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Analytical Method for Solving the Integro-Differential Equation Using the Chebyshev Summation Method

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Abstract:

The article aims to solve a special type of integro-differential equations of the second type. This article uses a collocation spectral method based on Chebyshev functions for differential-integral equation problems. Our focus is on the Chebyshev functional polynomials that represent the solution to the differential-integral problem. Finally, we will check the method's effectiveness with some numerical examples.

Keywords: The integro-differential equation, numerical methods, collocation method, combined functions.

1-Introduction

Ordinary differential equations and differential equations with partial derivatives are solved using different numerical approaches, or integral equations, linear or non-linear, time-dependent or non-time-dependent. Among these methods, finite differences, finite elements, and spectral methods can be mentioned. Spectral methods are critical due to their high accuracy and fast convergence. There are three primary categories of spectral approaches, each with advantages and disadvantages: Galerkin, Tau, and collocation (co-local). In articles [1, 2], non-linear Volterra equations with Chebyshev and single integral differential equations are discussed. Piecewise fuzzy interpolation and wavelet Haar are used for delayed differential and integral equations in articles [3, 4, 5]. Recently, there has been a lot of research on numerical solutions for both ordinary and partial differential equations [6-9]. The presentation of a quadrature approach for numerically solving the double integral in the physical sciences is based on combining two rules of the same accuracy level [10]. Solving some integral equations is possible using collocation methods [11]. Recently, there has been a lot of research on numerical solutions for ordinary and partial differential equations.

2- Collocation points method

In this method, we consider the distinct points $x_1, x_2, \dots, x_d \epsilon D$ and set the residual $r_n(x)$ equal to zero at these points:

$$r_n(x_i) = 0, \qquad i = 1,2,\dots,d, \tag{1}$$

In other words, by solving the device (1) of the unknown coefficients $\{c_j | j = 1,2,\dots,d\}$ is determined,

$$\sum_{j=1}^{d} c_j \left\{ \lambda \phi_j(x_i) - \int_D K(x_i.y)\phi_j(y)dy \right\} = f(x_i), \qquad i = 1,2,\dots,d, \tag{2}$$

In addition, the following is the operational mode representation of equation (6-3):

$$\mathcal{P}_n r_n = 0,$$

Which is equivalent to

$$\mathcal{P}_n(\lambda - \mathcal{K})u_n = \mathcal{P}_n f,$$

The selection of basic functions and collocation points is essential in this method. Orthogonal and orthogonal functions such as Chebyshev polynomials, Legendre polynomials, orthogonal Bernstein polynomials, etc., can be used. In addition, choosing different points from the same local points will greatly impact the accuracy of the approximate solution.

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3- Chebyshev polynomials

The following differential equation is called the "Chebyshev problem":

Where *n* is a real number. Chebyshev's equation can be written in a simpler form using the transformation $x = \cos t$. With the conversion as mentioned earlier, we have:

$$x = cost \Longrightarrow dx = -sint\ dt \Longrightarrow \frac{dt}{dx} = -\frac{1}{sint} \Longrightarrow y' = \frac{dy}{dx} = \frac{dt}{dx} * \frac{dy}{dt} = -\frac{1}{sint} * \frac{dy}{dt},$$

And also,

$$y'' = \frac{d^2y}{dx^2} = \frac{d}{dx} * \left(\frac{dy}{dx}\right) = \frac{d}{dt} * \frac{dt}{dx}\left(\frac{dy}{dx}\right) = 1/\sin^2 t \left[\left(-\frac{cost}{sint}\right) * \frac{dy}{dt} + \frac{d^2y}{dt^2}\right]$$

By inserting the derivative expressions in the differential equation, we can compose:

$$\left(1-\cos^2 t\right)\frac{1}{\sin^2 t}\left[\left(\frac{-\cos t}{\sin t}\right)\frac{dy}{dt} + \frac{d^2y}{dt^2}\right] - \cos t\left[-\frac{1}{\sin t}\frac{dy}{dt}\right] + n^2 y = 0,$$

$$\Rightarrow -\frac{\cos t}{\sin t}\frac{dy}{dt} + \frac{d^2y}{dt^2} + \frac{\cos t}{\sin t}\frac{dy}{dt} + n^2 y = 0$$

$$\left(1-x^2\right)y'' - x\ y' + n^2 y = 0$$

As a result, Chebyshev's differential equation becomes the following compressed form:

$$\frac{d^2y}{dt^2} + n^2 y = 0$$

$$T_m(t) = \emptyset_m\left(\frac{2}{t_f - t_0}(t - t_0 - 1\right) = \emptyset_m(A(t - t_0) - 1),$$

$\{T_m(t):\ m=0,1,2,...\}$ In Hilbert space is a complete orthogonal set which $w(t)=\dfrac{1}{\sqrt{1-t^2}}$ is a weight function.

4- Numerical solution of integral equations using Chebyshev polynomials

Consider the following definition of polynomial coefficients for the linear differential operator D of order:

$$D=\sum_{r=0}^{v}p_r(x)\frac{d^r}{dx_r} \qquad (3)$$

A polynomial $p_r(x)$ is also:

$$p_r(x)=\sum_{j=0}^{\alpha_r}p_{rj}\,x^j$$

In which α_r, the degree of polynomial $p_r(x)$ is said.
Assume that $y(x)$ is the precise answer to the following integral differential equation:

$$Dy(x)=f(x)+\lambda\int_a^b k(x,t)\cdot y(t)dt, x\in[a,b]$$

With

$$\sum_{k=1}^{v}\left[c_{jk}^{(1)}y^{(k-1)}(a)+c_{jk}^{(2)}y^{(k-1)}(b)\right]=d_j\ ,\ \ j=1,...,v$$

here $\lambda$the real parameter and $f(x), k(x,t)$functions of continuous are known, and the function$y(x)$ is unknown and constants $a, b, c_{jk}^{(1)}, c_{jk}^{(2)}, d_j$are known.

The solution of Fredholm integral, Volterra, and Volterra-Fredholm differential-integral problems of the second type below:

$$Dy(x)=f(x)+\lambda\int_{-1}^{1}k(x,t)y(t)dt,$$

$$Dy(x)=\ f(x)+\lambda\int_{-1}^{x}k(x,t)\,y\ (t)\,dt$$

$$Dy(x)=f(x)+\lambda\int_{-1}^{x}k_1(x,t)y(t)dt+\lambda_2\int_{-1}^{1}k_2(x,t)y(t)\ dt$$

It can be written as a interrupted Chebyshev series, so the approximate answer is:

$$y(x)\approx y_N(x)=\sum_{j=0}^{N}a_jT_j(x)$$

It can be written as the following matrix:

$$y(x)=T^T(x)A\,,$$

where

$$T(x)=\left[T_0(x),T_1(x),...,T_N(x)\right]^T\ ,\ A=\left[a_0,a_1,a_2,...,a_n\right]^T\ ,$$

Consequently, using the Theorem and substituting the previous Equations in the Fredholm equation, we can write:

$$A^T ZT\,(x) = f(x) + \lambda \int_{-1}^{1} K\,(x,t)[T^T\,(t)\,A\,]\;dt,,$$

We also consider Chebyshev's collocation points as follows:

$$x_i = \cos\left(\frac{i\pi}{N}\right) \;,\; i = 0,1,2,\ldots,N$$

This equation will consequently give rise to a novel set of algebraic equations:

$$A^T Z\,T\,(x_i) = f\,(x_i) + \lambda\int_{-1}^{1} K\,(x_i,t)\left[T^T(t)A\right]dt,$$

And so, the unknown coefficients $a_i, i = 0,1,2,\ldots,N$are found.

Definition 1. The integral differential equation has an approximation in the form of a polynomial:

$$Dy\,(x) = f(x) + \lambda \int_a^b k\,(x\,,t) * y\,(t)dt \;\; such\; that\;\; x \in [\,a\,,b],$$

With

$$\sum_{k=1}^{v}\left[c_{jk}^{(1)} y^{(k-1)}(a) + c_{jk}^{(2)} y^{(k-1)}(b)\right] = d_j \;,\; j = 1,\ldots,v$$

The reason being that the vectors $A = [a_0, a_1, a_2, \ldots, a_N]$ may be used to solve the system of linear equations:

In these relations, y(x) is the unknown function, while λ is the actual parameter, and f(x) and $k(x,t)$are known continuous functions and $a, b, c_{jk}^{(1)}, c_{jk}^{(2)}, d_j$are known constants.

Likewise, we may create a procedure for the issue specified in the interval [0, 1].

$$Dy\,(x) = f(x) + \lambda \int_0^1 k(x,t) * y(t)\,dt \;,\quad x \in \;[0\,,1]$$

The solution is given below when expressed in terms of the shifted Chebyshev polynomial $T_j^*(x)$

$$y\,(x) \approx y_N\,(x) = \sum_{j=0}^{N} a_j^* T_j^*(x) \quad, 0 \le x \le 1 \;,$$

where

$$T_j^*(x) = T_j\,(2x - 1)$$

In the same vein as the prior approach, which makes use of collocation locations specified by

$$x_i = \frac{1}{2}\left(1 + \cos\left(\frac{i\pi}{N}\right)\right) \;,\; i = 0,1,2,\ldots,N$$

A system of algebraic equations like this one can be derived

$$A^T \cdot Z^* \cdot T^*\,(x_i) = f(x_i) + \lambda \int_0^1 K\,(x_i\,,t) * \left[T^{*^T}(t)A\right] dt$$

$$i = 0\,,1\,,2\,,\ldots,N$$

where

$$T^*(x) = [T_0^*(x), T_1^*, \dots, T_N^*(x)]$$

$$Z^* = T^* * QT^{*-1} * T^* = \underline{T^*} = T^*\underline{X}$$

We find the unknown coefficients a_j by solving the nonlinear system. Volterra differential integral issues are also obtained in the same manner. This research directly computes the integrals rather than estimating them.

5- Results and numerical examples

Here we numerically solve integral differential equations using the approximation approach; in these cases, the precise solution contains boundary-dependent second derivatives that are not specified.

Example 1. Consider the Volterra-Fredholm integral-differential problem below:

$$\begin{cases} y'(x) = -2\sin(x) - x^2\sin(2x) - 2xco(2x) + 2\sin(2x) - 2e^x + 5e^{x-1} + 2x \\ \qquad + \int_0^x \cos(x+t)\, y(t)dt + \int_0^1 e^{x-t} y(t)dt, \\ y(0) = 0. \end{cases}$$

The exact answer here is:

$$y(x) = x^2$$

If we approximate y(x) according to the Chebyshev series as follows:

$$y(x) = \sum_{j=0}^{3} a_j T_j^*(x), \; 0 \le x \le 1$$

Using the information mentioned in the previous section, we get the approximate answer to the problem. In Table 5-1, we compared the numerical results of the problem with the method proposed by N=3, with the numerical results of the source [11].

Example 2. Consider the following Fredholm integral differential problem of the second type:

$$\begin{cases} y''(x) - y(x) = -\dfrac{1}{20}\int_0^1 t^{39} * y(t)dt - x^2 - 2x + \dfrac{2521}{688800}, \\ y(0) - y'(0) = 0 \\ y(1) - y'(1) = 9 \end{cases}$$

The exact answer here is:

$$y(x) = x^2 + 2x + 2$$

We compared the approach's absolute errors with the source method [11] and obtained numerical results using the suggested method with N=2.4. The findings are shown in Tables 2-4.

Table 5.1. Numerical results of example 1

X	Presented method	Method in [11]
0	2.391×10^{-11}	3.341×10^{-3}
0.1	1.232×10^{-12}	3.284×10^{-3}
0.2	2.745×10^{-11}	3.231×10^{-3}
0.3	1.112×10^{-11}	3.442×10^{-4}
0.4	3.343×10^{-12}	3.234×10^{-5}
0.5	6.212×10^{-12}	2.774×10^{-3}
0.6	8.181×10^{-12}	8.331×10^{-4}
0.7	1.323×10^{-11}	5.650×10^{-5}
0.8	4.378×10^{-11}	4.123×10^{-5}
0.9	3.456×10^{-11}	2.154×10^{-5}
1	6.878×10^{-11}	6.124×10^{-5}

Table 2. Numerical results of example 2

X	Presented method	Method in [11]
0	6.927×10^{-13}	4.317×10^{-4}
0.1	3.222×10^{-13}	1.924×10^{-6}
0.2	5.354×10^{-13}	2.179×10^{-5}
0.3	3.942×10^{-13}	4.472×10^{-7}
0.4	2.853×10^{-13}	3.234×10^{-5}
0.5	2.242×10^{-13}	5.061×10^{-7}
0.6	5.221×10^{-13}	8.000×10^{-6}
0.7	2.573×10^{-13}	1.924×10^{-6}
0.8	2.378×10^{-13}	3.343×10^{-4}
0.9	6.543×10^{-13}	4.162×10^{-5}
1	2.178×10^{-13}	2.124×10^{-4}

5- Conclusion

The method has two obvious advantages: satisfactory accuracy and fast computing speed. In this paper, we considered several types of one-dimensional integral differential equations. However, in general, integral equations are more complex in practical application problems. For example, the kernel is more complex, and the integral equation is multidimensional. Therefore, extending the presented method to more complex practical problems is essential, which can be a suggestion for future work.

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