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Strongly Radical g-Supplemented Modules

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Abstract:

In this work, strongly radical g-supplemented modules are defined and some properties of these modules are investigated. This concept is natural generalization of the concept of g-supplemented modules. It is proved that every srgs-module is g-semilocal. We study a weak g-supplement and proved weak g-supplemented and srgs-modules are independent from each other. We show that if M is an srgs-module then $M = T(M) + Rad_g(M)$.

Keywords: g-supplemented modules, strongly radical supplement modules, strongly radical g-supplemented modules, g-semilocal modules .

1-Introduction

In addition to all of the modules being unital left modules, all of the rings in this work will be associative rings with identity. Let M be an R -module and R be a ring. It is implied that H is a submodule of M by the notation $H \leq M$. $Rad(M)$ and $Soc(M)$ shall stand for the radical of M and the socle of M, respectively. If $X \cap H \neq 0$ for any non-zero submodules $H \leq M$, then the submodule $X \leq M$ is named essential in M , symbolized by $X \trianglelefteq M$. As a dual concept of an essential submodule, a submodule H of M is named small in M , indicated by $H \ll M$, if $M \neq H + W$ for every proper submodule W of M, then $H \ll M$ [2]. $Rad(M)$ is the sum of all small submodules of M or the intersection of all maximal submodules of M (see [6, 2.7], [1]). If every proper submodule of a module M is small in M, it is named hollow; if M is hollow and finitely generated, it is called local, see [2].

Let N and L be submodules of M. If L is minimal with respect to $M = N + L$, then L is said to be a supplement of N in M, or N is said to have a supplement L in M. If and only if $M = N + L$ and $N \cap L \ll L$, then a submodule L of M is a supplement of N in M. If there is a supplement in M for every submodule N of M, then M is named supplemented (see [13], [6] and [1]).

A submodule N of M has ample supplements in M if each submodule X of M with $M = N + X$ contains a supplement of N in M. M is named amply supplemented if all submodule in M has ample supplements in M. Hollow modules and semisimple modules are (amply) supplemented see [13, Section 41]. Zöschinger [16] has obtained detailed information about supplemented and related modules and referred to the module M as a radical supplemented module after studying it in such a way that $Rad(M)$ has a supplement in M. As a proper generalization A module M is said to be strongly radical supplemented module (shortly srs-module) by Büyükaşik and Türkmen [5] if each submodule that contains the radical $Rad(M)$ has a supplement. Similar to [5], in this article we define the concept of strongly radical g-supplemented modules, or simply srgs-modules, will be introduced and studied in this direction.

Zhou [14] introduced δ-small submodules, extended small submodules of a module M as follows. An R-module M is called singular if there exists R-modules $A \trianglelefteq B$ such that $M \cong B/A$. A submodule $H \leq M$ is named δ-small in M (symbolized by $H \ll_{\delta} M$) if whenever $M = H + X$ with $\frac{M}{X}$ singular, implies $M = X$. Obviously, any small submodule is δ-small in M.

A submodule H in M is named a δ-supplement to N in M if $M = N + H$ and $N \cap H$ is δ-small in H (as a result in M), and M is named δ-supplemented in case every submodule of M has a δ-supplement according to [7]. If for every $K \trianglelefteq M$ with $M = U + K$ we have $K = M$, then a submodule $U \leq M$ is called generalized small (abbreviated g-small); this is indicated by $U \ll_{\mathrm{g}} M$ [2] (from [15], it's named an e-small submodule of M and showed by $U \ll_{\mathrm{e}} M$). K is named a generalized maximal submodule of M if it is both an essential and maximal submodule of M. The generalized radical of M, represented by $Rad_{\mathrm{g}}(M)$, is the intersection of all generalized maximal submodules of M; from [15], it is represented by $Rad_e(M)$. If M have no generalized maximal submodules, after that the generalized radical of M is stated by $\mathrm{Rad}_{\mathrm{g}}(M) = M$. Let N and L be submodules of M. If $M = N + L$ and $M = N + K$ with $K \trianglelefteq L$ implies that $K = L$, or equivalently, $M = N + L$ and $N \cap L \ll_{\mathrm{g}} L$, then L is called a g-supplement of N in M. If each submodule in M contains a g-supplement, then M is named a g-supplemented module (see [7] and [11, Definition 2], where it is referred to as e-supplemented). Observe that a δ-supplemented module is g-supplemented. In this paper , we call a module M is strongly radical g-supplemented (or briefly srgs-module) if every submodule of M containing the radical $Rad(M)$ has a g-supplement in M. The remaining definitions in this paper are found in [4, 6, 13].

In Section 2 of this paper, we define the concept of strongly radical g-supplemented modules. Also, we give some properties of these module. We prove that, All factor modules and srgs-module homomorphic images are srgs-modules .

Section 3 devoted on srgs-modules over Dedekind domains. Here, we prove that if R is Dedekind domain and M is a srgs-module over R, then every g-supplement of $W \leq M$ is coatomic.

Also, in Section 3, we proved that if R is nonlocal Dedekind domain and M is an srgs-module then $M = T(M) + Rad_g(M)$.

Lemma 1.1. (see [15] and [9]). For an R-module M and for $W, H \leq M$, the following conditions hold.

(i) If $W \leq H$ and $H \ll_g M$, then $W \ll_g M$.

(ii) If $W \ll_g H$, then W is a g-small submodule of every submodule of M which contains H.

(iii) If $f: M \longrightarrow H$ is an R-module homomorphism and $W \ll_g M$, then $f(W) \ll_g H$.

(iv) If $W \ll_g X$ and $H \ll_g K$ for $X, K \leq M$, then $W + H \ll_g X + K$.

Corollary 1.2. (1) Let M be an R-module and $W \leq H \leq M$. If $H \ll_g M$, then $\frac{H}{W} \ll_g \frac{M}{W}$.

(2) Let M be an R-module, $W \ll_g M$ and $X \leq M$. Then $\frac{W+X}{X} \ll_g \frac{M}{X}$.

Lemma 1.3. [15, Lemma 5]. Let M be an R-module. Then $\text{Rad}_g(M) = \sum_{X \ll_g M} X$.

Lemma 1.4. The following assertions are hold for an R-module M.

(i) If M is an R-module, then $mR \ll_g M$ for every $m \in \text{Rad}_g(M)$.

(ii) If $H \leq M$, then $\text{Rad}_g H \leq \text{Rad}_g(M)$.

(iii) If $W, X \leq M$, then $\text{Rad}_g(W) + \text{Rad}_g(X) \leq \text{Rad}_g(W + X)$.

(iv) If $f: M \longrightarrow H$ is an -module homomorphism, then $f(\text{Rad}_g M) \leq \text{Rad}_g(H)$.

(v) If $X \leq M$, then $\frac{Rad_g(M+X)}{X} \leq \text{Rad}_g\left(\frac{M}{X}\right)$.

(vi) Let $M = \bigoplus_{i \in I} M_i$. Then $\text{Rad}_g(M) = \bigoplus_{i \in I} \text{Rad}_g(M_i)$.

Proof: (1), (2), (3), (4), (5) follows from Lema 1.1 and Lema 1.3 (we use [4, Lem. 5.19] as essential criteria for a module), in which (6) follows from (1) and (2) see [2]. □

Definition 1.5. [15] Suppose M is a module. Define

$Rad_g(M) = \cap \{H \trianglelefteq M \mid H \text{ is maximal in } M\}$. See besides [2].

2- Strongly Radical g-Supplemented Modules

In this section, we defined and study the concept of strongly radical g-supplemented modules (for short, srgs-module) . The main result here state : every factor module of srgs-module is srgs-module. Also, if R is a ring, we have $_RR$ is a srgs-module iff for any finitely generated R-module is a srgs-module.

Definition 2.1. A module M is said to be strongly radical g-supplemented module (or briefly srgs-module) if every submodule N of M with $\text{Rad}(M) \leq N$ has a g-supplement in M. In other words for any $N \leq M$ with $\text{Rad}(M) \leq N$, there exists $L \leq M$ such that $N + L = M$ and $N \cap L \ll_g L$.

Example 2.2. (1) [14, Example 4.3] Let F be a field, consider $I = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$ and $R = \{(x_1, \dots, x_n, x, x, \dots) : n \in \mathbb{N}, x_i \in M_2(F), x \in I\}$. Observe that under component-wise operations, R is a ring. Here, Rad $(R) = \text{Rad}_{\text{g}}\,(R) = \{(x_1, \dots, x_n, x, x, \dots) : n \in \mathbb{N}, x_i \in M_2(F), x \in J\}$, where $J = \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}$. R is not semiregular. Hence ${}_R R$ is not supplemented but ${}_R R$ is δ-supplemented so ${}_R R$ is g-supplemented. Hence, ${}_R R$ a srgs-module.

(2) [6, Example 20.12] Consider $\mathbb{Q}$ as a $\mathbb{Z}$-module. Since $\text{Rad}_{\text{g}}\,(\mathbb{Q}) = \text{Rad}\,(\mathbb{Q}) = \mathbb{Q}$, $\mathbb{Q}$ is a srgs-module. However, since $\mathbb{Q}$ is not supplemented and every non-zero submodule of $\mathbb{Q}$ is essential in $\mathbb{Q}$, $\mathbb{Q}$ is not g-supplemented .

(3) (see [7, Example 2.14] and [6, Example 17.10]) Let $R = \mathbb{Z}$ and $M = \frac{\mathbb{Q}}{\mathbb{Z}} = \bigoplus_{i=1}^{\infty} M_i$ with each $M_i = \mathbb{Z}_{p^\infty} = \{r \in \mathbb{Q} : p^n r \in \mathbb{Z} \text{ for some } n\}$, where p is a prime number. Then Rad $(M) = \text{Rad}_{\text{g}}\,(M) = \bigoplus_i \text{Rad}_{\text{g}}\,(M_i) = \bigoplus_i M_i = M$ is essential in M. But since the p-component of M is M that is not Artinian, M is not supplemented by [13, p. 370]. Since M is singular, M is not g-supplemented.

(4) [14, Example 4.1] Let F be a field and $F_i = F$ for all $i \in \mathbb{N}$. Consider $R = <\bigoplus_{i=1}^{\infty} F_i, 1_{\prod_{i=1}^{\infty} F_i}>$, which is an F-subalgebra of $\prod_{i=1}^{\infty} F_i$ generated by $\bigoplus_{i=1}^{\infty} F_i$ and $1_{\prod_{i=1}^{\infty} F_i}$.See that R is not semisimple, and the Jacobson radical, $J(R) = 0$. Therefroe, R is not semilocal besides so ${}_R R$ is not a srs-module. ${}_R R$ is a srgs-module [see therom 2.19 below].

Proposition 2.3. Every factor module and homomorphic image of a srgs-module are srgs-modules.

Proof: Let's $X \le H \le M$ with Rad $\left(\frac{M}{X}\right) \le \frac{H}{X}$. Since, $\frac{\text{Rad}\,(M+X)}{X} \le \text{Rad}\left(\frac{M}{X}\right)$, we have $\text{Rad}(M) \le H$. By, assumption, H has g-supplement W in M. Thus we have $H + W = M$ and $H \cap W \ll_{\text{g}} W$. Now it is easy to see that $\frac{H}{X} + \frac{W+X}{X} = \frac{M}{X}$ and $\frac{H}{X} \cap \frac{W+X}{X} = \frac{(H \cap W)+X}{X} \ll_{\text{g}} \frac{W+X}{X}$. Therefore, $\frac{W+X}{X}$ is a g-supplement of $\frac{H}{X}$ in $\frac{M}{X}$. The remain is clear. □

Lemma 2.4. Let M be an R-module and let M_1 and H be submodules of M with $\text{Rad}(M) \le H$. If M_1 is a srgs-module and $M_1 + H$ has a g-supplement in M, then H has a g-supplement.

Proof: Let X be a g-supplement of $M_1 + H$ in M. Then $X + (M_1 + H) = M$ with $X \cap (M_1 + H) \ll_{\text{g}} X$. Since, Rad $(M_1) \le$ Rad $(M) \le H$, we have $\text{Rad}(M_1) \le (X + H) \cap M_1$. Then $(X + H) \cap M_1$ contains a g-supplement (say) W in M_1, because M_1 is an srgs-module. Thus, $M = ((X + H) \cap M_1 + W) + H + X = ((X + H) \cap M_1) + W + (H + X) = W + (H + X) = H + (W + X)$. Since,$H + W \le H + M_1$, $X \cap (H + W) \le X \cap (M_1 + H) \ll_{\text{g}} X$, hence $H \cap (W + X) \le (H + X) \cap W + (H + W) \cap X \ll_{\text{g}} W + X$. So, $W + X$ is a g-supplement of H. □

Proposition 2.5. Let $M = M_1 + M_2$, where M_1 and M_2 are srgs-modules. Then M is a srgs-module.

Proof: Presume $H \le M$ with Rad $(M) \le H$. According to by Lemma 2.4, $M_1 + H$ contains g-supplement in M, but $M_1 + M_2 + H$ has the trivial g-supplement 0 in M. By the lemma 1.1 again, one has a g-supplement for H in M. □

Corollary 2.6. Every finite sum of srgs-modules is a srgs-module.

Assume that M is an R-module. Remember that if H is a homomorphic image of a direct sum of copies of M, then the R-module H is said to be M-generated.

Lemma 2.7. Let M be a srgs-module. Then every finitely M-generated module is srgs-module.
Proof: Clear from Proposition 2.3 and Corollary 2.6. □

In [8], A module M is named semilocal if $\frac{M}{Rad\,(M)}$ is a semisimple module.

Definition 2.8. [9] A module M is called g-semilocal if $\frac{M}{Rad_g\,(M)}$ is a semisimple module.

Proposition 2.9. Every srgs-module is g-semilocal.
Proof: Let's $\frac{N}{\mathrm{Rad_g}\,(M)}$ be a submodule of $\frac{M}{\mathrm{Rad_g}\,(M)}$. Clearly, $Rad(M) \leq \mathrm{Rad_g}\,(M) \leq N$. Since M is a srgs-module, there exists a submodule L in M such that $M = N + L$ and $N \cap L \ll_g L$. Since, $N \cap L \ll_g L$, by Lemma 1.1(4), $N \cap L \leq \mathrm{Rad_g}\,(M)$. Hence we have, $\frac{M}{\mathrm{Rad_g}\,(M)} = \frac{N+L}{\mathrm{Rad_g}\,(M)} = \frac{N}{\mathrm{Rad_g}(M)} + \frac{L+\mathrm{Rad_g}\,(M)}{\mathrm{Rad_g}\,(M)}$ and $\frac{N}{\mathrm{Rad_g}\,(M)} \cap \frac{(L+\mathrm{Rad_g}\,(M))}{\mathrm{Rad_g}\,(M)} = \frac{\mathrm{Rad_g}\,(M)+(N\cap L)}{\mathrm{Rad_g}\,(M)} = \frac{\mathrm{Rad_g}\,(M)}{\mathrm{Rad_g}\,(M)} = 0$. As a result, M is g-semilocal. □

Corollary 2.10. Let M be a srgs-module. Then $M = M_1 \oplus M_2$, where M_1 is semisimple, $\mathrm{Rad_g}\,(M) \trianglelefteq M_2$ and $\frac{M_2}{\mathrm{Rad_g}\,(M)}$ is semisimple.
Proof: Follows from Proposition 2.9 and [7, Proposition 2.1]. □

Definition 2.11. [9] A submodule $L \leq M$ is named a weak g-supplement of $N \leq M$ if $M = N + L$ and $N \cap L \ll_g M$. The module M is named weakly g-supplemented if every submodule of M has a weak g-supplement in M.

Example 2.12. (1) [12, Example 2.1] Let R be a local Dedekind domain, or DVR and K be $R's$ quotient field. Then, as can be seen in [4, Exercise 18. (2)], the left R-module W is injective. Let $M = \oplus_I W$, where I is an infinite index set, be a left R-module. Since R is noetherian, M is injective and $\mathrm{Rad_g}\,(M) = \mathrm{Rad}\,(M) = M$. Therefore M is a srgs-modul but it is not weakly g-supplemented.

(2) [9, Example 1] Let p and q be prime numbers and let $R = \mathbb{Z}_{p,q} = \left\{\frac{a}{b} \in \mathbb{Q}: p \nmid b, q \nmid b\right\}$ be the ring . Then R is a commutative domain with exactly two maximal ideals pR and qR and every non-zero ideal is essential in R. That, ${}_RR$ is weakly g-supplementd but is not a srgs-module.

Above We have seen that the concept of weakly g-supplemented modules and srgs-modules are quite independent from each other. However we have the following result.

Proposition 2.13. Presume M is a srgs-module with $\mathrm{Rad_g}\,(M) \ll_g M$. Then M is weakly g-supplemented.
Proof: Follows from Proposition 2.9 & [9, Lemma 13]. □

Note that $\mathbb{Z}$-module $M = \mathbb{Q} \oplus \frac{\mathbb{Z}}{p^2\mathbb{Z}}$ for any prime p, is srgs-module through Proposition 2.3 but not a g-supplemented module. Thus, we try to explore conditions for which a srgs-module will be a g-supplemented module. As can be seen from (see [11, Lemma 2.4]) any srgs-module M is g-supplementd if $\mathrm{Rad}_g(M)$ is semisimple. Actually we posses the following:

Proposition 2.14. Suppose M is a srgs-module and Rad (M) is a g-supplemented submodule. Then M is g-supplemented.

Proof: Let's N be a submodule of M. Presumably, Rad $(M+N)$ has a g-supplement U of M. Another time Rad (M) is g-supplementd, hence $(U+N) \cap$ Rad (M) has a g-supplement Y in Rad (M). So $U+Y$ is the required g-supplement of N in M. □

The outcomes that showed up for amply g-supplemented modules in [10, Theorem 5] generalizes to srgs-modules.

Corollary 2.15. Let's M be finitely generated. Then M is Artinian iff M is a srgs-module satisfying DCC on g-small submodules.

Currently we have the following using the same method as in proof (1) ⇒ (2) of [14, Lemma 1.2].

Lemma 2.16. Let A and B be two submodules of a module M with $M = A + B$. Then $A \oplus H$ is essential in M for some submodule H of B.

Proof: By Zorn's Lemma, for the property $A \cap H = 0$,there is always a submodule H of B maximal. Let $0 \neq m \in M$. We already presume $m \notin H$. By the maximality of H, we've $A \cap (H + Rm) \neq 0$. Take, $0 \neq a = h + rm \in A$, where $h \in H$ and $r \in R$. Then $rm = a - h \in A + H$. Since $A \cap H = 0$, we have $rm \neq 0$. Consequently, $(A \oplus H) \cap Rm \neq 0$. □

Observe that $\delta(R) = \mathrm{Rad}_g(R) :=$ the intersectoin of all essential maximal left ideals of R (see [14, Theorem 1.6]). Following [14, Definition 3.1 and Theorem 3.6]), a ring R is named δ-semiperfect if $\frac{R}{\delta(R)} = \frac{R}{\mathrm{Rad}_g(R)}$ is a semisimple ring and idempotents lift modulo $\delta(R) = \mathrm{Rad}_g(R)$. We have the next definition.

Definition 2.17. A ring R is named g-semiperfect if $\frac{R}{\mathrm{Rad}_g(R)}$ is a semisimple and idempotents lift modulo $\mathrm{Rad}_g(R)$.

We add a note here before declaring the next theorem.

Remark 2.18. For any two left ideals I and J of a ring R with $I \leq J$ such that $\frac{J}{I}$ is a singular module, so I not required to be essential in J.

For example, consider $R = \mathbb{Z} \oplus \frac{\mathbb{Z}}{2\mathbb{Z}}$. Then $I = 0 \oplus 0$ and $J = 0 \oplus \frac{\mathbb{Z}}{2\mathbb{Z}}$ are left ideals of R with $I \leq J$ and $\frac{J}{I}$ is singular R-module but I is not essential in J.

Theorem 2.19. Suppose that R is a ring with $\text{Rad}_g(R) \ll_g R$ and so that if any two left ideals $I \leq J$ of R satisfy the property that if $\frac{J}{I}$ singular then $I \trianglelefteq J$. Then ${}_RR$ is a srgs-module if and only if R is a g-semiperfect ring.

Proof: Through the use of [7, Theorem 3.3], we only show that every left ideal of R has a g-supplement in ${}_RR$. Let I be a left ideal of R. Since ${}_RR$ is a srgs-module, we have $I + Rad_g(\text{R}) + W = R$ with $(I + Rad_g(\text{R})) \cap W \ll_g W$ for some left ideal W of R. Now by Lemma 2.15 we can find a submodule H of $Rad_g(\text{R})$ such that $(I + W) \cap H = 0$ and $(I + W) \oplus H$ essential in R ,Thus, $R = I + (W \oplus H) + Rad_g(\text{R})$ implies that $R = I + (W \oplus H)$ (since, $Rad_g(\text{R}) \ll_g R$) and $I \cap (W \oplus H) \ll_g (W \oplus H)$. Therefore, $W + H$ is the required g-supplement of I in R. The other direction is obvious as in [14, Theorem 3.6]. □

Remark 2.20. Consider the ring made up of integers localized away from the ideal $6\mathbb{Z}$ (of $\mathbb{Z}$): $R = \mathbb{Z}_{(6)} = \left\{\frac{a}{b} \in \mathbb{Q}: a, b \in \mathbb{Z}, \gcd(b, 6) = 1\right\}$ (see [4, Exercise 27.(4)]). Since $\frac{R}{J(R)} \cong \frac{\mathbb{Z}}{6\mathbb{Z}}$ has four idempotents and R has only the trivial idempotents, this ring is a classic example of a ring where idempotents do not lift modulo the Jacobson radical (represented by $J(R)$). It can be seen that although ${}_RR$ is not a srgs-module, $\text{Rad}_g(R) = \delta(R) = J(R) = 6R$, $\text{Rad}_g(R) \ll_g R$ and $\frac{R}{\text{Rad}_g(R)}$ is semisimple. □

Recall that for a ring R the left socle of R, denoted by $\text{Soc}(R)$, is defined as the sum of all its minimal right ideals and can be shown to coincide with the intersection of all the essential right ideals of R. Moreover $\text{Soc}(R)$ is a two sided ideal of R (see [4, Proposition 9.7]). Following [14, Definition 3.1 and Theorem 3.8]), a ring R is called δ-perfect if $\frac{R}{\text{Soc}(R)}$ is left perfect and idempotents lift modulo $\text{Rad}_g(R)$.

Theorem 2.21. Let Λ be a countable set, R a ring such that $\delta(\bigoplus_{i\in\Lambda} R) \ll_g \bigoplus_{i\in\Lambda} R$ and so that for any two left ideals $I \leq J$ of R if $\frac{J}{I}$ singular then $I \trianglelefteq J$. Then, the statements that follow are equivalent:

(i) R is a δ-perfect ring.

(ii) Every left R-module is δ-supplemented.

(iii) Every left R-module is g-supplemented.

(iv) Every left R-module is storngly radical g-supplemented (srgs-module).

Proof: (1) ⇔ (2) follows from [7, Theorem 3.4].

(2) ⇒ (3) is clear form the reality that if H is a δ-small submodule of M, then H is a g-small submodule of M.

(3) ⇒ (4) is clear. So, it remains to see (4) ⇒ (1). By Theorem 2.19 R is δ-semiperfect. By [14, Theorems 3.7 and 3.8] we only need to show that $\text{Rad}\left(\frac{R}{\text{Soc}(R)}\right)\left(= \frac{\delta(R)}{\text{Soc}(R)}\text{ by [14, Corollary 1.7]}\right)$ is left K-nilpotent. For this we shall use the technique of [4, Lemma 28.1]. Let $F = \bigoplus_{\mathbb{N}} R$ be a free left R-module with basis $x_1, x_2, \dots, x_i, \dots, i \in \mathbb{N}$, and G the submodule of F spanned by $y_i = x_i - x_{i+1}a_i, i \in \mathbb{N}$, where $a_1, a_2, a_3, \dots,$ is a sequence of elements form $\delta(R) = \text{Rad}_g(R)$. Then, $F = G + \delta(F)$. By hypothesis, $\delta(F) \ll_g F$ and hence by Lemma 2.16, $F = G \oplus B$ for some submodule B of $\delta(F)$. By [4, Lemma 28.2], there exists $n \in \mathbb{N}$ such that $Ra_{n+1}a_n \cdots a_1 = Ra_n a_{n-1} \cdots a_1$.

Therefore, $ra_{n+1}a_na_{n-1}\cdots a_1 = a_na_{n-1}\cdots a_1$ for some r in R, and so $(1 - ra_{n+1})a_na_{n-1}\cdots a_1 = 0$. Therefore, $a_na_{n-1}\cdots a_1 \in \text{Soc}\,(R)$. Thus, Rad $\left(\frac{R}{\text{Soc}\,(R)}\right)$ is left K-nilpotent and R is left δ-perfect. □

3- srgs-Modules over Dedekind Domains

In this section, we study some properties and results on the srgs-modules over Dedekind domains.

Take R is an integral domain. The definition of torsion submodule of R-module M is

$$T(M) = \{m \in M : mr = 0 \text{ for some non-zero } r \in R\},$$

if $T(M) = M$, then a module M (over an integral domain) is named a torsion module.

As seen by the example below, over a nonlocal domain every torsion module need not be srgs-module.

Example 3.1. Let $\mathbb{Z}$ be the ring of integers and let p be a prime in $\mathbb{Z}$: Consider the $\mathbb{Z}$-module $M = \bigoplus_{n\geq 1} \mathbb{Z}_{p^n}$ where $\mathbb{Z}_{p^n} = \frac{\mathbb{Z}}{p^n\mathbb{Z}}$, Then M is a torsion module. To see that M is not a srgs-module, consider the submodule pM of M. Since $\frac{M}{pM}$ is a semisimple module, we've Rad $(M) \leq pM$. Now, it can be demonstrated that pM does not have a g-supplement in M, i.e., M is not a srgs-module, using the same method as in [5, Example 2.2].

Definition 3.2. [13, 16.6] A module M over an integral domain R is divisible if $M = rM$ for all non-zero $r \in R$.

Definition 3.3. [3] A module M over an arbitrary ring is coatomic if every proper submodule of M is contained in a maximal submodule of M.

Remark 3.4. [3] A module M is coatomic if and only if for all submodule H of M, Rad $\left(\frac{M}{H}\right) = \frac{M}{H}$ implies $H = M$.

Lemma 3.5. Let R be a Dedekind domain and M an R-module. If $H \ll_g M$, then H is coatomic.

Proof: Let H be a g-small submodule of M and take $X \leq H$ with Rad $\left(\frac{H}{X}\right) = \frac{H}{X}$. Then $\left(\frac{H}{X}\right)P = \frac{H}{X}$ for every maximal ideal P of R. $\frac{H}{X}$ is divisible since R is a Dedekind domain, making it an injective R-module. Consequently $\frac{H}{X} \oplus \frac{W}{X} = \frac{M}{X}$ for some $W \leq M$. Then $H + W = M$ which further implies that $H' \oplus W = M$ for some $H' \leq H$ (by Lemma 2.15) and $H = H' \oplus X$. But, by [14, Proposition 2.3] $H + W = M$ implies that $\frac{M}{W}$ is semisimple and hence $\frac{H}{X} \cong H'$ is semisimple. Therefore Rad $\left(\frac{H}{X}\right) = 0$, consequently $H = X$. Thus H is coatomic. □

Lemma 3.6. Let M be a sgrs-module over a Dedekind domain and N be a submodule of M with $\text{Rad}_g\,(M) \leq N$. Then, every g-supplement of N is coatomic.

Proof: By Proposition 2.8, $\frac{M}{\text{Rad}_g\,(M)}$ is semisimple. So, $\frac{M}{N}$ is semisimple as a factor module of $\frac{M}{\text{Rad}_g\,(M)}$. Presume L is g-supplement of N in M. Then, $M = N + L$ and $N \cap L \ll_g L$. Now in the following exact sequence $0 \longrightarrow N \cap L \longrightarrow L \longrightarrow \frac{L}{N\cap L} \longrightarrow 0$ both $N \cap L$, by Lemma 3.5 and $\frac{L}{N\cap L}\left(\cong \frac{M}{N}\right)$ are coatomic. By [15, Lemma 1.5 (a)], L is coatomic. □

Reduced groups are abelian groups ($\mathbb{Z}$-modules) that have no divisible subgroups other than 0 . Denote

$$P(M) := \sum \{X \leq M : L \text{ has no maximal submodules }\}.$$

Let R be a Dedekind domain. Then an R-module M has no non-zero divisible submodules iff $P(M) = 0$.

According to Zöschinger [16], if $P(M) = 0$,then an R-module M for for any ring R is a reduced module.

The following proposition is an analogue of [5, Proposition 3.2].

Proposition 3.7. Let R be a nonlocal domain and let M be a reduced R-module. If M is a srgs-module, then $M = T(M) + \text{Rad}_g(M)$.

Proof: Presume $T(M) + \text{Rad}_g(M) \neq M$. Since, $\text{Rad}_g(M) \subseteq T(M) + \text{Rad}_g(M)$, there exist $X \leq M$ such that $T(M) + \text{Rad}_g(M) + X = M$ and $X \cap (T(M) + \text{Rad}_g(M)) \ll_g X$. Now M being reduced we have a maximal submodule W of X such that $W' = T(M) + \text{Rad}_g(M) + W$ is a maximal submodule of M. (To see W' maximal in M, write $U = T(M) + \text{Rad}_g(M)$ and consider $W_0 \leq M$, since $U + W \leq W_0 \leq M$. Then W being maximal in X, we have either $X \cap W_0 = W$ or $X \cap W_0 = X$. But $X \cap W_0 = W$ implies that $W_0 = U + W$ and $X \cap W_0 = X$ implies that $W_0 = M$, as required). So W' has a g-supplement L in M. Now W' being maximal, one can locate a cyclic submodul L_0 of L such that $W' + L_0 = M$, and so $L_0 \cong \frac{R}{I}$ for some nonzero $I \leq R$. Therefore, L_0 is a torsion submodule of M, and so $L_0 \leq T(M)$. Hence, we have $= W' + L_0 = T(M) + \text{Rad}_g(M) + W + L_0 = T(M) + \text{Rad}_g(M) + W = W'$, a contradiction. So, $M = T(M) + \text{Rad}_g(M)$. □

The following three results appeared in a similar fashion in [11, Propositions 3.3, 3.4, and 3.5].

Proposition 3.8. Let R be a domain and M an R-module. Presume $M = T(M) + \text{Rad}(M)$ and $T(M)$ is g-supplemented. Then M is a srgs-module.

Proof: Let's H be a submodule of M since $Rad\ M \subseteq H$. Then $H = H \cap T(M) + Rad\ (M) = T(H) + Rad\ M$. Let X be a g-supplement of $T(H)$ in $T(M)$. Then $T(H) + X = T(M)$ and $T(H) \cap X \ll_g X$. Hence, $M = T(M) + Rad\ M = T(H) + X + Rad\ M \subseteq H + X$, and so $M = H + X$. Since X is a torsion one, we have $H \cap X = T(H) \cap X$. Therefore, X is a g-supplement of H of M. □

Let R be a Dedekind domaina and let M be an R-module and let R. ,The divisible part of M is $P(M)$, since R is a Dedekind domain. By [6, Lemma 4.4], $P(M)$ is (divisible) injective, and hence there exists a submodule H of M such that $M = P(M) \oplus H$. Here, H is called the reduced part of M. Note that $P(M) \subseteq Rad\ M$. By [5, Corollary 2.2], we know that $P(M)$ is an srs-module. By using these fact , we obtain the next result:

Proposition 3.9. Presume R is a Dedekind domain and M an R-module. Then M is an srgs-module iff the reduced part H of M is an srgs-module.

Proof: By Prop. 2.3, H ia an srgs-module as a homomorphic image of M. The converse follows from Propo. 2.5. □

Proposition 3.10. Presume R is a nonlocal Dedekind domain and M a srgs-module. Then $M = T(M) + Rad_g(M)$.

Proof: Let $M = P(M)\oplus H$ with H reduced. Then H is an srgs-module as a direct summand of M. By Proposition 3.7, we have $H = T(H) + Rad_{g}(H)$.Thus $M = P(M)\oplus H = P(M) + T(H) + Rad_{g}(H) \subseteq T(M) + Rad_{g}(M)$. As a result, $M = T(M) + Rad_{g}(M)$. □

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