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Minimal Intersection Graph of Submodules of Modules

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Abstract:

Let R be a commutative ring with $1 \neq 0$, and M be an R-module. The minimal intersection graph of M, denoted by $\Gamma(M)$, is a simple undirected graph whose vertices are proper non-zero submodules of M and any two distinct vertices F and Q are adjacent if and only if $F \cap Q$ be an minimal (= simple) submodule of M. In this article, we explore connectedness, clique number, split character, planarity, independence number and domination number of $\Gamma(M)$.

Keywords: Minimal intersection graph, Module, clique number, independence number, domination number.

1-Introduction

At first the idea of studying the intersection graph of algebraic structures, was appeared by J. Bosak in [8], where defined the intersection graph of proper subsemigroups of a semigroup in 1964. Inspired by his work, many mathematicians have been attracted by this topic and considered the intersection graph of algebraic structures, see for instance [3-6]. In 2015, S. Akbari et. al. in [1], defined the notion of the intersection graph of submodules of a module. In [2], the second author introduced the maximal submodule graph of a module. Recently, in 2021 [9], B. Barman, K.K. Rajkhowa introduced the notion of intersection minimal ideal graph of a ring R, the vertices set represent all non-trivial ideals of R, with edges connecting every pair of distinct vertices. Here, our main goal is to relate the combinatorial properties of the intersection graph $\Gamma(\mathcal{M})$ to the algebraic properties of the module M.

Here ,we study the smallest intersection graph corresponding to a module $\mathcal{M}$, designated as $\mathbb{\Gamma}(\mathcal{M})$. Let G a minimal unoriented graph along here vertex set $V(G)$ and edge set $E(G)$. If G does not contain any edge, then G is called null graph. The neighborhood in $\text{Ꝑ} \in V(G)$ is denoted by $N(\text{Ꝑ})$. By K_n, we mean the complete graph with n vertices. If the vertices of G can be partitioned into two disjoint sets W_1 and W_2 with every vertex of W_1 is adjacent to any vertex of W_1 and no two vertices belonging to same set are adjacent, then G is called a complete bipartite graph. As $|W_1| = m, |W_2| = n$, the complete bipartite graph is denoted by $K_{m,n}$. If one of the partite sets contains exactly one element, then the graph becomes a star graph. If G graph does not have K_5 or $K_{3,3}$ as its subgraph, then G is planar [7]. The girth of , denoted is girth(G) (or gr(G)), is the length of the shortest cycle in G. If there exists a path between any two distinct vertices, then G is connected. If υ, ν are two distinct vertices of G, then $d(\upsilon, \nu)$ is the length of the shortest path from υ to ν and $d(\upsilon, \nu) = \infty$, if there does not exist a path between υ and ν. The maximum distance among all the distances between every pair of vertices of G is called the diameter of G, denoted by diam(G). A clique is a complete subgraph of G. The number of vertices in the largest clique of G is named the clique number of G beside is denoted by $\omega(G)$. A subset B of $V(G)$ is called the independent set if no two vertices of B are adjacent. The cardinality of the largest independent set is said independence number and it is denoted by $\alpha(G)$. If $V(G)$ can be partitioned in an independent set and a clique then G is said to be an split. A set $D \subset V(G)$ said to be a dominating set if every vertex does not in B is adjacent to at least one of the members of B. The cardinality of smallest dominating set be the domination number of the graph G and is denoted by $\gamma(G)$. Let $\mathcal{M}$ be an R-module. The collection of all minimal (= simple) submodules of $\mathcal{M}$ is denoted by $min(\mathcal{M})$ and the collection of all maximal submodules of $\mathcal{M}$ be denoted by $max(\mathcal{M})$, in the corresponding order. The sum for each minimal (simple) submodules of M is called the socle of $\mathcal{M}$ which is denoted by $Soc(\mathcal{M})$. A submodule E of $\mathcal{M}$ is named essential if $E \cap F \neq 0$ for all $F \leq \mathcal{M}$. For any two submodules X and Y of $\mathcal{M}$, we have $\frac{X+Y}{X} \cong \frac{Y}{X \cap Y}$. A module $\mathcal{M}$ is an Artinian precisely when there exists no infinite strictly decreasing sequence of submodules. In Artinian module, every submodule contains a minimal submodule. In section 2, we study the connectivity property of $\Gamma(\mathcal{M})$. In section 3, we study on the clique and independence numbers of $\mathbb{\Gamma}(\mathcal{M})$. Any undefined terminology in modules can be found in [10, 11] and any undefined terminology in graphs can be found in [7].

2- Connectedness of $\mathbb{\Gamma}(\mathcal{M})$

In his work, R is a commutative ring with identity and $\mathcal{M}$ is a unitary R-module. By a non-trivial submodule of $\mathcal{M}$, we mean is a nonzero proper submodule of $\mathcal{M}$. In this part, we study the connectivity property of $\mathbb{\Gamma}(\mathcal{M})$.

Definition 2.1: Let $\mathcal{M}$ be a module over a ring R. The minimal intersection graph of $\mathcal{M}$, indicated by $\mathbb{\Gamma}(\mathcal{M})$ is a graph where each vertex represents a non-trivial submodule of $\mathcal{M}$ and there is an edge between two vertices U and V if and only if $U \cap V$ is a non-zero minimal (=simple) submodule of $\mathcal{M}$.

Lemma 2.2: The following hold in $\mathbb{\Gamma}(\mathcal{M})$:

1. Every non-minimal submodule for $\mathcal{M}$ is adjacent to at least one of the minimal submodules of $\mathcal{M}$.
2. If $Soc(M) \neq M$, then every member of $min(\mathcal{M})$ is adjacent to $Soc(\mathcal{M})$.

Remark 2.3: If $p, r \in min(\mathcal{M})$, then it is easy to observe that p and r are not adjacent in $\mathbb{\Gamma}(\mathcal{M})$. Thus the subgraph induced by the minimal submodules of $\mathcal{M}$ is null.

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Theorem 2.4: If υ, ν, r are distinct vertices in $\mathbb{\Gamma}(\mathcal{M})$ with $r \in min(\mathcal{M})$ and $\upsilon \cap \nu \neq r$, then the following hold:

i. $r \in N(\upsilon \cap \nu)$ if and only if $r \in N(\upsilon) \cap N(\nu)$.
ii. If $Soc(\mathcal{M}) \subsetneq \upsilon$, then $r \in N(\upsilon)$.
iii. If $\upsilon \notin min(\mathcal{M})$ and $\upsilon \subsetneq \nu$, then $\upsilon \notin N(\nu)$.

Proof: (i) If $r \in N(\upsilon) \cap N(\nu)$, then $r \cap \upsilon = r = r \cap \nu$. Clearly, $r \subsetneq \upsilon \cap \nu$, which infers that $r \in (\upsilon \cap \nu)$. Similarly the proof for the opposite direction can be established.
(ii) Since $r \subset Soc(\mathcal{M})$ beside $c(\mathcal{M}) \subsetneq \upsilon$, $r \subsetneq \upsilon$. This results in $r \cap \upsilon = r$. Therefore $r \in N(\upsilon)$.
(iii) If $\upsilon \subsetneq \nu$, now $\upsilon \cap \nu = \upsilon$. As $\upsilon \notin min(\mathcal{M})$, we get $\upsilon \notin N(\nu)$.

Proposition 2.5: If $T, F \notin min(\mathcal{M})$ and $\{T, F\} \in E(\mathbb{\Gamma}(\mathcal{M}))$, then there exists a unique $r \in min(\mathcal{M})$ with $r \in N(T) \cap N(F)$.

Proof: Suppose $\{T, F\} \in E(\mathbb{\Gamma}(M))$, then $T \cap F \in min(\mathcal{M})$. Clearly, $T \cap F$ is adjacent to both T so F . If it holes, suppose that there existence of an $r \in min(\mathcal{M})$ by $r \neq T \cap F$ and r is adjacent to both T and F. By Theorem 2.4, it is clear that $r \in N(T \cap F)$. So, $r \subsetneq T, F$. This gives $r \subset T \cap F$. Since $T \cap F$ is minimal, $r = T \cap F$. This completes the proof. □

Theorem 2.6: Every non-zero proper submodule of $\mathcal{M}$ is minimal if and only if $\mathbb{\Gamma}(\mathcal{M})$ is null graph.

Proof: Suppose that for each non-zero suitable submodule in $\mathcal{M}$ is minimal. Let us take two vertices T, F in $\mathbb{\Gamma}(\mathcal{M})$. Obviously, $T \cap F = 0$. So, T beside F are not next to each other in $\mathbb{\Gamma}(\mathcal{M})$. Since T and F are general, we state that $\Gamma(\mathcal{M})$ has no elements. In reverse, suppose that $\mathbb{\Gamma}(\mathcal{M})$ has no elements so $B \in V(\mathbb{\Gamma}(\mathcal{M}))$. Let $B \notin mir(\mathcal{M})$. Since $\mathcal{M}$ is Artinian, there is some $s \in min(\mathcal{M})$ with $s \subsetneq B$. This gives that s & B are adjacent, which contradicts the null character of $\mathbb{\Gamma}(\mathcal{M})$. Consequently any submodule of $\mathcal{M}$ is minimal. Hence the theorem.

Proposition 2.7: The graph $\mathbb{\Gamma}(\mathcal{M})$ is connected if and only if the sum of any two distinct minimal submodules of $\mathcal{M}$ is not $\mathcal{M}$, or $|min(\mathcal{M})| = 1$.

Proof: Suppose that $|min(\mathcal{M})| = 1$, therefore it is clear that $\mathbb{\Gamma}(\mathcal{M})$ is connected. Assume that $|min(\mathcal{M})| \neq 1$, and the sum of any two distinct minimal submodules of $\mathcal{M}$ is not $\mathcal{M}$. Take two vertices X and Y for $\mathbb{\Gamma}(\mathcal{M})$. Let $\{X, Y\} \in E(\mathbb{\Gamma}(\mathcal{M}))$, is now $X - Y$ a path. Suppose $\{X, Y\} \notin E(\mathbb{\Gamma}(\mathcal{M}))$. Then either $Z \subsetneq X \cap Y$ for some $Z \in min(\mathcal{M})$, or $X \cap Y = 0$. If $Z \subsetneq X \cap Y$, now $X - Z - Y$ is a path of $\mathbb{\Gamma}(\mathcal{M})$. If $X \cap Y = 0$, then the following three cases arise.
Case 1: Suppose X and Y are both minimal. Then $X - (X + Y) - Y$ is a path in $\mathbb{\Gamma}(\mathcal{M})$.
Case 2: If exactly one of X and Y is minimal, then without loss of generality, assume that $X \in min(M)$ and $Y \notin min(M)$. Since $\mathcal{M}$ is Artinian, there exists some $r \in min(\mathcal{M})$ such that $R \subsetneq Y$. Thus, we get the path $X - (R + X) - R - Y$.
Case 3: If both X and Y are not minimal, it follows that there is $R_1, R_2 \in min(\mathcal{M})$ such that $R_1 \subsetneq X$ also $R_2 \subsetneq Y$, respectively. If $R_1 = R_2$, then $X - R_1 - Y$ is a path. If $R_1 \neq R_2$, then $X - R_1 - (R_1 + R_2) - R_2 - Y$ is apath. Hence we deduce that $\mathbb{\Gamma}(\mathcal{M})$ is connected.

Conversely, consider that $\mathbb{\Gamma}(\mathcal{M})$ is connected. If it can be done, suppose that there exist two minimal submodules $T_1$and T_2 such that $T_1 + T_2 = \mathcal{M}$. Clearly, $\mathcal{M} = T_1 \oplus T_2$. Also, $\frac{\mathcal{M}}{T_1} \cong T_2$ and $\frac{\mathcal{M}}{T_2} \cong T_1$. Since $\mathcal{M}$ is a commutative Artinian module, T_1 and $T_2$are minimal as well as maximal submodules of $\mathcal{M}$. Assume that T_1 is neighboring some $\mathcal{S} \in V(\mathbb{\Gamma}(\mathcal{M}))$. now $T_1 \cap \mathcal{S} = T_1$, this means that $T_1 \subsetneq \mathcal{S}$. Since T_1 is maximal, we obtain $T_1 = \mathcal{S}$. This asserts that $T_1$is an isolated vertex a contradiction. This completes the proof.

Proposition 2.8: If $\mathbb{\Gamma}(\mathcal{M})$ is a connected graph, then $diam(\mathbb{\Gamma}(\mathcal{M})) \leq 4$.

Proof: Consider that $\mathbb{\Gamma}(\mathcal{M})$ is connected. If $min(\mathcal{M}) = 1$, therefore, clearly $diam(\mathbb{\Gamma}(\mathcal{M})) = 2$. Suppose that $|min(\mathcal{M})| \neq 1$. Assume $\{B, H\} \notin E(\mathbb{\Gamma}(\mathcal{M}))$. Therefor either $R \subsetneq B \cap H$ for some $R \in min(\mathcal{M})$ or $B \cap H = 0$. In the same way, as of Proposition 2.7, we can also determine that $d(B, H) = 2$ or 4. Hence $diam(\mathbb{\Gamma}(\mathcal{M})) \leq 4$. □

Theorem 2.9: If $\mathcal{S} = \mathcal{S}_1 \times \mathcal{S}_2 \times \ldots \times \mathcal{S}_r$, where $\mathcal{S}_i$ is a simple module, for $i = 1, 2, \ldots, r$. Then $diam(\mathbb{\Gamma}(\mathcal{S})) = 2$.

Proof: Let $\mathcal{S} = \mathcal{S}_1 \times \mathcal{S}_2 \times \ldots \times \mathcal{S}_r$, we are $\mathcal{S}_i$ is simple module, for $i = 1,2,\ldots,r$. Any submodule of $\mathcal{S}$ is of the from $A = \prod_{i=1}^{r} G_i$ where $G_i = 0$ or $\mathcal{S}_i$ and the minimal submodules of $\mathcal{S}$ is of the form $R_k = \prod_{i=1}^{r} G_i$ where $G_i = 0$ in $i \neq k$ beside $G_k = \mathcal{S}_k$. That $\mathcal{S}$ contains n minimal submodules. Think about tow not adjacent vertices L , T for $\mathbb{\Gamma}(\mathcal{S})$. Let L also T both include the same minimal submodule, then $d(T,F) = 2$. If not, then there exist $R_i$and R_j with $R_i \subset L, R_j \subset T, R_i \not\subset T$ and $R_j \not\subset L$. Now we consider the submodule $h = \prod_{l=1}^{r} G_l$, where $G_l = \mathcal{S}_l$, for $l = i,j$ plus 0 or else. This provides the way $L - h - T$. Hence, $diam(\mathbb{\Gamma}(\mathcal{S})) = 2$.

Theorem 2.10: If $Soc(\mathcal{M}) \neq \mathcal{M}$, then $girth(\mathbb{\Gamma}(\mathcal{M})) = 3$ or 4 whenever $\mathbb{\Gamma}(\mathcal{M})$ contains a cycle.

Proof: Let $Soc(\mathcal{M}) \neq \mathcal{M}$. Suppose that $\{B,H\} \in E(\mathbb{\Gamma}(\mathcal{M}))$. Clearly, at least one of B or H does not belong to $min(\mathcal{M})$. If $B,H \notin min(\mathcal{M})$, then $B - B\cap H - H - B$ is a cycle. In this case, $girth(\mathbb{\Gamma}(\mathcal{M})) = 3$. Consider that one of B or H be minimal. Without loss in general terms, take $B \in min(\mathcal{M}), H \notin min(\mathcal{M})$. In this case, there is some $\mathrm{Ƥ} \in min(\mathcal{M})$ So that $\mathrm{Ƥ} \subsetneq H$. Henceforth, we obtain the cycle $B - H - \mathrm{Ƥ} - Soc(\mathcal{M}) - B$. If so, $girth(\mathbb{\Gamma}(\mathcal{M})) = 4$. Their proof is complete. □

Theorem 2.11: Assume that $\mathcal{S} = \mathcal{S}_1 \times \mathcal{S}_2 \times \ldots \times \mathcal{S}_r$, where $\mathcal{S}_i$ is a simple module for $i = 1,2,\ldots,r$, then $girth(\mathbb{\Gamma}(\mathcal{M})) = 3$.

Proof: Let $\mathcal{S} = \mathcal{S}_1 \times \mathcal{S}_2 \times \ldots \times \mathcal{S}_r$, wherever $\mathcal{S}_i$ a simple module for $i = 1,2,\ldots,r$, any submodule of $\mathcal{S}$ is *of* the from $A = \prod_{i=1}^{r} G_i$ where$G_i = 0$ or $\mathcal{S}_i$. Let us consider the submodule $B = \prod_{i=1}^{r} G_i$ where $G_i = \mathcal{S}_i$, for $i = 1,2$ and otherwise $G_i = 0$; $C = \prod_{i=1}^{r} G_i$ where $G_i = \mathcal{S}_i$, for $i = 1,3$ beside otherwise $G_i = 0$; $D = \prod_{i=1}^{r} G_i$, where $G_i = \mathcal{S}_i$, for $i = 2,3$ in any other way $G_i = 0$. So, for any minimal submodule of F is in the form of $R_k = G_i$ in which $G_i = 0$, for $i \neq k$ and $G_k = \mathcal{S}_k$. So. $\mathcal{S}$ has r minimal submodules. Because $h\cap C = R_1, C\cap D = R_3$ and $h\cap D = R_2$, thus we get the cycle $h - C - D - h$. This concludes that $girth(\mathbb{\Gamma}(\mathcal{M}))$ =3.

Theorem 2.12: If $\mathbb{\Gamma}(\mathcal{M})$ is a complete, then $\mathcal{M}$ is module with $|min(\mathcal{M})| = 1$.

Proof: Assume $\mathbb{\Gamma}(\mathcal{M})$ is complete. If $\mathrm{Ƥ}, q \in min(\mathcal{M})$ and $\mathrm{Ƥ} \neq q$, now $\mathrm{Ƥ}\cap q = 0$ .This implies that $|min(\mathcal{M})| = 1$. □

Remark 2.13: We observe that D_{p_r} has exactly one minimal submodule, but $\mathbb{\Gamma}(D_{p_r})$ is not complete. Hence the converse of theorem 2.12, does not hold.

Theorem 2.14: If a chain is formed using the submodules of $\mathcal{M}$, then $\mathbb{\Gamma}(\mathcal{M})$ is a star.

Proof: If a chain is formed the submodules in $\mathcal{M}$, hence there is a $\mathrm{Ƥ} \in min(\mathcal{M})$ such that $\{\mathrm{Ƥ},v\} \in E(\mathbb{\Gamma}(\mathcal{M}))$, for evey $v \in V(\mathbb{\Gamma}(\mathcal{M}))$. should $B,H \in V(\mathbb{\Gamma}(\mathcal{M}))$ and $B \neq \mathrm{Ƥ}, H \neq \mathrm{Ƥ}$, as a result, it is apparent that B, H are not adjacent. Hence $\mathbb{\Gamma}(\mathcal{M})$ is star. □

Proposition 2.15: If $Soc(\mathcal{M}) \neq \mathcal{M}$. Then $\mathbb{\Gamma}(\mathcal{M})$ is compete bipartite iff any submodule of R is either essential or minimal.

Proof: Let v_1 and v_2 be the set of minimal submodules and essential submodules of $\mathcal{M}$, respectively. If $p,q \in v_1$, then $p \cap v = 0$. Thus any two vertices of v_1 are not adjacent. Also, if $S,H \in v_2$, then $Soc(\mathcal{M}) \subset S \cap H$. So any two vertices of V_2 are also not adjacent. Again, using Proposition 2.3, we get that every vertex in v_1 is adjacent to each vertex in ν_2. Thus $\mathbb{\Gamma}(\mathcal{M})$ is a complete bipartite graph. For the opposite direction, assume that $\mathbb{\Gamma}(\mathcal{M})$ is a complete bipartite graph. It is easy to prove that the vertex set $\nu(\mathbb{\Gamma}(M))$ can be partitioned into the two disjoint subsets $min(\mathcal{M})$ and $\{\upsilon \in \nu(\mathbb{\Gamma}(\mathcal{M})) : Soc(\mathcal{M}) \subset \upsilon\}$. This completes the proof. □

Theorem 2.16: If the sum of any two distinct minimal submodules of $\mathcal{M}$ is not $\mathcal{M}$, and $\mathcal{S}$ is a cut vertex of $\mathbb{\Gamma}(\mathcal{M})$. then $\mathcal{S} = B + H$. for some $B, H \in min(\mathcal{M})$.

Proof: If $\mathcal{S} \in min(\mathcal{M})$, as a result is clear. If $\mathcal{S} \notin min(\mathcal{M})$. Suppose υ, ν are two vertices in separate component C_1 beside C_2 of $\nu(\mathbb{\Gamma}(\mathcal{M})\backslash\{\mathcal{S}\})$, in the same order. We derive the following results:

Case I: If $\upsilon, \nu \in min(\mathcal{M})$, then $\upsilon + \nu \in N(\upsilon) \cap N(\nu)$. Thus $\mathcal{S} = \upsilon + \nu$, while $\mathcal{S}$ is a cut vertex.
Case II: If $\upsilon \in min(\mathcal{M})$ & $\nu \notin min(\mathcal{M})$, then there is some $B \in min(\mathcal{M})$ by $B \subsetneq \nu$. Thus B, ν belong to the same parts C_2. since $B + \upsilon \in N(B) \cap N(\upsilon)$, and all for B beside υ belong to two dissimilar components, also $\mathcal{S} = \upsilon + B$.
Case III: If $\upsilon, \nu \notin min(\mathcal{M})$, then there is some $B, H \in min(\mathcal{M})$ having $B \subsetneq \upsilon$, $H \subsetneq \nu$. Here, υ and B belong to the component $C_1$and ν and H associated with the other component C_2. As $B + H \in N(B) \cap N(H)$, beside B, H belong to C_2, therefore $\mathcal{S} = B + H$. The proof is complete.

3- Independence number, Clique number and Planarity of $\mathbb{\Gamma}(\mathcal{M})$

Theorem 3.1: In $\mathbb{\Gamma}(\mathcal{M})$, a clique is contained in the subgraph induced by $\{\upsilon \in V(\mathbb{\Gamma}(\mathcal{M})) : \nu \subset \upsilon\}$, for some $\nu \in min(\mathcal{M})$.

Proof: Assume that C be a clique of $\mathbb{\Gamma}(\mathcal{M})$. Since no two different minimal submodules be neighboring in $\mathbb{\Gamma}(\mathcal{M})$. So C have at most one minimal submodule. The complete-ness of C and Theorem 2.5, since that there is a unique $\nu \in min(\mathcal{M})$ in order that C is a subgraph induced by $\{\upsilon \in V(\mathbb{\Gamma}(\mathcal{M})) : \nu \subset \upsilon\}$. Hence the theorem. □

Theorem 3.2: If $\mathbb{\Gamma}(\mathcal{M})$ is not empty and $V(\mathbb{\Gamma}(\mathcal{M})) = min(\mathcal{M}) \cup max(\mathcal{M})$, then $\mathbb{\Gamma}(\mathcal{M})$ is split.

Proof: Consider the subgraph caused by $max(\mathcal{M})$ of $\mathbb{\Gamma}(\mathcal{M})$. Let $\upsilon, \nu \in max(\mathcal{M})$ with $\upsilon \neq \nu$. If it can be done, consider $\upsilon \cap \nu = 0$ now $\frac{\mathcal{M}}{\upsilon} \cong \nu$ beside $\frac{\mathcal{M}}{\nu} \cong \upsilon$. Thus simple modules [10] also υ, ν is minimal. Via Th. 5, $\mathbb{\Gamma}(\mathcal{M})$ is not empty, inconsistency. Then $\upsilon \cap \nu \neq 0$. Clearly $\upsilon \cap \nu \notin max(\mathcal{M})$. As a result $\upsilon \cap \nu \in min(\mathcal{M})$. This, the induced subgraph of $max(\mathcal{M})$ be complete. Also, by Remark 2.3, the induced sub graph of $min(\mathcal{M})$ is empty. Thus $\mathbb{\Gamma}(\mathcal{M})$ is split.

Theorem 3.3: If $v(\mathbb{\Gamma}(\mathcal{M})) = min(\mathcal{M}) \cup max(\mathcal{M})$ and $|max(\mathcal{M})| \leq 3$, then $\mathbb{\Gamma}(\mathcal{M})$ is planar.

Proof: If $v(\mathbb{\Gamma}(\mathcal{M})) = min(\mathcal{M}) \cup max(\mathcal{M})$, thus, as for Theorem 3.2, $\mathbb{\Gamma}(\mathcal{M})$ a split graph. As $|max(\mathcal{M})| \leq 3$, every subgraph included using 5 vertices be non-complete. Then, $\mathcal{S}_5$ be not contained of $\mathbb{\Gamma}(\mathcal{M})$. If feasible, if $\mathcal{S}_{3,3}$ is included in $\mathbb{\Gamma}(\mathcal{M})$ with part set W_1=$\{v_1, v_2, v_3\}$ and $W_2 = \{v_1, v_2, v_3\}$. It is clear that either $W_1 \subset min(\mathcal{M})$ or $W_2 \subset min(\mathcal{M})$. If we take $W_1 \subset min(\mathcal{M})$, then $W_2 \subset max(\mathcal{M})$, this contradiction the fact that any two maximal submodules are next to each other. Hence, $\mathbb{\Gamma}(\mathcal{M})$ is a planar graph. □

Theorem 3.4: If $|min(\mathcal{M})|$ is finite for an Artinian module $\mathcal{M}$, then $\alpha(\mathbb{\Gamma}(\mathcal{M})) = |min(\mathcal{M})|$.

Proof: Assume $min(\mathcal{M}) = \{m_1, m_2, \dots, m_r\}$. Clearly, $min(\mathcal{M})$ is a independent set, by using Remark 2.3. Therefor, $r \leq \alpha(\mathbb{\Gamma}(\mathcal{M}))$. Since $S = \{v_1, v_2, \dots, , v_l\}$ is a maximal independent set. So, $\alpha\big(\mathbb{\Gamma}(\mathcal{M})\big) = l$. For any $X \in B$, there exists some $m_i \in min(\mathcal{M})$ such that $m_i \subset X$. If $l > n$, then by Pigeonhole principle, there exist at least two vertices $v_i, v_j \in S$ which contain the same minimal submodule. This implies that v_i, v_j are adjacent, a contradiction to the reality thus S be independent set. Hence $l = n$, that is $\alpha\big(\mathbb{\Gamma}(\mathcal{M})\big) = n$.

Theorem 3.5: If $\mathcal{M}$ is an Artinian module with a unique minimal submodule and S is simple module, then $\gamma(\mathbb{\Gamma}(\mathcal{M} \times \mathcal{S})) = 1$.

Proof: It is clear.

Proposition 3.6: Let $\mathcal{S}_1$and $\mathcal{S}_2$ be two simple modules, then $\gamma((\mathcal{S}_1 \times \mathcal{S}_2)) = 2$.

Proof: It is clear.

Theorem 3.7: Let $\mathcal{S} = \mathcal{S}_1 \times \mathcal{S}_2 \times \dots \times \mathcal{S}_r$, where $\mathcal{S}_i$ is a simple module for $i = 1, 2, \dots, r$. Then $\gamma(\mathbb{\Gamma}(\mathcal{S})) \leq r$.

Proof: Let $\mathcal{S} = \mathcal{S}_1 \times \mathcal{S}_2 \times \dots \times \mathcal{S}_r$, where$\mathcal{S}_i$ is simple module of $i = 1, 2, \dots, r$. Any submodule of $\mathcal{S}$ is of the from $A = \prod_{i=1}^{r} G_i$ where $G_i = 0$ or $\mathcal{S}_i$ and the minimal submodules of $\mathcal{S}$ is for in $R_k = \prod_{i=1}^{r} G_i$ where $G_i = 0$ for $i \neq k, G_k = \mathcal{S}_k$. Also, $\mathcal{S}$ having r minimal submodules. Since the set $B = \{R_i : i = 1, 2, \dots, r\}$. The set B dominates all the vertices of the graph. So, $\gamma(\mathbb{\Gamma}(\mathcal{M} \times \mathcal{S})) \leq r$.

The following example provides that the equality does not hold necessarily in Theorem 3.7.

Example 3.8: If $\mathcal{S} = \mathcal{S}_1 \times \mathcal{S}_2 \times \mathcal{S}_3$, where $\mathcal{S}_i$ is a simple module for $i = 1, 2, \dots, r$, then $V\big(\mathbb{\Gamma}(\mathcal{S})\big) = \{\mathcal{S}_1 \times 0 \times 0, \mathcal{S}_1 \times \mathcal{S}_2 \times 0, 0 \times \mathcal{S}_2 \times 0, 0 \times \mathcal{S}_2 \times \mathcal{S}_3, \mathcal{S}_1 \times 0 \times \mathcal{S}_3, 0 \times \mathcal{S}_3\}$. Now consider the set $B = \{\mathcal{S}_1 \times \mathcal{S}_2 \times 0, 0 \times \mathcal{S}_2 \times \mathcal{S}_3\}$. Every vertex of $\mathbb{\Gamma}(\mathcal{S})$ is adjacent at least one of the vertices of B. Hence $\gamma\big(\mathbb{\Gamma}(\mathcal{S})\big) = 2(< 3)$.

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