,\beta} g(\tau). \tag{9}$$

Remark 2.7. [14]: According to Theorem 2.6, the MK derivative of order α of some elementary functions can be expressed as

i- $\displaystyle {}^{MK}D^{\alpha,\beta}\left[\frac{\beta}{(\alpha+\beta-1)\Gamma(\alpha)}\tau^{\alpha+\beta-1}\right]=1 .$

ii- $\displaystyle {}^{MK}D^{\alpha,\beta}\left[e^{\frac{\beta}{(\alpha+\beta-1)\Gamma(\alpha)}\tau^{\alpha+\beta-1}}\right]=e^{\frac{\beta}{(\alpha+\beta-1)\Gamma(\alpha)}\tau^{\alpha+\beta-1}} .$

iii- $\displaystyle {}^{MK}D^{\alpha,\beta}\left[\sin\left(\frac{\beta}{(\alpha+\beta-1)\Gamma(\alpha)}\tau^{\alpha+\beta-1})\right)\right]=\cos\left(\frac{\beta}{(\alpha+\beta-1)\Gamma(\alpha)}\tau^{\alpha+\beta-1})\right) .$

iv- $\displaystyle {}^{MK}D^{\alpha,\beta}\left[\cos\left(\frac{\beta}{(\alpha+\beta-1)\Gamma(\alpha)}\tau^{\alpha+\beta-1})\right)\right]=-\sin\left(\frac{\beta}{(\alpha+\beta-1)\Gamma(\alpha)}\tau^{\alpha+\beta-1})\right) .$

Remark 2.8. [8,9,12]: From the differentiability property of the MK derivative, which is α,β-differentiability, and by assuming that $g(\tau)>0$, then Equation (9) can be represented as

$$ {}^{MK}D^{\alpha,\beta}(f\circ g)(\tau) = \frac{1}{M(\alpha,\beta,\sigma)} g(\tau)^{\alpha+\beta-2}\, {}^{MK}D^{\alpha,\beta} f(g(\tau))\, {}^{MK}D^{\alpha,\beta} g(\tau). \tag{10}$$

Where $M(\alpha,\beta,\sigma) = \dfrac{\Gamma(\sigma-\beta+1)}{\beta\Gamma(\sigma-\alpha-\beta+2)}$ with $\sigma>-1$.

The MK α,β-integral of a function f starting at $\lambda\ge 0$, can be recalled as formulated in [9].

Definition 2.9. [8]: ${}^{MK}I^{\lambda}_{\alpha,\beta}(f)(\tau) = \dfrac{1}{M(\alpha,\beta,\sigma)}\displaystyle\int_{\lambda}^{\tau}\frac{f(t)}{t^{2-\alpha-\beta}}\cdot dt$, such that this integral is the well-known Riemann improper integral, $M(\alpha,\beta,\sigma) = \dfrac{\Gamma(\sigma-\beta+1)}{\beta\Gamma(\sigma-\alpha-\beta+2)}$, with $0<\beta\le 1$, and $\sigma>-1$.

From Definition 2.9, we obtain the following:

Theorem 2.10. [12]: ${}^{MK}D^{\alpha,\beta}\, {}^{MK}I^{\lambda}_{\alpha,\beta}(f)(\tau) = f(\tau)$, for $\tau\ge n$, , such that f is any continuous function in the domain of ${}^{MK}I^{\lambda}_{\alpha,\beta}$.

Theorem 2.11. [8,12]: Suppose that $\lambda>0$, $0<\alpha$, $\beta\le 1$, $\sigma>-1$, and f is a continuous real-valued function (RVF) on $[\lambda,\kappa]$ Let V be any RVF with the property: ${}^{MK}D^{\alpha,\beta} V(\tau) = f(\tau)$, for all $\tau\in[\lambda,\kappa]$.

Then

$$ {}^{MK}I^{\lambda}_{\alpha,\beta}(f)(\kappa) = V(\kappa) - V(\lambda) . \tag{11}$$

3- Theorems and properties of Fractal–Fractional Elzaki transform

In this section we defined a new notion of fractal–fractional Elzaki transform ; in this work, it is referred to as Martínez–Kaabar fractal–fractional Elzaki transform, we will use the term MK Elzaki transform in all upcoming results . Analogously, MK Elzaki transform was defined and some properties were given as the following:

Definition 3.1. [14]: Let that $0<\alpha\ ,\beta\le 1$, $\sigma>-1,$, and $f\ :[0,\infty)\to R$ is an RVF. Then, the MK Elzaki transform of order $\alpha\ ,\beta$ is known as

$$E_{MK}\left[f\ (\tau):\rho)\right]=\frac{\rho}{M\ (\alpha,\beta,\sigma)}\int_0^\infty e^{-\frac{1}{M\ (\alpha,\beta,\sigma)}\frac{\tau^{\alpha+\beta-1}}{\rho(\alpha+\beta-1)}}f\ (\tau)\frac{d\,\tau}{\tau^{2-\alpha-\beta}}=T_{\alpha,\beta}(\rho)\ . \tag{12}$$

provided that integral exists.

Theorem 3.2. [17]: Let $0<\alpha\ ,\beta\le 1$, with $\alpha+\beta-1>0$, $\sigma>-1,$ and $\omega\in R\ ,\rho>0$ So, we obtain

i- If $f\ (\tau)=\omega$ then $E_{MK}\left[\omega:\rho)\right]=\omega\rho^2$

ii- If $f\ (\tau)=\tau^j$ then $E_{MK}\left[\tau^j\ :\rho)\right]=\rho^{\left[2+\frac{j}{(\alpha+\beta-1)}\right]}M\ (\alpha,\beta,\sigma)(\alpha+\beta-1)^{\frac{j}{(\alpha+\beta-1)}}\,\Gamma\left(1+\frac{j}{(\alpha+\beta-1)}\right)$

iii- If $f\ (\tau)=e^{\frac{\omega\ \tau^{\alpha+\beta-1}}{M\ (\alpha,\beta,\sigma)(\alpha+\beta-1)}}$ then $E_{MK}\left[e^{\frac{\omega\ \tau^{\alpha+\beta-1}}{M\ (\alpha,\beta,\sigma)(\alpha+\beta-1)}}:\rho\right]=\frac{\rho^2}{1-\omega\rho}$

iv- If $f\ (\tau)=\sin\left(\frac{\omega\ \tau^{\alpha+\beta-1}}{M\ (\alpha,\beta,\sigma)(\alpha+\beta-1)}\right)$ then $E_{MK}\left[\sin\left(\frac{\omega\ \tau^{\alpha+\beta-1}}{M\ (\alpha,\beta,\sigma)(\alpha+\beta-1)}\right):\rho\right]=\frac{\omega\rho^3}{1+\omega^2\rho^2}$

v- If $f\ (\tau)=\cos\left(\frac{\omega\ \tau^{\alpha+\beta-1}}{M\ (\alpha,\beta,\sigma)(\alpha+\beta-1)}\right)$ then $E_{MK}\left[\cos\left(\frac{\omega\ \tau^{\alpha+\beta-1}}{M\ (\alpha,\beta,\sigma)(\alpha+\beta-1)}\right):\rho\right]=\frac{\rho^2}{1+\omega^2\rho^2}$

Proof:

i- $$E_{MK}\left[\omega:\rho)\right]=\frac{\rho}{M\ (\alpha,\beta,\sigma)}\int_0^\infty e^{-\frac{1}{M\ (\alpha,\beta,\sigma)}\frac{\tau^{\alpha+\beta-1}}{\rho(\alpha+\beta-1)}}(\omega)\frac{d\,\tau}{\tau^{2-\alpha-\beta}}$$

$$=\omega\rho\left[-\rho e^{-\frac{1}{M\ (\alpha,\beta,\sigma)}\frac{\tau^{\alpha+\beta-1}}{\rho(\alpha+\beta-1)}}\right]_0^\infty=\omega\rho^2$$

ii $$E_{MK}\left[\tau^j\ :\rho)\right]=\frac{\rho}{M\ (\alpha,\beta,\sigma)}\int_0^\infty e^{-\frac{1}{M\ (\alpha,\beta,\sigma)}\frac{\tau^{\alpha+\beta-1}}{\rho(\alpha+\beta-1)}}\left(\tau^j\right)\frac{d\,\tau}{\tau^{2-\alpha-\beta}}$$

$$=\frac{\rho}{M\ (\alpha,\beta,\sigma)}\int_0^\infty e^{-\frac{1}{M\ (\alpha,\beta,\sigma)}\frac{\tau^{(\alpha+\beta-1)+j}}{\rho(\alpha+\beta-1)}}\ \frac{d\,\tau}{\tau^{2-\alpha-\beta}}$$

$$=\frac{\left[M\ (\alpha,\beta,\sigma)(\alpha+\beta-1)\right]^{\frac{j}{(\alpha+\beta-1)}}\rho}{M\ (\alpha,\beta,\sigma)}\int_0^\infty e^{-\frac{1}{M\ (\alpha,\beta,\sigma)}\frac{\tau^{(\alpha+\beta-1)+j}}{\rho(\alpha+\beta-1)}}\left[M\ (\alpha,\beta,\sigma)(\alpha+\beta-1)\right]^{\frac{-j}{(\alpha+\beta-1)}}\frac{d\,\tau}{\tau^{2-\alpha-\beta}}$$

$$=\left[M\ (\alpha,\beta,\sigma)(\alpha+\beta-1)\right]^{\frac{j}{(\alpha+\beta-1)}}\rho^{2+\frac{j}{\alpha+\beta-1}}\Gamma\left(1+\frac{j}{\alpha+\beta-1}\right)$$

iii- $$E_{MK}\left[e^{\frac{\omega\ \tau^{\alpha+\beta-1}}{M\ (\alpha,\beta,\sigma)(\alpha+\beta-1)}}:\rho\right]=\frac{\rho}{M\ (\alpha,\beta,\sigma)}\int_0^\infty e^{-\frac{1}{M\ (\alpha,\beta,\sigma)}\frac{\tau^{\alpha+\beta-1}}{\rho(\alpha+\beta-1)}}\left(e^{\frac{\omega\ \tau^{\alpha+\beta-1}}{M\ (\alpha,\beta,\sigma)(\alpha+\beta-1)}}\right)\frac{d\,\tau}{\tau^{2-\alpha-\beta}}$$

$$=\frac{\rho}{M\ (\alpha,\beta,\sigma)}\int_0^\infty e^{-\frac{1}{M\ (\alpha,\beta,\sigma)}\frac{\tau^{\alpha+\beta-1}(1-\omega\rho)}{\rho(\alpha+\beta-1)}}\ \frac{d\,\tau}{\tau^{2-\alpha-\beta}}$$

$$=\rho\left[-\frac{\rho}{1-\omega\rho}e^{-\frac{1}{M\ (\alpha,\beta,\sigma)}\frac{\tau^{\alpha+\beta-1}(1-\omega\rho)}{\rho(\alpha+\beta-1)}}\right]_0^\infty=\frac{\rho^2}{1-\omega\rho}$$

iv- $E_{MK}\left[\sin\left(\frac{\omega\,\tau^{\alpha+\beta-1}}{M(\alpha,\beta,\sigma)(\alpha+\beta-1)}\right):\rho\right]=E_{MK}\left[\frac{e^{i\frac{\omega\,\tau^{\alpha+\beta-1}}{M(\alpha,\beta,\sigma)(\alpha+\beta-1)}}-e^{-i\frac{\omega\,\tau^{\alpha+\beta-1}}{M(\alpha,\beta,\sigma)(\alpha+\beta-1)}}}{2i}\right]$

$$=\frac{1}{2i}\left(E_{MK}\left[e^{i\frac{\omega\,\tau^{\alpha+\beta-1}}{M(\alpha,\beta,\sigma)(\alpha+\beta-1)}}\right]-\left[e^{-i\frac{\omega\,\tau^{\alpha+\beta-1}}{M(\alpha,\beta,\sigma)(\alpha+\beta-1)}}\right]\right)=\frac{1}{2i}\left(\frac{\rho^2}{1-i\,\omega\rho}-\frac{\rho^2}{1+i\,\omega\rho}\right)=\frac{\omega\rho^3}{1+\omega^2\rho^2}$$

v- $E_{MK}\left[\cos\left(\frac{\omega\,\tau^{\alpha+\beta-1}}{M(\alpha,\beta,\sigma)(\alpha+\beta-1)}\right):\rho\right]=E_{MK}\left[\frac{e^{i\frac{\omega\,\tau^{\alpha+\beta-1}}{M(\alpha,\beta,\sigma)(\alpha+\beta-1)}}+e^{-i\frac{\omega\,\tau^{\alpha+\beta-1}}{M(\alpha,\beta,\sigma)(\alpha+\beta-1)}}}{2}\right]$

$$=\frac{1}{2}\left(E_{MK}\left[e^{i\frac{\omega\,\tau^{\alpha+\beta-1}}{M(\alpha,\beta,\sigma)(\alpha+\beta-1)}}\right]+\left[e^{-i\frac{\omega\,\tau^{\alpha+\beta-1}}{M(\alpha,\beta,\sigma)(\alpha+\beta-1)}}\right]\right)=\frac{1}{2}\left(\frac{\rho^2}{1-i\,\omega\rho}+\frac{\rho^2}{1+i\,\omega\rho}\right)=\frac{\rho^2}{1+\omega^2\rho^2}$$

Theorem 3.3. [7,10]: (linearity of the MK Elzaki transform)

Let that $0<\alpha\,,\beta\le 1$, , with $\alpha+\beta-1>0$, $\sigma>-1$, $f:[0,\infty)\to R$ are RVFs, and $\theta_1,\theta_2\in R$. If $E_{MK}[f(\tau)](\rho)$ and $E_{MK}[g(\tau)](\rho)$ exist, then

$$E_{MK}[\theta_1 f(\tau)+\theta_2 g(\tau)](\rho)=\theta_1 E_{MK}[f(\tau)](\rho)+\theta_2 E_{MK}[g(\tau)](\rho)$$

Where q_1 and q_2 are constants

Proof:

$$E_{MK}[\theta_1 f(\tau)+\theta_2 g(\tau)](\rho)=\frac{\rho}{M(\alpha,\beta,\sigma)}\int_0^{\infty}[\theta_1 f(\tau)+\theta_2 g(\tau)]e^{-\frac{1}{M(\alpha,\beta,\sigma)}\frac{\tau^{\alpha+\beta-1}}{\rho(\alpha+\beta-1)}}\frac{d\tau}{\tau^{2-\alpha-\beta}}$$

$$=\frac{\rho}{M(\alpha,\beta,\sigma)}\int_0^{\infty}[\theta_1 f(\tau)]e^{-\frac{1}{M(\alpha,\beta,\sigma)}\frac{\tau^{\alpha+\beta-1}}{\rho(\alpha+\beta-1)}}\frac{d\tau}{\tau^{2-\alpha-\beta}}+\frac{\rho}{M(\alpha,\beta,\sigma)}\int_0^{\infty}[\theta_2 g(\tau)]e^{-\frac{1}{M(\alpha,\beta,\sigma)}\frac{\tau^{\alpha+\beta-1}}{\rho(\alpha+\beta-1)}}\frac{d\tau}{\tau^{2-\alpha-\beta}}$$

$$=\frac{\theta_1\,\rho}{M(\alpha,\beta,\sigma)}\int_0^{\infty}e^{-\frac{1}{M(\alpha,\beta,\sigma)}\frac{\tau^{\alpha+\beta-1}}{\rho(\alpha+\beta-1)}}f(\tau)\frac{d\tau}{\tau^{2-\alpha-\beta}}+\frac{\theta_2\,\rho}{M(\alpha,\beta,\sigma)}\int_0^{\infty}e^{-\frac{1}{M(\alpha,\beta,\sigma)}\frac{\tau^{\alpha+\beta-1}}{\rho(\alpha+\beta-1)}}g(\tau)\frac{d\tau}{\tau^{2-\alpha-\beta}}$$

$$=\theta_1 E_{MK}[f(\tau)](\rho)+\theta_2 E_{MK}[g(\tau)](\rho)$$

Theorem 3.4. [8,11]: **(** Convolution theorem)

Let that $0<\alpha\,,\beta\le 1$, , with $\alpha+\beta-1>0$, $\sigma>-1$, $f:[0,\infty)\to R$ are RVFs, and If $E_{MK}\left[f(\tau^{\alpha+\beta-1})\right](\rho)$ and $E_{MK}[g(\tau)](\rho)$ exist, then we have

$$E_{MK}[f*g](\rho)=\frac{1}{\rho}E_{MK}\left[f(\tau^{\alpha+\beta-1})\right](\rho).E_{MK}[g(\tau)](\rho). \tag{13}$$

Where $$(f*g)(\tau)=\frac{1}{M(\alpha,\beta,\sigma)}\int_0^{\tau}f(\tau^{\alpha+\beta-1}-x^{\alpha+\beta-1})g(x)\frac{dx}{x^{2-\alpha-\beta}}\,. \tag{14}$$

Proof: By applying MK Elzaki transform to Equation (12), we have

$$E_{MK}[f*g](\rho)=\frac{\rho}{M(\alpha,\beta,\sigma)}\int_0^{\infty}e^{-\frac{1}{M(\alpha,\beta,\sigma)}\frac{\tau^{\alpha+\beta-1}}{\rho(\alpha+\beta-1)}}\left[\frac{1}{M(\alpha,\beta,\sigma)}\int_0^{\tau}f(\tau^{\alpha+\beta-1}-x^{\alpha+\beta-1})g(x)\frac{dx}{x^{2-\alpha-\beta}}\right]\frac{d\tau}{\tau^{2-\alpha-\beta}}$$

$$=\frac{\rho}{M(\alpha,\beta,\sigma)}\int_0^{\infty}e^{-\frac{1}{M(\alpha,\beta,\sigma)}\frac{x^{\alpha+\beta-1}}{\rho(\alpha+\beta-1)}}g(x)\left[\frac{1}{M(\alpha,\beta,\sigma)}\int_x^{\infty}e^{-\frac{1}{M(\alpha,\beta,\sigma)}\frac{(\tau^{\alpha+\beta-1}-x^{\alpha+\beta-1})}{\rho(\alpha+\beta-1)}}f\left(\frac{\tau^{\alpha+\beta-1}-x^{\alpha+\beta-1}}{\alpha+\beta-1}\right)\frac{d\tau}{\tau^{2-\alpha-\beta}}\right]\frac{dx}{x^{2-\alpha-\beta}}$$

Now, consider the change of variables, $\tau^{\alpha+\beta-1}-x^{\alpha+\beta-1}=\nu^{\alpha+\beta-1}$

$$E_{MK}\left[f*g\right](\rho)=\frac{\rho}{M\left(\alpha,\beta,\sigma\right)}\int_0^\infty e^{-\frac{1}{M(\alpha,\beta,\sigma)}\frac{x^{\alpha+\beta-1}}{\rho(\alpha+\beta-1)}}g(x)\left[\frac{1}{M\left(\alpha,\beta,\sigma\right)}\int_x^\infty e^{-\frac{1}{M(\alpha,\beta,\sigma)}\frac{\nu^{\alpha+\beta-1}}{\rho(\alpha+\beta-1)}}f\left(\nu^{\alpha+\beta-1}\right)\frac{d\nu}{\nu^{2-\alpha-\beta}}\right]\frac{dx}{x^{2-\alpha-\beta}}$$

$$=\frac{1}{\rho}E_{MK}\left[f\left(\tau^{\alpha+\beta-1}\right)\right](\rho).E_{MK}\left[g(\tau)\right](\rho)$$

Theorem 3.5. [19]: Suppose that $f(\tau)$ is continuous and ${}^{MK}D^{\alpha,\beta}f(\tau)$ is piece-wise continuous for all $\tau>0$. Then, $E_{MK}\left[{}^{MK}D^{\alpha,\beta}f(\tau)\right](\rho)$ exists , and moreover

$$E_{MK}\left[{}^{MK}D^{\alpha,\beta}f(\tau):\rho\right]=\frac{1}{\rho}T_{\alpha,\beta}(\rho)-\rho f(0)$$

Proof: By using Equation (12), we obtain

$$E_{MK}\left[{}^{MK}D^{\alpha,\beta}f(\tau):\rho\right]=\frac{\rho}{M\left(\alpha,\beta,\sigma\right)}\int_0^\infty e^{-\frac{1}{M(\alpha,\beta,\sigma)}\frac{\tau^{\alpha+\beta-1}}{\rho(\alpha+\beta-1)}}\left[{}^{MK}D^{\alpha,\beta}f(\tau)\right]\frac{d\tau}{\tau^{2-\alpha-\beta}}$$

Applying the integration by parts, we have

$$E_{MK}\left[{}^{MK}D^{\alpha,\beta}f(\tau):\rho\right]=\rho\left[e^{-\frac{1}{M(\alpha,\beta,\sigma)}\frac{\tau^{\alpha+\beta-1}}{\rho(\alpha+\beta-1)}}f(\tau)\right]_0^\infty$$

$$+\frac{1}{M\left(\alpha,\beta,\sigma\right)}\int_0^\infty e^{-\frac{1}{M(\alpha,\beta,\sigma)}\frac{\tau^{\alpha+\beta-1}}{\rho(\alpha+\beta-1)}}f(\tau)\frac{d\tau}{\tau^{2-\alpha-\beta}}$$

$$=\rho\left[0-f(0)\right]+\frac{1}{M\left(\alpha,\beta,\sigma\right)}\int_0^\infty e^{-\frac{1}{M(\alpha,\beta,\sigma)}\frac{\tau^{\alpha+\beta-1}}{\rho(\alpha+\beta-1)}}f(\tau)\frac{d\tau}{\tau^{2-\alpha-\beta}}$$

$$=\frac{1}{\rho}T_{\alpha,\beta}(\rho)-\rho f(0)$$

Corollary 3.6. [17]: Assume that $f(\tau), {}^{MK}D^{\alpha,\beta}f(\tau),...., {}^{MK}D^{\alpha,\beta}_{(n-1)}f(\tau)$ are continuous, and ${}^{MK}D^{\alpha,\beta}_{(n)}f(\tau)$ is piece-wise continuous for all $\tau>0$. Suppose further that $f(\tau), {}^{MK}D^{\alpha,\beta}f(\tau),...., {}^{MK}D^{\alpha,\beta}_{(n-1)}f(\tau)$ are GFrFrEO. Then, $E_{MK}\left[{}^{MK}D^{\alpha,\beta}f(\tau)\right](\rho)$ exists , is given by

$$E_{MK}\left[{}^{MK}D^{\alpha,\beta}_{(n)}f(\tau):\rho\right]=\frac{1}{\rho^n}T_{\alpha,\beta}(\rho)-\sum_{k=0}^{n-1}\rho^{2-n+k}\ {}^{MK}D^{\alpha,\beta}_{(k)}f(0) \quad (15)$$

Hence, ${}^{MK}D^{\alpha,\beta}_{(n)}f(\tau)$ means the application of the MK $\alpha,\beta-$derivative n times.

Theorem 3.7. [8]: Assume that g is a GFrFrEO and continuous for $\tau\geq 0$. Then, we obtain

$$E_{MK}\left[\frac{1}{M\left(\alpha,\beta,\sigma\right)}\int_0^\tau f(x)\frac{dx}{x^{2-\alpha-\beta}}:\rho\right]=\rho E_{MK}\left[f(x):\rho\right]$$

Proof: By using Theorem 3.4, If we take $g(\tau)=1$ and $E_{MK}\left[(1):\rho\right]=\rho^2$ from Theorem 3.2, our result follows easily.

4- Applications

In this section, the technique of MK Elzaki transform is a powerful classical technique to solve Martínez–Kaabar MK Abel Integral Equation [8].

Consider the following MK Abel Integral Equation in Eq.(1) :

$$f(\tau)=\frac{1}{M\left(\alpha,\beta,\sigma\right)}\int_0^\tau V\left(\tau^{\alpha+\beta-1}-x^{\alpha+\beta-1}\right)W(x)\frac{dx}{x^{2-\alpha-\beta}}$$

where $V\left(\tau^{\alpha+\beta-1}-x^{\alpha+\beta-1}\right)$ is the kernel, $W(x)$ is an unknown function, $f(\tau)$ is a perturbation known function, and $V\left(\tau^{\alpha+\beta-1}-x^{\alpha+\beta-1}\right)=\frac{1}{\sqrt{\tau^{\alpha+\beta-1}-x^{\alpha+\beta-1}}}$, then it is named the α,β-difference kernel. So, we obtain

$$f(\tau)+[V*W](\tau)=f(\tau)+\frac{1}{M(\alpha,\beta,\sigma)}\int_0^\tau \frac{1}{\sqrt{\tau^{\alpha+\beta-1}-x^{\alpha+\beta-1}}}W(x)\frac{dx}{x^{2-\alpha-\beta}}=0, \tag{16}$$

by taking the MK Elzaki transform on both sides of Equation (16), by applying Theorem 3.7 we obtain

$$E_{MK}[f(\tau)](\rho)=E_{MK}\left[\tau^{\frac{\alpha+\beta-1}{2}}*W(\tau)\right](\rho)$$

$$E_{MK}[f(\tau)](\rho)=\sqrt{\frac{\pi\rho}{M(\alpha,\beta,\sigma)(\alpha+\beta-1)}}.E_{MK}[W(\tau)](\rho)$$

$$E_{MK}[W(\tau)](\rho)=\sqrt{\frac{M(\alpha,\beta,\sigma)(\alpha+\beta-1)}{\pi\rho}}.E_{MK}[f(\tau)](\rho)$$

$$=\frac{M(\alpha,\beta,\sigma)(\alpha+\beta-1)}{\pi\rho^2}\sqrt{\frac{\pi}{M(\alpha,\beta,\sigma)(\alpha+\beta-1)}}.\rho^{\frac{3}{2}}E_{MK}[f(\tau)](\rho)$$

$$=\frac{M(\alpha,\beta,\sigma)(\alpha+\beta-1)}{\pi\rho^2}\left[E_{MK}\left[\tau^{\frac{\alpha+\beta-1}{2}}*f(\tau)\right](\rho)\right]$$

$$=\frac{M(\alpha,\beta,\sigma)(\alpha+\beta-1)}{\pi\rho^2}.E_{MK}\left[\frac{1}{M(\alpha,\beta,\sigma)}\int_0^\tau \frac{1}{\sqrt{\tau^{\alpha+\beta-1}-x^{\alpha+\beta-1}}}f(x)\frac{dx}{x^{2-\alpha-\beta}}\right]$$

We can write this equation as

$$E_{MK}[W(\tau)](\rho)=\frac{M(\alpha,\beta,\sigma)(\alpha+\beta-1)}{\pi\rho^2}.E_{MK}\left[F(\tau)\right](\rho) \tag{17}$$

Where

$$F(\tau)=\frac{1}{M(\alpha,\beta,\sigma)}\int_0^\tau \frac{1}{\sqrt{\tau^{\alpha+\beta-1}-x^{\alpha+\beta-1}}}f(x)\frac{dx}{x^{2-\alpha-\beta}} \tag{18}$$

Now, by applying Theorem 3.5 on Equation (18), we obtain

$$E_{MK}\left[{}^{MK}D^{\alpha,\beta}F(\tau)\right](\rho)=\frac{1}{\rho}E_{MK}\left[F(\tau)\right](\rho)-F(0)$$

$$\Rightarrow E_{MK}\left[F(\tau)\right](\rho)=\rho E_{MK}\left[{}^{MK}D^{\alpha,\beta}F(\tau)\right](\rho) \tag{19}$$

From Equations (17) and (19), we obtain

$$E_{MK}[W(\tau)](\rho)=\frac{M(\alpha,\beta,\sigma)(\alpha+\beta-1)}{\pi\rho}E_{MK}\left[{}^{MK}D^{\alpha,\beta}F(\tau)\right](\rho) \tag{20}$$

Applying inverse MK Elzaki transform, we get

$$W(\tau)=\frac{M(\alpha,\beta,\sigma)(\alpha+\beta-1)}{\pi\rho}\,{}^{MK}D^{\alpha,\beta}F(\tau) \tag{21}$$

Using Equations (18) and (21), we obtain

$$W(\tau)=\frac{M(\alpha,\beta,\sigma)(\alpha+\beta-1)}{\pi\rho}\,{}^{MK}D^{\alpha,\beta}\left[\frac{1}{M(\alpha,\beta,\sigma)}\int_0^\tau \frac{1}{\sqrt{\tau^{\alpha+\beta-1}-x^{\alpha+\beta-1}}}f(x)\frac{dx}{x^{2-\alpha-\beta}}\right]$$

which is the required solution of the Equation (16).

Example 4.1. Consider the following MK FrFr Abel Integral Equation :

$$\frac{3\pi}{8}\tau^{2(\alpha+\beta-1)}=\frac{1}{M(\alpha,\beta,\sigma)}\int_0^\tau \frac{1}{\sqrt{\tau^{\alpha+\beta-1}-x^{\alpha+\beta-1}}}W(x)\frac{dx}{x^{2-\alpha-\beta}}\ . \tag{22}$$

Solution:
Taking MK Elzaki transform Eq.(22)

$$\frac{3\pi\left(M\left(\alpha,\beta,\sigma\right)\left(\alpha+\beta-1\right)\right)^{2}}{4}\rho^{4}=\frac{1}{\rho}\sqrt{\frac{\pi}{M\left(\alpha,\beta,\sigma\right)\left(\alpha+\beta-1\right)}}.\rho^{\frac{3}{2}}.E_{MK}\left[W\left(\tau\right):\rho\right]$$

$$E_{MK}\left[W\left(\tau\right):\rho\right]=\frac{3\pi\left(M\left(\alpha,\beta,\sigma\right)\left(\alpha+\beta-1\right)\right)^{2}\sqrt{M\left(\alpha,\beta,\sigma\right)\left(\alpha+\beta-1\right)}}{8\pi^{\frac{1}{2}}}\rho^{\frac{7}{2}}$$

$$E_{MK}\left[W\left(\tau\right):\rho\right]=\frac{3\pi^{\frac{1}{2}}\left(M\left(\alpha,\beta,\sigma\right)\left(\alpha+\beta-1\right)\right)^{\frac{5}{2}}}{4}\rho^{\frac{7}{2}}$$

Applying inverse MK Elzaki transform, we get

$$W\left(\tau\right)=M\left(\alpha,\beta,\sigma\right)\left(\alpha+\beta-1\right)\tau^{\frac{3(\alpha+\beta-1)}{2}}$$

which is the exact solution of the equation. This is the same solution obtained for the equation when solved by the application of the MK Laplace transform in [8].

Example 4.2. Consider the following MK FrFr Abel Integral Equation :

$$2\tau^{\frac{\alpha+\beta-1}{2}}+\frac{8}{3}\tau^{\frac{3(\alpha+\beta-1)}{2}}=\frac{1}{M\left(\alpha,\beta,\sigma\right)}\int_{0}^{\tau}\frac{1}{\sqrt{\tau^{\alpha+\beta-1}-x^{\alpha+\beta-1}}}w\left(x\right)\frac{dx}{x^{2-\alpha-\beta}}\ . \tag{23}$$

Solution:
Taking MK Elzaki transform Eq.(23)

$$\pi\left(M\left(\alpha,\beta,\sigma\right)\left(\alpha+\beta-1\right)\right)^{\frac{1}{2}}\rho^{\frac{5}{2}}+2\left(M\left(\alpha,\beta,\sigma\right)\left(\alpha+\beta-1\right)\right)^{\frac{3}{2}}\rho^{\frac{7}{2}}=\frac{1}{\rho}\sqrt{\frac{\pi}{M\left(\alpha,\beta,\sigma\right)\left(\alpha+\beta-1\right)}}.\rho^{\frac{3}{2}}.E_{MK}\left[W\left(\tau\right):\rho\right]$$

$$E_{MK}\left[W\left(\tau\right):\rho\right]=\frac{\pi\left[\left(M\left(\alpha,\beta,\sigma\right)\left(\alpha+\beta-1\right)\right)^{\frac{1}{2}}\rho^{\frac{5}{2}}+2\left(M\left(\alpha,\beta,\sigma\right)\left(\alpha+\beta-1\right)\right)^{\frac{3}{2}}\rho^{\frac{7}{2}}\right]\sqrt{M\left(\alpha,\beta,\sigma\right)\left(\alpha+\beta-1\right)}}{\pi^{\frac{1}{2}}}\rho^{-\frac{1}{2}}$$

$$E_{MK}\left[W\left(\tau\right):\rho\right]=\pi^{\frac{1}{2}}M\left(\alpha,\beta,\sigma\right)\left(\alpha+\beta-1\right)\rho^{2}+2\pi^{\frac{1}{2}}\left(M\left(\alpha,\beta,\sigma\right)\left(\alpha+\beta-1\right)\right)^{2}\rho^{3}$$

Applying inverse MK Elzaki transform, we get

$$W\left(\tau\right)=M\left(\alpha,\beta,\sigma\right)\left(\alpha+\beta-1\right)\left[1+2\tau^{\alpha+\beta-1}\right]$$

which is the exact solution of the equation. This is the same solution obtained for the equation when solved by the application of the MK Laplace transform in [8].

Conclusion

In this work discussed the definitions and theorems of the Martinez–Kaabar calculus, and has been proposed a novel Martinez–Kaabar fractal–fractional (MK FrFr) Elzaki transform, called MK Elzaki transform, to which important theorems and properties of this relatively new technique transform were applied. The transform have been applied to the MK FrFr Abel Integral Equation, the use of this technique has been demonstrated by several interesting problems. The study of a Martinez–Kaabar fractal–fractional (MK FrFr) Elzaki transform has succeeded in finding distinct solutions. Our results allow us to conclude that this transform, being of the local type, provides a simple tool for analytic solutions to many different engineering and natural science problems. a Martinez–Kaabar fractal–fractional (MK FrFr) Elzaki transform provide the mantle and the MK FrFr Abel Integral Equation. In future studies, the newly proposed technique will be expanded to include Martinez–Kaabar fractal–fractional (MK FrFr) double and triple Elzaki transform ,with further applications in mathematics and general physics .

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