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Applying The Quasi-Hadamard Product For Some Multivalent Functions That Implicitly Contain The Generalized Komatu Integral Operator

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Abstract:

Discussing and studying the principal results of the quasi-Hadamard product is the core goal of this paper, which we study for some multivalent functions with negative coefficients through the definition of a distinct class of multivalent functions with negative coefficients using the generalized linear operator, which implicitly contains the generalized Komatu integral operator. We also present most important basic properties of this distinct class of coefficient bounds and some results of the quasi-Hadamard product. It is worth noting that this paper is nothing but a generalization or continuation of what was presented by the distinguished researchers previously. Mentioning some of these previous studies related to the subject as precisely as possible. Let us not forget that all our work is within the famous unit disc $\mathcal{U}$.

Keywords: Coefficient bounds, Komatu integral operator, Multivalent functions, Quasi-Hadamard, Unit disk.

1-Introduction

Let's start by taking the class $\mathfrak{D}_n(p)$, which is the class of all complex multivalent analytic functions inside the famous unit disc $\mathcal{U} = \{z \in \mathbb{C}: |z| < 1\}$ see [1], which can be expressed as follows:

$$f(z) = z^p + \sum_{k=p+n}^{\infty} a_k z^k. \tag{1}$$

Provided that p, n are natural numbers. The second step is to take a subclass of the main class $\mathfrak{D}_n(p)$, let it be $\mathfrak{I}_n(p)$, which contains all the functions that can be written in the following form:

$$f(z) = z^p - \sum_{k=p+n}^{\infty} a_k z^k, \tag{2}$$

provided that a_k is positive or zero and n, p are natural number.

Now, let us take the generalized Komatu integral operator, which was studied in [1] and is defined on the fundamental class $\mathfrak{I}_n(p)$ in the following form:

$$\mathcal{K}_{n,p}^{\xi} f(z) = \frac{(b+p)^{\xi}}{\Gamma(\xi) z^{b}} \int_0^z t^{b-1} \left(\log \frac{z}{t}\right)^{\xi-1} f(t)\ dt, \tag{3}$$

provided that $\mathcal{K}_{n,p}^{0} f(z) = f(z)$, and also every element of the class $\mathfrak{I}_n(p)$ can be written in the following form:

$$\mathcal{K}_{n,p}^{\xi} f(z) = z^{p} - \sum_{k=p+n}^{\infty} \left(\frac{b+p}{b+k}\right)^{\xi} a_k z^k. \tag{4}$$

Through studding the previous articles, Salim, in [2], used the generalized Komatu integral operator to define the generalized linear operator $\mathfrak{A}_{b,p,\tau}^{\ell,\xi}$ in the following manner:

$$\mathfrak{A}_{b,p,\tau}^{0,0} f(z) = f(z),$$

$$\mathfrak{A}_{b,p,\tau}^{1,\xi} f(z) = (1-\tau)\mathcal{K}_{b,p}^{\xi} f(z) + \frac{\tau z}{p}\left(\mathcal{K}_{b,p}^{\xi} f(z)\right)', \tag{5}$$

$$= \mathfrak{A}_{b,p,\tau}^{\xi} f(z),$$

$$\mathfrak{A}_{b,p,\tau}^{2,\xi} f(z) = \mathfrak{A}_{b,p,\tau}^{\xi}\left(\mathfrak{A}_{b,p,\tau}^{1,\xi} f(z)\right),$$

we continue in this manner and

$$\mathfrak{A}_{b,p,\tau}^{\ell,\xi} f(z) = \mathfrak{A}_{b,p,\tau}^{\xi}\left(\mathfrak{A}_{b,p,\tau}^{\ell-1,\xi} f(z)\right),$$

provided $b > -p,\ \xi > 0, \tau \geq 0, \ell \in \mathbb{N}$. So, naturally, using equations (4), (5), every element or function of class $\mathfrak{I}_n(p)$ can be written in the following form:

$$\mathfrak{A}_{b,p,\tau}^{\ell,\xi} f(z) = z^{p} - \sum_{k=p+n}^{\infty} \left\{\left(\frac{b+p}{b+k}\right)^{\xi}\left(1 + \frac{\tau(k-p)}{p}\right)\right\}^{\ell} a_k z^k. \tag{6}$$

In fact, many distinguished researchers in recent years have used the integral operator in their researchers, such as [3- 7[. In addition, by giving us specific values for the constants in the generalized linear operator $\mathfrak{A}_{b,p,\tau}^{\ell,\xi}$, it gives us well-known operators that have been studied by researchers in previous years. For example, if we take $\ell = 1, \tau = 0$, $\mathfrak{A}_{b,p,\tau}^{\ell,\xi}$ is reduced to generalized Komatu operator [8]. Also, this operator is reduced to the integral operator, which was identified in the research [9] when $\ell = 1, b = c, p = 1, \tau = 0$. And if $\ell = 1, b = 1, \tau = 0$, it is exactly the integral operator that was presented in the research [10] . It is nothing but the Bernardi-Libra-Livingston integral operator in the research [11] in case $\ell = 1, \xi = 1\ b = c, p = 1, \tau = 0$. If we fix $\ell = 1, b = 1, p = 1, \tau = 0$, the result is the integral operator that was studied in 12]. And we get the generalized Salagean operator [13] when $\xi = 0\ b = c, p = 1$. Finally, it' is the Salagean's operator itself when fixing $\xi = 0\ b = c, p = 1, \tau = 0$ [14]. Honestly, these are the operators that were noticed during the study of the topic, and there may be other operators that were not noticed.

The next important step is to use the generalized linear operator $\mathfrak{A}_{b,p,\tau}^{\ell,\xi}$ to define the distinct class of multivalent and analytic functions in the following manner:

Definition 1.1. Any element of the class $\mathfrak{I}_n(p)$ belongs to the distinct class $\mathfrak{R}_p(\ell,\xi,b,p,\tau,n,\omega,\sigma)$ if it satisfies the following conditions:

$$\left|\frac{\frac{\left(\mathfrak{A}^{\ell,\xi}_{b,p,\tau}f(z)\right)'}{z^{p-1}}-p}{\frac{\left(\mathfrak{A}^{\ell,\xi}_{b,p,\tau}f(z)\right)'}{z^{p-1}}+p-2\omega}\right|<\sigma, \tag{7}$$

all of this is subject to the following condition

$$0\le\omega<p,0<\sigma\le 1,b>-p,\qquad \xi>0,\tau\ge 0,\ell\in\mathbb{N},\qquad z\in\mathcal{U}.$$

Note that this class reduces to classes previously studied by researchers by taking the constants to specific values. Here, we review some of them, such as when taking $\sigma=1,\ell=0$, it is the class that was identified in [15], this is also the class in the research [16] when taking $\sigma=1,\ell=0,p=1$. By fixing $\sigma=1,\ell=0,p=1$, this is exactly the class that [17] dealt with. The last case is the class in [18] and that is when $\sigma=1,\ell=0,p=1,n=1$. There are certainly other classes that were not noticed during the research process, and we are satisfied with only those mentioned.

We must always remember that multivalent analytic functions with negative coefficients is a topic that many researchers have previously addressed in their research, most notably in recent years [19- 23].

In the following sections of this article, we present some of the more radical results, including the identification of coefficients, some properties of the quasi-Hadamard product, and a generalization of some of the results presented in previous research.

2- Estimate or Determine Coefficients:

In the following theorem, we define the necessary and sufficient conditions for an element of the fundamental class $\mathfrak{I}_n(p)$ to belong to the distinct class $\mathfrak{R}_p(\ell,\xi,b,p,\tau,n,\omega,\sigma)$, with a detailed proof.

Theorem 2.1. The function f, whose form is (2), belongs to the fundamental class $\mathfrak{R}_p(\ell,\xi,b,p,\tau,n,\omega,\sigma)$, if and only if the following condition is met:

$$\sum_{k=p+n}^{\infty}\left\{\left(\frac{b+p}{b+k}\right)^{\xi}\left(1+\frac{\tau(k-p)}{p}\right)\right\}^{\ell}k(1+\sigma)a_k\le 2\sigma(p-\omega), \tag{8}$$

on condition $0\le\omega<p,0<\sigma\le 1,b>-p,\ \xi>0,\tau\ge 0,\ell\in\mathbb{N},\ z\in\mathcal{U}$.
The result is radical for functions of the form:

$$f(z)=z^p-\frac{2\sigma(p-\omega)}{\left\{\left(\frac{b+p}{b+k}\right)^{\xi}\left(1+\frac{\tau(k-p)}{p}\right)\right\}^{\ell}k(1+\sigma)}z^k. \tag{9}$$

Proof. To prove the necessary part, we take

$$\left|\frac{\left(\mathfrak{A}^{\ell,\xi}_{b,p,\tau}f(z)\right)'}{z^{p-1}}-p\right|-\sigma\left|\frac{\left(\mathfrak{A}^{\ell,\xi}_{b,p,\tau}f(z)\right)'}{z^{p-1}}+p-2\omega\right|=$$

$$\left|-\sum_{k=p+n}^{\infty}k\left\{\left(\frac{b+p}{b+k}\right)^{\xi}\left(1+\frac{\tau(k-p)}{p}\right)\right\}^{\ell}a_kz^{k-p}\right|$$

$$-\sigma\left|(p-\omega)-\sum_{k=p+n}^{\infty}k\left\{\left(\frac{b+p}{b+k}\right)^{\xi}\left(1+\frac{\tau(k-p)}{p}\right)\right\}^{\ell}\right|a_k|z^{k-p}|$$

$$\le\sum_{k=p+n}^{\infty}\left\{\left(\frac{b+p}{b+k}\right)^{\xi}\left(1+\frac{\tau(k-p)}{p}\right)\right\}^{\ell}k(1+\sigma)|a_k|-2\sigma(p-\omega)\le 0, \tag{10}$$

because $|z| = 1$, and using the assumption.
Based on the maximum modulus theorem, we obtain

$$f \in \Re_{p}(\ell, \xi, b, p, \tau, n, \omega, \sigma).$$

To prove the second part, i.e., the sufficient condition, we begin with

$$\left| \frac{\frac{\left(\mathfrak{A}^{\ell,\xi}_{b,p,\tau} f(z)\right)'}{z^{p-1}} - p}{\frac{\left(\mathfrak{A}^{\ell,\xi}_{b,p,\tau} f(z)\right)'}{z^{p-1}} + p - 2\omega} \right| = \left| \frac{-\sum_{k=p+n}^{\infty} \left\{ \left(\frac{b+p}{b+k}\right)^{\xi} \left(1 + \frac{\tau(k-p)}{p}\right) \right\}^{\ell} k\, |a_k| z^{k-p}}{2(p-\omega) - \sum_{k=p+n}^{\infty} \left\{ \left(\frac{b+p}{b+k}\right)^{\xi} \left(1 + \frac{\tau(k-p)}{p}\right) \right\}^{\ell} k\, |a_k| z^{k-p}} \right| < \sigma$$

From this, we conclude that:

$$\mathcal{Re}\left\{ \frac{-\sum_{k=p+n}^{\infty} \left\{ \left(\frac{b+p}{b+k}\right)^{\xi} \left(1 + \frac{\tau(k-p)}{p}\right) \right\}^{\ell} k\, |a_k| z^{k-p}}{2(p-\omega) - \sum_{k=p+n}^{\infty} \left\{ \left(\frac{b+p}{b+k}\right)^{\xi} \left(1 + \frac{\tau(k-p)}{p}\right) \right\}^{\ell} k |a_k| z^{k-p}} \right\} < \sigma. \tag{11}$$

Now we choose values of z on the real axis such that the value $\frac{\left(\mathfrak{A}^{\ell,\xi}_{b,p,\tau} f(z)\right)'}{z^{p-1}}$ is real. By taking the real values of z as it approaches one, we obtain:

$$\sum_{k=p+n}^{\infty} \left\{ \left(\frac{b+p}{b+k}\right)^{\xi} \left(1 + \frac{\tau(k-p)}{p}\right) \right\}^{\ell} k(1+\sigma)\, a_k \leq 2\sigma(p-\omega), \tag{12}$$

which is the required sufficient condition.
So, we have completed the proof of the theorem in detail.
It is clear that functions of the form

$$f(z) = z^{p} - \frac{2\sigma(p-\omega)}{\left\{ \left(\frac{b+p}{b+k}\right)^{\xi} \left(1 + \frac{\tau(k-p)}{p}\right) \right\}^{\ell} k(1+\sigma)} z^{k}, \tag{13}$$

are extreme, such that the limitation of the result is proven then.

3- Principal Consequences of Quasi-Hadamard Product:

In this section, we study the most important results of the quasi-Hadamard product for functions in the special class $\Re_{p}(\ell, \xi, b, p, \tau, n, \omega, \sigma)$, which we review through the theorems, their results, and an explanation of their proof in the rest of this article.

Theorem 3.1. Let's take the function $f_i(z) \in \Re_{p}(\ell, \xi, b, p, \tau, n, \omega_i, \sigma), i = 1, 2, \dots, q$, then the quasi-Hadamard product $(f_1 * f_2 * f_3 * \dots * f_q)(z)$ is an element of the class $\Re_{p}(\ell, \xi, b, p, \tau, n, \varphi, \sigma)$ when,

$$\varphi = p - \frac{\prod_{i=1}^{q} 2\sigma(p-\omega_i)}{2\sigma\{(1+\sigma)(p+n)\}^{q-1} \left\{ \left(\frac{b+p}{b+p+n}\right)^{\xi} \left(1 + \frac{\tau n}{p}\right) \right\}^{\ell}}, \tag{14}$$

under the condition

$$0 \leq \omega_i < p, 0 < \sigma \leq 1, b > -p, \quad \xi > 0, \tau \geq 0, \ell \in \mathbb{N}, \quad z \in \mathcal{U},$$

functions in the form

$$f_i(z) = z^{p} - \frac{2\sigma(p-\omega_i)}{\left\{ \left(\frac{b+p}{b+p+n}\right)^{\xi} \left(1 + \frac{\tau n}{p}\right) \right\}^{\ell} (p+n)(1+\sigma)} z^{p+n}, i = 1, 2, \dots, q. \tag{15}$$

are extreme functions that show the extremeness of the result.

Proof. To prove the theorem, we use the method of mathematical induction. So, the first step is to take $q = 1$, then $\varphi = \omega_i$, and the result is true. As for $q = 2$

$$\sum_{k=p+n}^{\infty} \frac{k\left\{\left(\frac{b+p}{b+k}\right)^{\xi}\left(1+\frac{\tau\,(k-p)}{p}\right)\right\}^{\ell}(1+\sigma)}{2\sigma(p-\omega_i)} a_{k,i} \leq 1, \;\; (i = 1, 2) \tag{16}$$

is true, according to Theorem 2.1.

From this, we obtain

$$\sum_{k=p+n}^{\infty} \frac{k\left\{\left(\frac{b+p}{b+k}\right)^{\xi}\left(1+\frac{\tau\,(k-p)}{p}\right)\right\}^{\ell}(1+\sigma)}{\sqrt{\prod_{i=1}^{2} 2\sigma(p-\omega_i)}} \sqrt{a_{k,1}a_{k,2}} \leq 1. \tag{17}$$

Our goal now is to determine the value of φ so that it is the largest and satisfies the following inequality

$$\sum_{k=p+n}^{\infty} \frac{k\left\{\left(\frac{b+p}{b+k}\right)^{\xi}\left(1+\frac{\tau\,(k-p)}{p}\right)\right\}^{\ell}(1+\sigma)}{2\sigma(p-\varphi)} a_{k,1}a_{k,2} \leq 1, \tag{18}$$

in other words, satisfy

$$\frac{\sqrt{a_{k,1}a_{k,2}}}{2\sigma(p-\varphi)} \leq \frac{1}{\sqrt{\prod_{i=1}^{2} 2\sigma(p-\omega_i)}}, \qquad (k \geq p+n), \tag{19}$$

if we apply inequality (17), we arrive at the inequality

$$\frac{1}{2\sigma(p-\varphi)} \leq \frac{k\left\{\left(\frac{b+p}{b+k}\right)^{\xi}\left(1+\frac{\tau\,(k-p)}{p}\right)\right\}^{\ell}(1+\sigma)}{\prod_{i=1}^{2} 2\sigma(p-\omega_i)}, \qquad (k \geq p+n), \tag{20}$$

which in turn gives us

$$\varphi \leq p - \frac{\prod_{i=1}^{2} 2\sigma(p-\omega_i)}{2\sigma k\left\{\left(\frac{b+p}{b+k}\right)^{\xi}\left(1+\frac{\tau\,(k-p)}{p}\right)\right\}^{\ell}(1+\sigma)}, \qquad (k \geq p+n), \tag{21}$$

now, we assume that the right-hand side of the previous inequality is the function $\chi(k)$ which immediately gives us the desired result for $q = 2$

$$\varphi \leq \chi(p+n) = p - \frac{\prod_{i=1}^{2} 2\sigma(p-\omega_i)}{2\sigma(p+n)\left\{\left(\frac{b+p}{b+p+n}\right)^{\xi}\left(1+\frac{\tau\,n}{p}\right)\right\}^{\ell}(1+\sigma)}, \tag{22}$$

because the value of the function $\chi(k)$ is positive or zero for all values of k form $p + 1$and greater.

The third step in applying the method of mathematical induction is to assume that it is true for q. Therefore, the following relationship is true

$$\nu = p - \frac{4\sigma^2(p-\varphi)(p-\omega_{q+1})}{2\sigma(p+n)\left\{\left(\frac{b+p}{b+p+n}\right)^{\xi}\left(1+\frac{\tau n}{p}\right)\right\}^{\ell}(1+\sigma)}, \tag{23}$$

we simplify it and obtain

$$\nu = p - \frac{\prod_{i=1}^{q+1} 2\sigma(p-\omega_i)}{2\sigma\{(1+\sigma)(p+n)\}^{q}\left\{\left(\frac{b+p}{b+p+n}\right)^{\xi}\left(1+\frac{\tau n}{p}\right)\right\}^{\ell}}, \tag{24}$$

which shows us that it is true for the positive integer $q+1$

Finally, we have proven that it is true for every positive integer q.

To prove the limitation of the function $f_i(z)$ in (15), we take the quasi-Hadamard product of them in the following form

$$(f_1 * f_2 * f_3 * \dots * f_q)(z) = z^{p} - \left[\prod_{i=1}^{q} \frac{2\sigma(p-\omega_i)}{2\sigma(p+n)\left\{\left(\frac{b+p}{b+p+n}\right)^{\xi}\left(1+\frac{\tau n}{p}\right)\right\}^{\ell}(1+\sigma)}\right] z^{p+n}. \tag{25}$$

Naturally, we have the following equation true

$$\sum_{k=p+n}^{\infty} \frac{k\left\{\left(\frac{b+p}{b+k}\right)^{\xi}\left(1+\frac{\tau(k-p)}{p}\right)\right\}^{\ell}(1+\sigma)}{2\sigma(p-\varphi)}\left[\prod_{i=1}^{q} \frac{2\sigma(p-\omega_i)}{2\sigma(p+n)\left\{\left(\frac{b+p}{b+p+n}\right)^{\xi}\left(1+\frac{\tau n}{p}\right)\right\}^{\ell}(1+\sigma)}\right] = 1. \tag{26}$$

This is exactly the proof of the theorem in full.

During the research and exploration of previous studies on this topic, we noticed that by assigning specific values to the parameters in the previous theorem, we obtaine results, and these results are nothing more than a generalization of studies presented in previous years by some distinguished researchers. The first of these results we obtained by fixing the values of $\omega_i = \omega, i = 1, 2, \dots q$ in the previous theorem in the following form:

Corollary 3.2. If we take the functions $f_i(z) \in \Re_p(\ell, \xi, b, p, \tau, n, \omega, \sigma), i = 1, 2, \dots, q$, then their quasi-Hadamard product $(f_1 * f_2 * f_3 * \dots * f_q)(z) \in \Re_p(\ell, \xi, b, p, \tau, n, \varphi, \sigma)$, this is only true when

$$\varphi = p - \frac{\{2\sigma(p-\omega)\}^{q}}{2\sigma\{(1+\sigma)(p+n)\}^{q-1}\left\{\left(\frac{b+p}{b+p+n}\right)^{\xi}\left(1+\frac{\tau n}{p}\right)\right\}^{\ell}}. \tag{27}$$

Functions of the form

$$f_i(z) = z^{p} - \frac{2\sigma(p-\omega)}{\left\{\left(\frac{b+p}{b+p+n}\right)^{\xi}\left(1+\frac{\tau n}{p}\right)\right\}^{\ell}(p+n)(1+\sigma)} z^{p+n}, i = 1, 2, \dots, q, \tag{28}$$

achieve extremity and sharpness of the result.

If we fix the values $\sigma = 1, \ell = 0$ in Theorem 3.1 directly, we get:

Corollary 3.3. If we take the functions $f_i(z) \in \Re_p(0, \xi, b, p, \tau, n, \omega_i, 1), i = 1, 2, \dots, q$, then their quasi-Hadamard product $(f_1 * f_2 * f_3 * \dots * f_q)(z) \in \Re_p(0, \xi, b, p, \tau, n, \varphi)$, this is only true when

$$\varphi = p - \frac{\prod_{i=1}^{q}(p - \omega_i)}{(p + n)^{q-1}}. \tag{29}$$

Functions of the form

$$f_i(z) = z^p - \frac{p - \omega_i}{p + n} z^{p+n}, i = 1, 2, \dots, q. \tag{30}$$

Achieve extremity and sharpness of the result.

Here, let's pause for a moment to clarify that this result is a generalization of a theorem presented in [24], and if we fix the value of $n = 1$ in Corollary 3.2, it is a generalization of a theorem presented in [16] as follows:

Corollary 3.4. If we take the functions $f_i(z) \in \Re_p(\ell, \xi, b, p, \tau, 1, \omega, \sigma), i = 1, 2, \dots, q$, then their quasi-Hadamard product $(f_1 * f_2 * f_3 * \dots * f_q)(z) \in \Re_p(\ell, \xi, b, p, \tau, 1, \varphi, \sigma)$, this is only true when

$$\varphi = p - \frac{\{2\sigma(p - \omega_i)\}^q}{2\sigma\{(1 + \sigma)(p + 1)\}^{q-1}\left\{\left(\frac{b+p}{b+p+1}\right)^{\xi}\left(1 + \frac{\tau}{p}\right)\right\}^{\ell}}. \tag{31}$$

Functions of the form

$$f_i(z) = z^p - \frac{2\sigma(p - \omega)}{\left\{\left(\frac{b+p}{b+p+1}\right)^{\xi}\left(1 + \frac{\tau}{p}\right)\right\}^{\ell}(p + 1)(1 + \sigma)} z^{p+1}, i = 1, 2, \dots, q. \tag{32}$$

Achieve extremity and sharpness of the result.

If we take the values of $n = \sigma = 1, \ell = 0$ in the Corollary 3.2, gives us

Corollary 3.5. If we take the functions $f_i(z) \in \Re_p(0, \xi, b, p, \tau, 1, \omega, 1), i = 1, 2, \dots, q$, then their quasi-Hadamard product $(f_1 * f_2 * f_3 * \dots * f_q)(z) \in \Re_p(0, \xi, b, p, \tau, 1, \varphi, 1)$, this is only true when

$$\varphi = p - \frac{(p - \omega_i)^q}{(p + 1)^{q-1}}. \tag{33}$$

Functions of the form

$$f_i(z) = z^p - \frac{(p - \omega_i)}{(p + 1)} z^{p+1}, i = 1, 2, \dots, q. \tag{34}$$

Achieve extremity and sharpness of the result.

It is a generalization of a theorem that has been presented in [16], [25].

Finally, we fix the values $p = \sigma = 1, \ell = 0$ in Corollary 3.2, which in turn gives us:

Corollary 3.6. If we take the functions $f_i(z) \in \Re_p(0, \xi, b, 1, \tau, n, \omega, 1), i = 1, 2, \dots, q$, then their quasi-Hadamard product $(f_1 * f_2 * f_3 * \dots * f_q)(z) \in \Re_p(0, \xi, b, 1, \tau, n, \varphi, 1)$, this is only true when

$$\varphi = 1 - \frac{(1-\omega)^q}{(1+n)^{q-1}}. \tag{35}$$

Functions of the form

$$f_i(z) = z - \frac{(1-\omega)}{(1+n)} z^{1+n}, i = 1, 2, \dots, q. \tag{36}$$

Achieve extremity and sharpness of the result.

We concluded that this result is a generalization of a result previously studied in [17].

We would like to introduce another important property of the quasi-Hadamard product and present the proof, discussing its results and linking it to previous studies related to it, since from our point of view, reviewing and understanding the history of the subject is the cornerstone for constructing and arriving at new, original results.

Theorem 3.7. If we take the functions $f_i(z)$ in the class $\Re_p(\ell, \xi, b, p, \tau, n, \omega_i, \sigma)$ for each $i = 1, 2, \dots, q$, then the function

$$g(z) = z^p - \sum_{k=p+n}^{\infty} \left\{ \sum_{i=1}^{q} a_{k,i}^2 \right\} z^k,$$

is an element of the class $\Re_p(\ell, \xi, b, p, \tau, n, \varphi, \sigma)$. This occurs when

$$\varphi = p - \frac{q\{2\sigma(p-\omega_e)\}^2}{2\sigma(1+\sigma)(p+n)\left\{\left(\frac{b+p}{b+p+n}\right)^{\xi}\left(1+\frac{\tau n}{p}\right)\right\}^{\ell}}, \tag{37}$$

ω_e is the minimum of the values $\omega_1, \omega_2, \dots, \omega_q$

and we obtain the peak of the result for the functions of the form:

$$f_i(z) = z^p - \frac{2\sigma(p-\omega_i)}{\left\{\left(\frac{b+p}{b+p+n}\right)^{\xi}\left(1+\frac{\tau n}{p}\right)\right\}^{\ell}(p+n)(1+\sigma)} z^{p+n}, i = 1, 2, \dots, q. \tag{38}$$

Proof. Based on the hypothesis and the coefficients bounds, the following inequality is true

$$\sum_{k=p+n}^{\infty} \left(\frac{\left\{\left(\frac{b+p}{b+k}\right)^{\xi}\left(1+\frac{\tau(k-p)}{p}\right)\right\}^{\ell} k(1+\sigma)}{2\sigma(p-\omega_i)} \right)^2 a_{k,i}^2$$

$$\leq \left(\sum_{k=p+n}^{\infty} \frac{\left\{\left(\frac{b+p}{b+k}\right)^{\xi}\left(1+\frac{\tau(k-p)}{p}\right)\right\}^{\ell} k(1+\sigma)}{2\sigma(p-\omega_i)} a_{k,i} \right)^2$$

$$\leq 1.$$

In other words

$$\sum_{k=p+n}^{\infty} \frac{1}{q}\left(\frac{\left\{\left(\frac{b+p}{b+k}\right)^{\xi}\left(1+\frac{\tau(k-p)}{p}\right)\right\}^{\ell} k(1+\sigma)}{2\sigma(p-\omega_i)}\right)^2 \left(\sum_{i=1}^{q} a_{k,i}^2\right) \leq 1. \quad (39)$$

Our primary goal is to determine the maximum value of φ so that it is the largest, and verify

$$\sum_{k=p+n}^{\infty} \left(\frac{\left\{\left(\frac{b+p}{b+k}\right)^{\xi}\left(1+\frac{\tau(k-p)}{p}\right)\right\}^{\ell} k(1+\sigma)}{2\sigma(p-\varphi)}\right)\left(\sum_{i=1}^{q} a_{k,i}^2\right) \leq 1. \quad (40)$$

Finally, we have

$$\varphi \leq p - \frac{q\left(2\sigma(p-\omega_e)\right)^2}{2\sigma\left\{\left(\frac{b+p}{b+k}\right)^{\xi}\left(1+\frac{\tau(k-p)}{p}\right)\right\}^{\ell} k(1+\sigma)}, \quad (41)$$

For each $k \geq p + n$, and

$$\varphi \leq p - \frac{q\left(2\sigma(p-\omega_e)\right)^2}{2\sigma\left\{\left(\frac{b+p}{b+k}\right)^{\xi}\left(1+\frac{\tau(k-p)}{p}\right)\right\}^{\ell} (p+n)(1+\sigma)}, \quad (42)$$

Thus, the proof has been successfully completed.

In the same view as the Theorem 3.1 and its results, we will give specific values to the constants in the previous theorem to give us results, some of which are nothing but generalizations of theorems or results already proposed and studied in previous research, which we will present more precisely as follows:

In the previous theorem, if we set the values at $\omega_i = \omega$, we arrive at:

Corollary 3.8. If we take the functions $f_i(z)$ in the class $\Re_p(\ell, \xi, b, p, \tau, n, \omega, \sigma)$ for each $i = 1, 2, \ldots, q$, then the function

$$g(z) = z^p - \sum_{k=p+n}^{\infty} \left\{\sum_{i=1}^{q} a_{k,i}^2\right\} z^k,$$

is an element of the class$\Re_{p}(\ell,\xi,b,p,\tau,n,\varphi,\sigma)$. This occurs when

$$\varphi = p - \frac{q\{2\sigma(p-\omega)\}^2}{2\sigma(1+\sigma)(p+n)\left\{\left(\frac{b+p}{b+p+n}\right)^{\xi}\left(1+\frac{\tau n}{p}\right)\right\}^{\ell}}, \tag{43}$$

and we obtain the peak of the result for the functions of the form:

$$f_i(z) = z^p - \frac{2\sigma(p-\omega)}{\left\{\left(\frac{b+p}{b+p+n}\right)^{\xi}\left(1+\frac{\tau n}{p}\right)\right\}^{\ell}(p+n)(1+\sigma)} z^{p+n}, i = 1,2,\dots,q. \tag{44}$$

Also, if we set the values at $\sigma = 1, \ell = 0$ in Theorem 3.7, we conclude that

Corollary 3.9. If we take the functions $f_i(z)$ in the class $\Re_{p}(0,\xi,b,p,\tau,n,\omega,1)$ for each $i = 1,2,\dots,q$, then the function

$$g(z) = z^p - \sum_{k=p+n}^{\infty}\left\{\sum_{i=1}^{q} a_{k,i}^2\right\} z^k,$$

is an element of the class$\Re_{p}(0,\xi,b,p,\tau,n,\varphi,1)$. This occurs when

$$\varphi = p - \frac{q(p-\omega_e)^2}{(p+n)}, \tag{45}$$

ω_e is the minimum of the values $\omega_1,\omega_2,\dots,\omega_q$,

and we obtain the peak of the result for the functions of the form:

$$f_i(z) = z^p - \frac{p-\omega_i}{p+n} z^{p+n}, i = 1,2,\dots,q. \tag{46}$$

This result is a generalization of a theorem that was proven in [24]

In Corollary 3.8, if we fix the values $\sigma = 1$, we get:

Corollary 3. 10: If we take the functions $f_i(z)$ in the class $\Re_{p}(\ell,\xi,b,p,\tau,n,\omega,1)$ for each $i = 1,2,\dots,q$, then the function

$$g(z) = z^p - \sum_{k=p+n}^{\infty}\left\{\sum_{i=1}^{q} a_{k,i}^2\right\} z^k,$$

is an element of the class$\Re_{p}(\ell,\xi,b,p,\tau,n,\varphi,1)$. This occurs when

$$\varphi = p - \frac{q(p-\omega)^2}{(p+n)\left\{\left(\frac{b+p}{b+p+n}\right)^{\xi}\left(1+\frac{\tau n}{p}\right)\right\}^{\ell}}, \tag{47}$$

and we obtain the peak of the result for the functions of the form:

$$f_i(z) = z^p - \frac{p-\omega}{p+n} z^{p+n}, i = 1,2,\dots,q. \tag{48}$$

Also in Corollary 3.8, if we fix the value $n = 1$, we will get the follows corollary

Corollary 3.11. If we take the functions $f_i(z)$ in the class $\Re_p(\ell, \xi, b, p, \tau, 1, \omega, \sigma)$ for each $i = 1,2,\dots,q$, then the function

$$g(z) = z^p - \sum_{k=p+n}^{\infty} \left\{ \sum_{i=1}^{q} a_{k,i}^2 \right\} z^k,$$

is an element of the class $\Re_p(\ell, \xi, b, p, \tau, 1, \varphi, \sigma)$. This occurs when

$$\varphi = p - \frac{q\{2\sigma(p-\omega)\}^2}{2\sigma(1+\sigma)(p+1)\left\{\left(\frac{b+p}{b+p+1}\right)^{\xi}\left(1+\frac{\tau}{p}\right)\right\}^{\ell}}, \tag{49}$$

and we obtain the peak of the result for the functions of the form:

$$f_i(z) = z^p - \frac{2\sigma(p-\omega)}{\left\{\left(\frac{b+p}{b+p+1}\right)^{\xi}\left(1+\frac{\tau}{p}\right)\right\}^{\ell}(p+1)(1+\sigma)} z^{p+1}, i = 1,2,\dots,q. \tag{50}$$

Again in Corollary 3.8 , if we fix the values $\sigma = n = 1, \ell = 0$, we get

Corollary 3.12. If we take the functions $f_i(z)$ in the class $\Re_p(0, \xi, b, p, \tau, 1, \omega, 1)$ for each $i = 1,2,\dots,q$, then the function

$$g(z) = z^p - \sum_{k=p+n}^{\infty} \left\{ \sum_{i=1}^{q} a_{k,i}^2 \right\} z^k,$$

is an element of the class$\Re_p(0, \xi, b, p, \tau, 1, \varphi, 1)$. This occurs when

$$\varphi = p - \frac{q(p-\omega)^2}{(p+n)}, \tag{51}$$

and we obtain the peak of the result for the functions of the form:

$$f_i(z) = z^p - \frac{2\sigma(p-\omega)}{(p+1)} z^{p+n}, i = 1,2,\dots,q. \tag{52}$$

It is exactly a generalization of a result presented in [16].

And in Corollary 3.8 , if we fix the values $\sigma = p = 1, \ell = 0$, we get:

Corollary 3.13. If we take the functions $f_i(z)$ in the class $\Re_p(0,\xi,b,1,\tau,n,\omega,1)$ for each $i = 1,2,\dots,q$, then the function

$$g(z) = z^p - \sum_{k=p+n}^{\infty}\left\{\sum_{i=1}^{q} a_{k,i}^2\right\} z^k,$$

is an element of the class$\Re_p(0,\xi,b,1,\tau,n,\varphi,1)$. This occurs when

$$\varphi = 1 - \frac{q(1-\omega)^2}{1+n}, \quad (53)$$

and we obtain the peak of the result for the functions of the form:

$$f_i(z) = z - \frac{1-\omega}{1+n} z^{1+n}, i = 1,2,\dots,q\,. \quad (54)$$

Finally, this result is a generalization of the theorem presented in [17].

In the end, we hope that we have presented this article accurately and understandably, and the previous studies related to it as well as possible.

Conclusions:
In this research article, we study some of the important properties of the quasi-Hadamard product of multivalent functions with negative coefficients contained in the new distinctive class defined by the generalized linear operator, which implicitly contains the generalized Komatu integral operator, and also discuss and prove its basict geometric properties within the unit disc $\mathcal{U}$.

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