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Some results related with b-intuitionistic fuzzy normed spaces

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Abstract:
In this paper, we introduce some application related to an b-intuitionistic fuzzy normed space and we define the Cartesian product of two b-intuitionistic fuzzy normed spaces, then we prove that the Cartesian product of two b-intuitionistic fuzzy normed space is also b-intuitionistic fuzzy normed space.
Finally, we prove the completeness of the Cartesian product of two complete b-intuitionistic fuzzy normed spaces.

Keywords: b-intuitionistic fuzzy normed space, Cartesian product and complete b-intuitionistic fuzzy normed spaces

1. Introduction:

The concept of b-metric space was introduced by I.A. Bakhtin, in1989, [3]. In 2018, K. Tiwary, K. Sarkar and T. Gain [12], prove some common fixed point theorems for four mappings using some control functions in b-metric spaces. T. Dosenovic, A. Javaheri, S. Sedghi and N. Shobe, in 2017 [6], proved a coupled coincidence fixed point theorem in complete b-fuzzy metric space. In 2018, K.P.R. Rao and A. Sombabu [10], obtain some unique common coupled fixed point theorems in dislocated quasi fuzzy b-metric spaces. The concept of fuzzy normed space has introduced by Katsara, In 1984[1]. In 2010, M. Mursaleen, V. Karakaya and S. A. Mohiuddine [9] define and study the concepts of Schauder basis, separability, and approximation property in intuitionistic fuzzy normed spaces and establish some results related to these concepts. The concepts of fuzzy completeness, fuzzy minimality, fuzzy biorthogonality, fuzzy basicity, and fuzzy space of coefficients are introduced by B.T. Bilalov, S.M. Farahani, and F.A. Guliyeva, in 2012[2]. In this paper, we define b-intuitionistic fuzzy metric space, b-intuitionistic fuzzy normed space and prove some results about them.

2-Preliminaries:

Definition (2.1) [7]:

The 3-tuple $(X, M, *)$ is called a fuzzy metric space if X is an arbitrary set (non-empty set), $*$ is a continuous t-norm and μ is a fuzzy set on $X \times X \times (0,\infty)$ $(i.e.\ M : X \times X \times (0,\infty) \longrightarrow [0,1])$ satisfying the following conditions:

for all $x, y, z \in X$ and $t, s > 0$

(IFM.1) $M(x, y, t) > 0$

(IFM.2) $M(x,y,t) = 1 \iff x = y$ **(IFM.3)** $M(x,y,t) = M(y,x,t)$
(IFM.4) $M(x,y,t+s) \geq M(x,z,t) * M(z,y,s)$
(IFM.5) $M(x,y,t): (0,\infty) \to [0,1]$ is continuous.

Definition (2.2) [5]:

Let M be a nonempty set and areal number $b \geq 1$. A map $d: M^2 \to R^*$ is said to be a b-metric if for every m,n,k in M,

$(\boldsymbol{bM1})$ $d(m,n) = 0 \iff m = n;$
$(\boldsymbol{bM2})$ $d(m,n) = d(n,m);$
$(\boldsymbol{bM3})$ $d(m,n) \leq b[d(m,k) + d(k,n)]$, for all m,n,k in M, and $b \geq 1$, (M,d) is called a b-metric space (in short: b.M.S) which is an extension of usual metric space. Clearly a b-metric space implies a metric if $b = 1$.

Remark: note that a (usual) metric space is evidently a b-metric space.

Example (2.3) [4]:

Let R be the set of real numbers and $d(m,n) = |m-n|$, d is a usual metric, let $p(m,n) = \left(d(m,n)\right)^3 = |m-n|^3$ is a b-metric space on M with $b = 4$, but not a metric on M.

Definition (2.4) [8]:

Suppose that (M,d) be (b,M.S), then a sequence $\{p_n\}$ in M is said to be :

1) Converge sequence if there is a point p in M and for every $\epsilon > 0$ there is an integer N such that $n \geq N$ implies that $d(p_n,p) < \epsilon$.
2) Cauchy sequence if for every $\epsilon > 0$ there is an integer N such that $n,m \geq N$ implies $d(p_n,p_m) < \epsilon$.

The (b.M.S) is complete if every Cauchy sequence in M is converges to some point $p \in M$.

Definition (2.5) [11]:

The 3-tuple $(X,M,*)$ is called a b-fuzzy metric space (in short, FbM) if X is an arbitrary (non-empty) set, $*$ is a continuous t-norm and M is a fuzzy set on $X^2 \times (0,\infty)$, satisfying the following conditions for each $x,y,z \in X, t,s > 0$ and a given real number $b \geq 1$,

FbM1: $M(x,y,0) > 0$;
FbM2: $M(x,y,t) = 1$ if and only if $x = y$;
FbM3: $M(x,y,t) = M(y,x,t)$;
FbM4: $M(x,z,t+s) \geq M\left(x,y,\frac{t}{b}\right) * M\left(y,z,\frac{s}{b}\right)$;

FbM5:if $M(x,y,\cdot):(0,\infty)\to[0,1]$ is continuous.

Definition (2.6) [11]:

Let $(X,M,*)$ be a b-fuzzy metric space and $t>0$ be a real number, we define an open ball and a closed ball with center x and radius$r$, $0<r<1$, $\forall t>0$ as follows

$B(x,r,t)=\{y\in X; M(x,y,t)>1-r\}$
$B[x,r,t]=\{y\in X; M(x,y,t)\geq 1-r\}$.

Definition (2.7) [11]:

Let $(X,M,*)$ be a fuzzy b-metric space, then:

A sequence $\{x_n\}$ in X is said to be convergent to $x\in X$ (in short, Fbconvergent) if for each $\epsilon\in(0,1)$ and $t>0$ there exist $n_0\in Z^+$ such that $M(x_n,x,t)>1-\epsilon$ for all $n\geq n_0$ (or equivalently $\lim_{n\to\infty}M(x_n,x,t)=1$) .

A sequence $\{x_n\}$ in X is said to be fuzzy Cauchy if for each $\epsilon\in(0,1)$ and $t>0$, there exists $n_0\in Z^+$ such that $M(x_n-x_m,t)>1-\epsilon$ for all $n,m\geq n_0$.
(or equivalently, $\lim_{n,m\to\infty}M(x_n-x_m,t)=1$

A b-fuzzy metric space $(X,M,*)$ is said to be complete if every Fb-Cauchy sequence in X is Fb-convergent sequence.

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3. b-intuitionistic fuzzy normed space **Definition (3.1):**

The 5 tuple $(X,M,N,*,\diamond)$ is called an b-intuitionistic fuzzy metric (in short IbFMS) if X is arbitrary set (non empty set), $*$ is continuous t-norm, $\diamond$ is continuous t-conorm and M,N are fuzzy sets in $X\times X\times(0,\infty)$ $i.e\ M,N:X\times X\times(0,\infty)\to[0,1]$ satisfying the following conditions for all $x,y,z\in X$ $t,s>0$

$(\boldsymbol{IbFM.1})\ M(x,y,t)+N(x,y,t)\leq 1$

$(\boldsymbol{IbFM.2})\ M(x,y,t)>0$

$(\boldsymbol{IbFM.3})\ M(x,y,t)=1\ \ \forall t>0$ if and only if $x=y$

$(\boldsymbol{IbFM.4})\ M(x,y,t)=M(y,x,t)$

$(\boldsymbol{IbFM.5}) M(x,z,t+s)\geq M\left(x,y,\frac{t}{b}\right)*M\left(y,z,\frac{s}{b}\right),\quad \forall b\geq 1$

$(\boldsymbol{IbFM.6})$ if $M(x,y,\cdot):(0,\infty)\to[0,1]$ is left continuous

$(\boldsymbol{IbFM.7})\ N(x,y,t)<1$

$(\boldsymbol{IbFM.8})\ N(x,y,t)=0$ if and only if $x=y$

$(\boldsymbol{IbFM.9})\ N(x,y,t)=N(y,x.t)$

$(\boldsymbol{IbFM.10})\ N(x,z,(t+s))\leq N\left(x,y,\frac{t}{b}\right)\diamond N\left(y,z,\frac{s}{b}\right),\quad \forall b\geq 1$

$(\boldsymbol{IbFM.11})\ if\ N(x,y,\cdot):(0,\infty)\to[0,1]$ is left continuous.

Definition (3.2):

The 5-tuple $(X,\mu,v,*,\Diamond)$ is said to be an b-intuitionistic fuzzy normed space (In short, IbFNS) if X be a linear space over the field K, $*$ is a continuous t-norm, $\Diamond$ is a continuous t-conorm , and μ,v are a fuzzy set in $X\times(0,\infty)$ (*i.e.* $\mu,v: X\times(0,\infty)\rightarrow$ $[0,1]$) satisfying the following conditions: for all $x,y,h\in X,\ t,s>0$,

(IbFNS.1) $\mu(x,t)+v(x,t)\le 1$

(IbFNS.2) $\mu(x,t)>0$,

(IbFNS.3) $\mu(x,t)=1 \Leftrightarrow x=0$,

(IbFNS.4) $\mu(\beta x,t)=\mu\left(x,\frac{t}{|\beta|}\right),\quad \forall\beta\in K/\{0\}$

(IbFNS.5) $\mu(x+y,(t+s)\ge\mu\left(x,\frac{t}{b}\right)*\mu\left(y,\frac{s}{b}\right),\quad \forall\, b\ge 1$

(IbFNS.6) if $\mu(x,\cdot):(0,\infty)\rightarrow[0,1]$ is a continuous and $\lim_{t\to\infty}\mu(x,t)=1$ **(IbFNS.7)** $v(x,t)<1$,

(IbFNS.8) $v(x,t)=0 \Leftrightarrow x=0$,

(IbFNS.9) $v(\beta x,t)=v\left(x,\frac{t}{|\beta|}\right),\quad \forall\beta\in K/\{0\}$,

(IbFNS.10) $v(x+y,(t+s))\le v\left(x,\frac{t}{b}\right)\ \Diamond\ v\left(y,\frac{s}{b}\right)$,

(IbFNS.11) if $v(x,\cdot):(0,\infty)\rightarrow[0,1]$ is a continuous and $\lim_{t\to\infty}v(x,t)=1$

Furthermore, assume that $(X,\mu,v,*,\Diamond)$ satisfying the following conditions:

(IbFNS.12) $\alpha*\alpha=\alpha$ and $\alpha\Diamond\alpha=\alpha,\quad \forall\alpha\in[0,1]$,

(IbFNS.13) $\mu(t,x)>0$ and $v(x,t)<1,\ \ \forall\, t>0 \Longrightarrow x=0$.

Definition (3.3):

Let $(X,\mu,v,*,\Diamond)$ be an b-intuitionistic fuzzy normed space and $\{x_n\}$ be a sequence of X, then:

(1) A sequence $\{x_n\}$ is said to be converges to x w.r.t. (μ,v), if for each $\alpha\in(0,1)$ and $t>0$, there exists $n_0\in Z^+$ such that $\mu(x_n-x,t)>1-\alpha$ and $v(x_n-x,t)<\alpha$ for every $n\ge n_0$.
(or equivalently $\lim_{t\to\infty}\mu(x_n-x,t)=1$) and $\lim_{t\to\infty}v(x_n-x,t)=0,$ as $n\to\infty$).

(2) A sequence $\{x_n\}$ is said to be Cauchy sequence w.r.t. (μ,v), if for each $\alpha\in(0,1)$ and $t>0$, there exists $n_0\in Z^+$ such that $\mu(x_n-x_m,t)>1-\alpha$ and $v(x_n-x_m,t)<\alpha$ for every $n,m\ge n_0$.
(or equivalently $\lim_{t\to\infty}\mu(x_n-x,t)=1$ and $\lim_{t\to\infty}\mu(x_n-x_m,t)=0$, $as\, n\to\infty$).

(3) An b-intuitionistic fuzzy normed space $(X,M,N,*,\diamond)$ is said to be complete if every IbFN-cauchy sequence in X is IbFN-convergent sequence.

Definition (3.4):

Let $(X,\mu,v,*,\Diamond)$ be an b-intuitionistic fuzzy normed space. The open ball $B(x,r,t)$ and the closed ball $B[x,r,t]$ with center at $x\in X$ and radius $0<r<1,\ \ t>0$ are defined as follows:

$$B_{(M,N)}(x,r,t) = \{y \in X: \mu(x-y\,,t) \geq 1-r \ \ and \ \ v(x-y\,,t) < r\}$$
$$B_{(M,N)}[x,r,t] = \{y \in X: \mu(x-y\,,t) \geq 1-r \ \ and \ \ v(x-y\,,t) \leq r\}.$$

Definition (3.5):

Let $(X,\mu,v,*,\lozenge)$ be an b-intuitionistic fuzzy normed space and A be a subset of X . Then:

(1) A is said to be open set if for each $x \in A$, there exists $t > 0$ and $0 < r < 1$ such that $B_{(M,N)}(x,r,t) \subseteq A$.

(2) A is said to be closed set if for any sequence $\{x_n\}$ in A and converges to x then $x \in A$.

Example (3.6):

Let $(X,\|.\|)$ be a b-normed space, and let $a * b = a.b$, $a \lozenge b = min\{1, a+b\}$ for all $a,b \in [0,1]$,let $\mu, v: X \times (0,\infty) \to [0,1]$ defined as follows :

$\mu(x\,,t) = \frac{t}{t+\|x\|}$ and $v(x\,,t) = \frac{\|x\|}{t+\|x\|}$, $t > 0$, $x \in X$ then $(X,\mu,v,*,\lozenge)$ is a b- intuitionistic normed space.

In this case, $(X,\mu,v,*,\lozenge)$ is said the induced b-intuitionistic fuzzy normed space.

Proof:

$\mu(x,t) + v(x,t) \leq 1$

1) Let $x \in X$,

since $\|x\| \geq 0$ and $t > 0 \Longrightarrow \frac{t}{t+\|x\|} > 0$,

since $\mu(x,t) = \frac{t}{t+\|x\|}$

$\Longrightarrow \mu(x,t) > 0$

2) Let $x \in X$ and $x = 0 \Longrightarrow \|x\| = 0$

Since $\mu(x,t) = \frac{t}{t+\|x\|} = \frac{t}{t} = 1$

If $\mu(x,t) = 1 \Longrightarrow \frac{t}{t+\|x\|} = 1 \Longrightarrow \|x\| = 0 \Longrightarrow x = 0$

Or

$$\mu(x,t) = 1 \Leftrightarrow \frac{t}{t+\|x\|} = 1 \Leftrightarrow \|x\| = 0 \Leftrightarrow x = 0$$

3) Let $x \in X$ and $\alpha \in F$, $\alpha \neq 0$

$$\mu(\alpha x,t) = \frac{t}{t+\|\alpha x\|} = \frac{t}{t+|\alpha|\|x\|} = \frac{\frac{t}{|\alpha|}}{\frac{t}{|\alpha|}+\|x\|} = \mu\left(x,\frac{t}{|\alpha|}\right)$$

$$\Longrightarrow \mu(\alpha x,t) = \mu\left(x,\frac{t}{|\alpha|}\right)$$

4) Let $x,y \in X$, $t,s > 0$

$$\mu\left(x,\frac{t}{b}\right) * \mu\left(y,\frac{s}{b}\right) = \frac{\frac{t}{b}}{\frac{t}{b}+\|x\|} \cdot \frac{\frac{s}{b}}{\frac{s}{b}+\|y\|} = \frac{1}{1+\frac{\|x\|}{t}} \cdot \frac{1}{1+\frac{\|y\|}{s}}$$

$\overline{b}$ $\overline{b}$

$$\leq \frac{1}{1+\frac{\|x\|}{\frac{t}{b}+\frac{s}{b}}} \cdot \frac{1}{1+\frac{\|y\|}{\frac{t}{b}+\frac{s}{b}}}$$

$$\leq \frac{1}{1+\frac{\|x\|+\|y\|}{\frac{t}{b}+\frac{s}{b}}} = \frac{t+s}{t+s+b(\|x\|+\|y\|)}$$

$$\leq \frac{t+s}{t+s+\|x+y\|} = \mu(x+y,t+s)$$

$$\Longrightarrow \mu(x+y,t+s) \geq \mu\left(x,\frac{t}{b}\right) * \mu\left(y,\frac{s}{b}\right)$$

5) $\|x\| \geq 0$ and $t > 0 \Longrightarrow \frac{\|x\|}{t+\|x\|} > 0 \Longrightarrow v(x,t) > 0$

$v(x,t) = 0 \Leftrightarrow \frac{\|x\|}{t+\|x\|} = 0 \Leftrightarrow \|x\| = 0 \Leftrightarrow x = 0$ 6)

$v(\alpha x,t) = \frac{\|\alpha x\|}{t+\|\alpha x\|} = \frac{|\alpha|\|x\|}{t+|\alpha|\|x\|} = \frac{\|x\|}{\frac{t}{|\alpha|}+\|x\|}$, $\alpha \neq 0$ 7)

$$= v\left(x,\frac{t}{|\alpha|}\right) \Longrightarrow v(\alpha x,t) = v\left(x,\frac{t}{|\alpha|}\right)$$

$v(x+y,t+s) = \frac{\|x+y\|}{s+t+\|x+y\|} \leq \frac{b\|x\|+b\|y\|}{(t+s)+b\|x\|+b\|y\|}$ 8)

$$= \frac{b\|x\|}{(t+s)+b\|x\|+b\|y\|} + \frac{b\|y\|}{(t+s)+b\|x\|+b\|y\|}$$

$$\leq \frac{b\|x\|}{t+b\|x\|} + \frac{b\|y\|}{s+b\|y\|} = \frac{\|x\|}{\frac{t}{b}+\|x\|} + \frac{\|y\|}{\frac{s}{b}+\|y\|}$$

$$= v\left(x,\frac{t}{b}\right) + v\left(y,\frac{s}{b}\right)$$

Since $v(x+y,t+s) \leq 1$

$$\Longrightarrow v(x+y,t+s) \leq \min\left\{1, v\left(x,\frac{t}{b}\right) + v\left(y,\frac{s}{b}\right)\right\}$$

$$\Longrightarrow v(x+y,t+s) \leq v\left(x,\frac{t}{b}\right) \diamond v\left(y,\frac{s}{b}\right)$$

9) $\mu(x,\cdot):(0,\infty) \to [0,1]$ is a continuous and $\lim_{t\to\infty} \mu(x,t) = 1$

$v(x,\cdot):(0,\infty) \to [0,1]$ is a continuous and $\lim_{t\to\infty} v(x,t) = 1$

Example (3.7):

Let $(X,M,N,*,\Diamond)$ be an b-intuitionistic fuzzy normed space and M,N defined by

$M(x,y,t) = \mu(x-y,t)$ and $N(x,y,t) = v(x-y,t)$, then $(X,M,N,*,\Diamond)$ be the b-intuitionistic fuzzy metric.

Proof:

Let $(X,M,N,*,\Diamond)$ be an b-intuitionistic fuzzy normed space

Define the intuitionistic fuzzy metric space by $M(x,y,t)=\mu(x-y\,,t)$ and $N(x,y,t)=v(x-y,t)$ for every $x,y\in X$

the intuitionistic fuzzy metric axioms are satisfied

(1) $M(x,y,t)=\mu(x-y\,,t)>0$ and $N(x,y,t)=v(x-y,t)<1$

(2) $M(x,y,t)=\mu(x-y\,,t)=1$ if and only if $x-y=0$, hence $x=y$ and $N(x,y,t)=v(x-y\,,t)=0$ if and only if $x-y=0$, hence $x=y$

(3) $M(x,y,t)=\mu(x-y\,,t)$
$=\mu\,(y-x\,,t)$
$=M(y,x,t)$

So $M(x,y,t)=M(y,x,t)$ and $N(x,y,t)=v(x-y\,,t)$
$=v\,(y-x\,,t)$
$=N(y,x,t)$

So $N(x,y,t)=N(y,x,t)$

$\forall x,y,z\in X,\quad s,t\geq 0$ (4)

$M(x,y,t+s)=\mu(x-y\,,t+s)$
$=\mu(x-y-z+z,t+s)$
$\geq\mu\left(x-z,\frac{t}{b}\right)*\mu\left(z-y\,,\frac{s}{b}\right)$,

Therefore $M(x,y,t+s)\geq M\left(x,z,\frac{t}{b}\right)*M\left(z,y,\frac{s}{b}\right)$

$N(x,y,t+s)=v(x-y,t+s)$
$=v(x-z+z-y,t+s)$
$\leq v(x-z,\frac{t}{b})\,\Diamond\,v(z-y,\frac{s}{b})$
$\leq N(x,z,\frac{t}{b})\,\Diamond\,N(z,y,\frac{s}{b})$

(5) Since μ,v are continuous then M,N are continuous .

($(X,M,N,*,\Diamond)$ is said to be the b-intuitionistic fuzzy metric space induced by the b-intuitionistic fuzzy normed $(X,\mu,v,)$).

Theorem (3.8):

(1) Let M,N be an b-intuitionistic fuzzy metric induced by a norm on intuitionistic fuzzy vector X, then:

(i) $M(x+z,y+z,t)=M(x,y,t)$ and
$N(x+z,y+z,t)=N(x,y,t)$

(ii) $M(r.x,r.y,t)=M(x,y,\frac{t}{|r|})$ and
$N(r.x,r.y,t)=N(x,y,\frac{t}{|r|})$

(2) Let M,N be an b-intuitionistic fuzzy metric on a fuzzy vector X such that (i) and (ii) holds, then M induced a norm on a fuzzy vector X.

Proof:

(1) Let M,N be an b-intuitionistic fuzzy metric induced by a norm μ,v on intuitionistic fuzzy vector X such that, $\forall\,x,y,z\in X$,
$M(x,y,t)=\mu(x-y\,,t),N(x,y,t)=v(x-y\,,t)$,

(i) $M(x+z,y+z,t) = \mu(x+z-(y+z),t\,) = \mu(x-y,t) = M(x,y,t)$
$N(x+z,y+z,t) = v(x+z-(y+z),t\,) = v(x-y,t) = N(x,y,t)$

(ii) $M(r.x,r.y,t) = \mu(r.x-r.y,t) = \mu(r(x-y),t\,) = \mu\left(x-y,\frac{t}{|r|}\right)$
$= M\left(x,y,\frac{t}{|r|}\right),$

and
$N(r.x,r.y,t) = v(r.x-r.y,t) = v(r(x-y),t\,) = v\left(x-y,\frac{t}{|r|}\right)$
$= N\left(x,y,\frac{t}{|r|}\right)$

(2) Suppose that the condition (i) and (ii) holds
Let $\mu, v: X \times (0,\infty) \to [0,1]$
$\mu(x,t) = M(x,0,t)$,and $v(x,t) = N(x,0,t),\ \ \forall\, x \in X,$ where 0 be a zero vector,
We have
(N1) $\mu(x,t) = M(x,0,t) > 0$ and
$v(x,t) = N(x,0,t) < 1$
(N2) $\mu\,(x,t) = M(x,0,t) = 1 \Leftrightarrow x = 0$, and
$v(x,t) = N(x,0,t) = 0 \Leftrightarrow x = 0$
(N3) $\mu\,(r.x,t) = M(r.x,0,t)$
$= M(r.x,r.0,t) = M\left(x,0,\frac{t}{|r|}\right)$
$= \mu\left(x,\frac{t}{|r|}\right),$

and
$v(r.x,t) = N(r.x,0,t)$
$= N(r.x,r.0,t)$
$= N\left(x,0,\frac{t}{|r|}\right)$
$= v\left(x,\frac{t}{|r|}\right)$

(N4) $\mu\,(x+y,t+s) = M(x+y,0\,,t+s)$

$= M(x+y,-y+y,t+s)$

$= M(x,-y,t+s)$

$\geq M\left(x,z,\frac{t}{b}\right) * M(z,-y,s)$

$\geq \mu\left(x-z,\frac{t}{b}\right) * \mu\left(z+y,\frac{s}{b}\right)$

$\geq \mu\left(x,\frac{t}{2b}\right) * \mu\left(z,\frac{t}{2b|-1|}\right) * \mu\left(z,\frac{s}{2b}\right) * \mu\left(x,\frac{s}{2b}\right)$

$\geq \mu\left(x,\frac{t}{2b}\right) * \mu\left(z,\frac{t}{2b}\right) * \mu\left(z,\frac{s}{2b}\right) * \mu\left(x,\frac{s}{2b}\right)$

and
$v(x+y,t+s) = N(x+y,-y+y\,,t+s)$
$= N(x,-y,t+s)$

$\geq N\left(x, z, \frac{t}{b}\right) \lozenge N(z, -y, s)$

$\geq v\left(x - z, \frac{t}{b}\right) \lozenge v\left(z + y, \frac{s}{b}\right)$

$\geq v\left(x, \frac{t}{2b}\right) \lozenge v\left(z, \frac{t}{2b|-1|}\right) \lozenge v\left(z, \frac{s}{2b}\right) \lozenge v\left(x, \frac{s}{2b}\right)$

$\geq v\left(x, \frac{t}{2b}\right) \lozenge v\left(z, \frac{t}{2b}\right) \lozenge v\left(z, \frac{s}{2b}\right) \lozenge v\left(x, \frac{s}{2b}\right)$

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4. Cartesian product of two b-intuitionistic fuzzy normed spaces:

Definition (4.1):

Let $(X, M_1, N_1, *, \diamond)$ and $(Y, M_2, N_2, *, \diamond)$ be two b-intuitionistic fuzzy normed. The Cartesian product of $(X, M_1, N_1, *, \diamond)$ and $(Y, M_2, N_2, *, \diamond)$ is the product space $(X \times Y, M, N, *, \diamond)$ where
$X \times Y$ is the Cartesian product of the sets X and Y and μ is a function $M: ((X \times Y) \times (0, \infty)) \to [0,1]$ is given by: $M((x, y), t) = M_1(x, t) * M_2(y, t)$ for all $(x, y) \in X \times Y$ and $t, s > 0$. $N: ((X \times Y) \times (0, \infty)) \to [0,1]$ is given by: $N((x, y), t) = N_1(x, t) \diamond N_2(y, t)$ for all $(x, y) \in X \times Y$ and $t > 0$.

Theorem (4.2):

Let $(X, M_1, N_1, *, \diamond)$ and $(Y, M_2, N_2, *, \diamond)$ be two b-intuitionistic fuzzy normed. Then $(X \times Y, M, N, *, \diamond)$ is an b-intuitionistic fuzzy normed.

Proof:

Since $(X, M_1, N_1, *, \diamond)$ and $(Y, M_2, N_2, *, \diamond)$ be two b-intuitionistic fuzzy normed space, let $(x, y), (r, h) \in X \times Y$, we have

1- Since $M_1(x, t) > 0$ and $M_2(y, t) > 0$, $\forall t > 0$ Since $N_1(x, t) > 0$ and $N_2(y, t) > 0$, $\forall t > 0$
$\Rightarrow M((x, y), t) > 0$, $N((x, y), t) > 0$

2- Since $M_1(x, t) = 1 \Leftrightarrow x = 0$, also $M_2(y, t) = 1 \Leftrightarrow y = 0$ Since $N_1(x, t) = 1 \Leftrightarrow x = 0$, also $N_2(y, t) = 1 \Leftrightarrow y = 0$
$\Rightarrow M((x, y), t) = 1 \Leftrightarrow x = 0$, $N((x, y), t) = 1 \Leftrightarrow x = 0$

3- Since $M_1(\alpha x, t) = M_1\left(x, \frac{t}{|\alpha|}\right)$ and $M_2(\alpha y, t) = M_2\left(y, \frac{t}{|\alpha|}\right)$, $\forall t > 0$

$$\begin{aligned} \Rightarrow M(\alpha(x, y), t) &= M((\alpha x, \alpha y), t) \\ &= M_1(\alpha x, t) * M_2(\alpha y, t) \\ &= M_1\left(x, \frac{t}{|\alpha|}\right) * M_2\left(y, \frac{t}{|\alpha|}\right) \\ &= M_1\left((x, y), \frac{t}{|\alpha|}\right) \end{aligned}$$

Since $N_1(\alpha x, t) = N_1\left(x, \frac{t}{|\alpha|}\right)$ and $N_2(\alpha y, t) = N_2\left(y, \frac{t}{|\alpha|}\right)$, $\forall t > 0$

$$\Rightarrow N(\alpha(x, y), t) = N((\alpha x, \alpha y), t)$$

$$= N_1(\alpha x, t) \diamond N_2(\alpha y, t)$$
$$= N_1\left(x, \frac{t}{|\alpha|}\right) \diamond N_2\left(y, \frac{t}{|\alpha|}\right)$$
$$= N_1\left((x, y), \frac{t}{|\alpha|}\right)$$

4- Since $M_1(x + r, (t + s)) \geq M_1\left(x, \frac{t}{b}\right) * M_1\left(r, \frac{s}{b}\right)$ and

$M_2(y + h, (t + s)) \geq M_2\left(y, \frac{t}{b}\right) * M_2\left(h, \frac{s}{b}\right)$

$$\Longrightarrow M((x, y) + (r, h), (t + s)) = M((x + r, y + h), (t + s))$$
$$\geq M_1\left(x, \frac{t}{b}\right) * M_1\left(r, \frac{s}{b}\right) * M_2\left(y, \frac{t}{b}\right) * M_2\left(h, \frac{s}{b}\right)$$
$$\geq M_1\left(x, \frac{t}{b}\right) * M_2\left(y, \frac{t}{b}\right) * M_1\left(r, \frac{s}{b}\right) * M_2\left(h, \frac{s}{b}\right)$$
$$\geq M\left((x, y), \frac{t}{b}\right) * M\left((r, h), \frac{s}{b}\right)$$

Since $N_1(x + r, (t + s)) \leq N_1\left(x, \frac{t}{b}\right) * N_1\left(r, \frac{s}{b}\right)$ and

$N_2(y + h, (t + s)) \leq N_2\left(y, \frac{t}{b}\right) * N_2\left(h, \frac{s}{b}\right)$

$$\Longrightarrow N((x, y) + (r, h), (t + s)) = N((x + r, y + h), (t + s))$$
$$\leq N_1\left(x, \frac{t}{b}\right) \diamond N_1\left(r, \frac{s}{b}\right) \diamond N_2\left(y, \frac{t}{b}\right) \diamond N_2\left(h, \frac{s}{b}\right)$$
$$\leq N_1\left(x, \frac{t}{b}\right) \diamond N_2\left(y, \frac{t}{b}\right) \diamond N_1\left(r, \frac{s}{b}\right) \diamond N_2\left(h, \frac{s}{b}\right)$$
$$\leq N\left((x, y), \frac{t}{b}\right) \diamond N\left((r, h), \frac{s}{b}\right)$$

5- Since $M_1(x, t): (0, \infty) \to [0,1]$ is continuous and $M_2(y, t): (0, \infty) \to [0,1]$ is continuous.

Since $N_1(x, t): (0, \infty) \to [0,1]$ is continuous and $N_2(y, t): (0, \infty) \to [0,1]$ is continuous.

Then $M((x, y), t): (0, \infty) \to [0,1]$, $N((x, y), t): (0, \infty) \to [0,1]$ is continuous.

6- Since $\lim_{t\to\infty} M_1(x, t) = 1$ and $\lim_{t\to\infty} M_2(y, t) = 1$ Since $\lim_{t\to\infty} N_1(x, t) = 1$ and $\lim_{t\to\infty} N_2(y, t) = 1$

Then $\lim_{t\to\infty} M((x, y), t) = 1$, $\lim_{t\to\infty} M((x, y), t) = 1$

Theorem (4.3):

Let $\{x_n\}$ be a sequence in an b-intuitionistic fuzzy normed space $(X, M_1, N_1, *, \diamond)$ converge to x in X and $\{y_n\}$ is a sequence in an b-intuitionistic fuzzy normed space $(Y, M_2, N_2, *, \diamond)$ converge to y in Y. Then $\{(x_n, y_n)\}$ is a sequence in an bintuitionistic fuzzy normed space $(X \times Y, M, N, *, \diamond)$ converge to (x, y) in $X \times Y$.

Proof:

We show that for each $\alpha \in (0,1)$ and $t > 0$, there exist $n_0 \in Z^+$ such that $M((x_n, y_n) - (x, y), t) > 1 - \alpha$ and $N((x_n, y_n) - (x, y), t) < \alpha$ for all $n \geq n_0$

By theorem (3.3.23) $(X \times Y, M, N, *, \diamond)$ is b-intuitionistic fuzzy normed spaces

Since $\{x_n\}$ be a sequence in $(X, M_1, N_1, *, \diamond)$ convergence to x

Then for each $\alpha \in (0,1)$ and $t > 0$, there exist $n_0 \in Z^+$ such that

$M_1\left(x_n - x, \frac{t}{2b}\right) > 1 - \alpha$, $N_1\left(x_n - x, \frac{t}{2b}\right) < \alpha$, for all $n \geq n_0$, $\forall b \geq 1$

Since $\{y_n\}$ be a sequence in $(Y, M_2, N_2, *, \diamond)$ convergence to y

Then for each $\alpha \in (0,1)$ and $t > 0$, there exist $n_0 \in Z^+$ such that

$M_2\left(y_n - y, \frac{t}{2b}\right) > 1 - \alpha$, $N_2\left(y_n - y, \frac{t}{2b}\right) < \alpha$, for all $n \geq n_0$, $\forall b \geq 1$

Then that for each $\alpha \in (0,1)$ and $t > 0$, there exist $n_0 \in Z^+$ such that

$$M((x_n, y_n) - (x, y), t) \geq M_1\left(x_n - x, \frac{t}{2b}\right) * M_2\left(y_n - y, \frac{t}{2b}\right)$$
$$\geq (1-\alpha) * (1-\alpha) = (1-\alpha)$$
$$N((x_n, y_n) - (x, y), t) < N_1\left(x_n - x, \frac{t}{2b}\right) \diamond N_2\left(y_n - y, \frac{t}{2b}\right)$$
$$< \alpha \diamond \alpha = \alpha \text{ , for all } n \geq n_0 \text{ , } \forall b \geq 1$$

Thus $\{(x_n, y_n)\}$ converges to (x, y).

Theorem (4.4):

If $\{x_n\}$ be a Cauchy sequence in an b-intuitionistic fuzzy normed space $(X, M_1, N_1, *, \diamond)$ and $\{y_n\}$ is a Cauchy sequence in an b-intuitionistic fuzzy normed space $(Y, M_2, N_2, *, \diamond)$ then $\{(x_n, y_n)\}$ is a Cauchy sequence in an b-intuitionistic fuzzy normed space $(X \times Y, M, N, *, \diamond)$.

Proof:

By theorem (3.3.23) , $(X \times Y, M, N, *, \diamond)$ is b-intuitionistic fuzzy normed space

Since $\{x_n\}$ be a Cauchy sequence in b-intuitionistic fuzzy normed space $(X, M_1, N_1, *, \diamond)$

Then for each $\alpha \in (0,1)$ and $t > 0$, there exist $n_0 \in Z^+$ such that

$$M_1\left(x_n - x_m, \frac{t}{2b}\right) > 1 - \alpha \text{ , } N_1\left(x_n - x_m, \frac{t}{2b}\right) < \alpha \text{ , for all } n, m \geq n_0 \text{ , } \forall b \geq 1$$

Since $\{y_n\}$ is a Cauchy sequence in b-intuitionistic fuzzy normed space $(Y, M_2, N_2, *, \diamond)$

Then for each $\alpha \in (0,1)$ and $t > 0$, there exist $n_0 \in Z^+$ such that

$$M_2\left(y_n - y_m, \frac{t}{2b}\right) > 1 - \alpha \text{ , } N_2\left(y_n - y_m, \frac{t}{2b}\right) < \alpha$$

Then $M((x_n, y_n) - (x_m, y_m), t) \geq M_1\left(x_n - x_m, \frac{t}{2b}\right) * M_2\left(y_n - y_m, \frac{t}{2b}\right)$

$$\geq (1-\alpha) * (1-\alpha) = (1-\alpha)$$
$$N((x_n, y_n) - (x_m, y_m), t) < N_1\left(x_n - x_m, \frac{t}{2b}\right) \diamond N_2\left(y_n - y_m, \frac{t}{2b}\right)$$
$$< \alpha \diamond \alpha = \alpha \text{ , for all } n, m \geq n_0 \text{ , } \forall b \geq 1$$

Thus $\{(x_n, y_n)\}$ is a Cauchy sequence in $(X \times Y, M, N, *, \diamond)$.

Theorem (4.5):

If $(X, M_1, N_1, *, \diamond)$ and $(Y, M_2, N_2, *, \diamond)$ are complete b-intuitionistic fuzzy normed spaces then the product $(X \times Y, M, N, *, \diamond)$ is complete b-intuitionistic fuzzy normed space.

Proof:

Let $\{(x_n, y_n)\}$ be a Cauchy sequence in $X \times Y$

Since $(X, M_1, N_1,*,\diamond)$ and $(Y, M_2, N_2,*,\diamond)$ are complete b-intuitionistic fuzzy normed spaces. Then $\exists x$ in X and y in Y such that $\{x_n\}$ convergent to x and $\{y_n\}$ convergent to y So for each $\alpha \in (0,1)$ and $t > 0$, there exist $n_0 \in Z^+$ such that

$M_1\left(x_n - x, \frac{t}{2b}\right) > 1-\alpha$ and $M_2\left(y_n - y, \frac{t}{2b}\right) > 1-\alpha$

$N_1\left(x_n - x, \frac{t}{2b}\right) < \alpha$ and $N_2\left(y_n - y, \frac{t}{2b}\right) < \alpha$, for all $n \geq n_0$, $\forall b \geq 1$

Now

Then for each $\alpha \in (0,1)$ and $t > 0$, there exist $n_0 \in Z^+$ such that

$$M((x_n, y_n) - (x,y), t) \geq M_1\left(x_n - x, \frac{t}{2b}\right) * M_2\left(y_n - y, \frac{t}{2b}\right)$$
$$\geq (1-\alpha) * (1-\alpha) = (1-\alpha)$$
$$N((x_n, y_n) - (x,y), t) < N_1\left(x_n - x, \frac{t}{2b}\right) \diamond N_2\left(y_n - y, \frac{t}{2b}\right)$$
$$< \alpha \diamond \alpha = \alpha \text{ , for all } n \geq n_0 \text{ , } \forall b \geq 1$$

Then $\{(x_n, y_n)\}$ convergent to (x,y) in $X \times Y$.

Theorem (4.6):

If $(X \times Y, M, N,*,\diamond)$ is an b-intuitionistic fuzzy normed space, then $(X, M_1, N_1,*,\diamond)$ and $(Y, M_2, N_2,*,\diamond)$ are b-intuitionistic fuzzy normed space by defining $M_1(x,y) = M((x,0),t)$ and $M_2(y,t) = M((0,y),t)$ for all $x \in X$, $y \in Y$ and $t > 0$. $N_1(x,y) = N((x,0),t)$ and $N_2(y,t) = N((0,y),t)$ for all $x \in X$, $y \in Y$ and $t > 0$.

Proof:

1- $M_1(x,y) = M((x,0),t) > 0$, $\forall x \in X$

$N_1(x,y) = N((x,0),t) > 0$

$\Rightarrow M_1(x,y) > 0$, $N_1(x,y) > 0$

2- For all $t > 0, 1 = M_1(x,t) = M((x,0),t) \Leftrightarrow x = 0$

$1 = N_1(x,t) = N((x,0),t) \Leftrightarrow x = 0$

$\Rightarrow M_1(x,t) = 1 \Leftrightarrow x = 0$, $N_1(x,t) = 1 \Leftrightarrow x = 0$

3- For all $t > 0, M_1(\alpha x, t) = M(\alpha(x,0),t) = M\left((x,0), \frac{t}{|\alpha|}\right) = M_1\left(x, \frac{t}{|\alpha|}\right)$

$$N_1(\alpha x, t) = N(\alpha(x,0),t) = N\left((x,0), \frac{t}{|\alpha|}\right) = N_1\left(x, \frac{t}{|\alpha|}\right)$$

4- $M_1(x+y,(t+s)) = M((x+y,0), t+s)$
$$= M((x,0)+(y,0), t+s)$$
$$\geq M\left((x,0), \frac{t}{b}\right) * M\left((y,0), \frac{t}{b}\right)$$
$$\geq M_1\left(x, \frac{t}{b}\right) * M_1\left(y, \frac{t}{b}\right)$$

$N_1(x+y,(t+s)) = N((x+y,0), t+s)$

$$= N\big((x,0)+(y,0),t+s\big)$$
$$\leq N\left((x\,,0),\tfrac{t}{b}\right) \diamond N\left((y\,,0),\tfrac{t}{b}\right)$$
$$\leq N_1\left(x\,,\tfrac{t}{b}\right) \diamond N_1\left(y\,,\tfrac{t}{b}\right)$$

5- $M_1(x,.) = M\big((x,0),.\big)$, $N_1(x,.) = N\big((x,0),.\big)$ are continuous from $(0,\infty)$ to $[0,1]$ for all $x \in X$

6- $\lim_{t\to\infty} M_1(x,y) = \lim_{t\to\infty} M\big((x,0),t\big) = 1$

$\lim_{t\to\infty} N_1(x,y) = \lim_{t\to\infty} N\big((x,0),t\big) = 1$

Then $(X, M_1, N_1,*,\diamond)$ is b-intuitionistic fuzzy normed space

Similarly we can prove that $(Y, M_2, N_2,*,\diamond)$ is b-intuitionistic fuzzy normed space.

Theorem (4.7):

If $(X\times Y, M, N,*,\diamond)$ be a complete b-intuitionistic fuzzy normed space , then $(X, M_1, N_1,*,\diamond)$ and $(Y, M_2, N_2,*,\diamond)$ are complete b-intuitionistic fuzzy normed spaces.

Proof:

$(X, M_1, N_1,*,\diamond)$ and $(Y, M_2, N_2,*,\diamond)$ are complete b-intuitionistic fuzzy normed spaces by theorem (3.3.27)

Let $\{x_n\}$ be a Cauchy sequence in $(X, M_1, N_1,*,\diamond)$

Then $\{(x_n,0)\}$ be a Cauchy sequence in $X\times Y$

But $X\times Y$ is complete b-intuitionistic fuzzy normed space

Then there is $(x,0)$ in $X\times Y$ such that $\{(x_n,0)\}$ convergent to $(x,0)$

Now , $\lim_{n\to\infty} M_1(x_n-x,t) = \lim_{n\to\infty} M\big((x_n-x,0),t\big) = 1$

$\lim_{n\to\infty} N_1(x_n-x,t) = \lim_{n\to\infty} N\big((x_n-x,0),t\big) = 1$

Then $(X, M_1, N_1,*,\diamond)$ is complete b-intuitionistic fuzzy normed space

Similarly we can prove that $(Y, M_2, N_2,*,\diamond)$ is complete b-intuitionistic fuzzy normed space.

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