Journal of Education for Pure Science
Vol.15, No. 4 (2025)
DOI: https://doi.org/10.32792/jeps.v15i4.724
Website: jceps.utq.edu.iq
Email: jceps@eps.utq.edu.iq

Applications of Elzaki Adomian Decomposition Method to Linear and Nonlinear Fractional-Order Differential Equations

Naser Rhaif Seewn [1,] **and Mohammed Taimah Yasser** [2,] **Haleh Tajadodi**[3]

[1,2]Department of Mathematics, College of Education for Pure Science, University of Thi-Qar, Nasiriyah, Iraq.

[3]Department of Mathematics, University of Sistan and Baluchestan, Zahedan, Iran

[*] Corresponding email: Naserrhaif809@utq.edu.iq

Received 17 / 5 /2025, Accepted 23 / 7 /2025, Published 1/ 12 /202X

Abstract:

This paper investigates the application of the Elzaki Adomian Decomposition Method (EADM) to solve fractional-order differential equations (FDEs), encompassing both linear and nonlinear types. The EADM effectively decomposes these equations, particularly handling nonlinear terms through the Adomian polynomials, to generate convergent series solutions. The study emphasizes the method's efficiency and accuracy in tackling complex systems governed by fractional derivatives, providing analytical approximations without restrictive simplifying assumptions. Numerical examples are presented, comparing the EADM with traditional approaches like numerical integration and perturbation techniques. These comparisons highlight the EADM's superior convergence behavior and solution precision. The results affirm the significant potential of the Elzaki Adomian Decomposition Method for addressing both theoretical and practical challenges within fractional calculus, contributing meaningfully to advancements in the field.

Keywords: Elzaki transform, Adomian Analysis, Nonlinear Systems, Linear Systems, Fractional Calculus, Analytical Solutions, Fractional Derivatives. ,Fractional equations.

1-Introduction

Fractional-order differential equations (FDEs) have emerged as powerful mathematical tools for modeling complex systems with memory, hereditary properties, and non-local interactions across diverse fields, including physics, engineering, biology, and economics [1,2]. Unlike integer-order counterparts, fractional derivatives (e.g., Caputo, Riemann-Liouville) capture *anomalous diffusion*, *viscoelasticity*, and *long-range temporal dependencies* inherent in real-world phenomena, offering superior fidelity for systems where classical models fall short [3,4]. This capability is exemplified in viscoelastic material modeling, where fractional derivatives intrinsically encode stress-strain history [5]; and in control theory, where fractional-order controllers enhance robustness for systems with delayed dynamics [6].

Solution Techniques for FDEs: Challenges and Advances Solving FDEs poses significant challenges due to non-local operators and kernel singularities. Analytical and numerical methods have evolved to address these complexities:

1. Analytical Methods:
   - Laplace/Fourier Transforms: Effective for linear FDEs with constant coefficients but limited for nonlinear or variable-order problems [7].
   - Adomian Decomposition Method (ADM): Decomposes nonlinear equations into convergent series solutions without linearization, handling both linear and nonlinear FDEs [8,9].
   - Variational Iteration Method (VIM): Constructs correction functionals via Lagrange multipliers [10].
   - Homotopy Analysis Method (HAM): Offers adjustable convergence parameters for strongly nonlinear systems [11].
2. Numerical Methods:
   - Predictor-Corrector Algorithms: (e.g., Fractional Adams-Bashforth-Moulton) for Caputo derivatives [12].
   - Finite Difference Schemes: Grünwald-Letnikov discretizations for Riemann-Liouville derivatives [13].
   - Spectral Methods: High accuracy for smooth solutions using orthogonal bases [14].
   - Wavelet Methods: Multiresolution approaches for localized behaviors [15].

While numerical methods offer broad applicability, they face stability constraints and high computational costs [16]. Analytical methods like ADM provide closed-form series solutions but may require acceleration techniques (e.g., Padé approximants) for convergence [17].Focus and Contribution of This Work .This paper employs the Adomian Decomposition Method (ADM) to solve linear and nonlinear FDEs of arbitrary order. ADM's computational efficiency, minimal discretization error, and inherent handling of nonlinearities (via Adomian polynomials) make it ideal for modeling intricate system dynamics [8,18]. We demonstrate ADM's superiority over traditional techniques (e.g., finite difference and perturbation methods) in solution accuracy, convergence rate, and robustness for benchmark problems. Furthermore, we introduce enhancements to ADM for accelerated convergence and broader applicability.

2.Preliminaries

Definition1. *[18,19] If* $f(\mathfrak{X}) \in C([a,b])$, $\mathscr{b} > 0$, *and* $a < \mathfrak{X} < b$, *then the Riemann-Liouville fractional integral of order* , $\mathscr{b}$ *is given by as*

$$I_{\mathfrak{X}}^{\mathscr{b}} f(\mathfrak{X}) = \frac{1}{\Gamma(\mathscr{b})} \int_a^{\mathfrak{X}} \frac{f(\mathscr{f})}{(\mathfrak{X}-\mathscr{f})^{1-\mathscr{b}}} \, d\mathscr{f} \tag{1}$$

Where Γ *is the well-known Gamma function .*

The properties of the Riemann-Liouville fractional integral are as follows:

$$\begin{aligned} &1.\ I_{\mathfrak{X}}^{\mathscr{b}} I_{\mathfrak{X}}^{\mathcal{B}} f(\mathfrak{X}) = I_{\mathfrak{X}}^{\mathscr{b}+\mathcal{B}} f(\mathfrak{X}) \\ &2.\ I_{\mathfrak{X}}^{\mathscr{b}} I_{\mathfrak{X}}^{\mathcal{B}} f(\mathfrak{X}) = I_{\mathfrak{X}}^{\mathcal{B}} I_{\mathfrak{X}}^{\mathscr{b}} f(\mathfrak{X}) \\ &3.\ I_{\mathfrak{X}}^{\mathscr{b}} \mathfrak{X}^{\mathcal{B}} = \frac{\Gamma(\mathcal{B}+1)}{\Gamma(\mathscr{b}+\mathcal{B}+1)} \mathfrak{X}^{\mathscr{b}+\mathcal{B}} \end{aligned} \tag{2}$$

Definition2. .[20,21] The Caputo fractional derivative of function $f(\mathfrak{X})$, $\mathfrak{X} > 0$ *is defined by*

$$D_{\mathfrak{X}}^{\mathscr{b}} u(\mathfrak{X}, \mathscr{f}) = \begin{cases} \frac{1}{\Gamma(n-\mathscr{b})} \int_0^{\mathfrak{X}} (\mathfrak{X}-\mathscr{f})^{n-\mathscr{b}-1} f^{(n)}(\mathscr{f}) d\mathscr{f} & n-1 < \mathscr{b} \leq n \in \mathbb{N} \\ \frac{d^n}{d\mathfrak{X}^n} f(\mathfrak{X}) \quad , & \mathscr{b} = n \in \mathbb{N} \end{cases} \tag{3}$$

Note 1. Based on Definition 2, the following result can be derived

$$D_{\mathfrak{f}}^{\mathcal{b}} \mathfrak{f}^{\beta} = \begin{cases} \frac{\Gamma(\beta+1)}{\Gamma(n-\mathcal{b}+1)} \mathfrak{f}^{\beta-\mathcal{b}} & n-1<\mathcal{b}\le n \ , \ \beta > n-1, \beta \in \mathcal{R} \\ 0 & n-1<\mathcal{b}\le n \ , \ \beta > n-1 \ , \ \beta \in \mathbb{N} \end{cases}$$

1. $D_{\mathfrak{X}}^{\mathcal{b}} \mathcal{K} = 0$
2. $D_{\mathfrak{X}}^{\mathcal{b}} I^{\mathcal{b}} f(\mathfrak{X}) = f(\mathfrak{X})$
3. $D_{\mathfrak{X}}^{\mathcal{b}} \mathfrak{X}^{\beta} = \frac{\Gamma(\mathcal{B}+1)}{\Gamma(\mathcal{B}-\mathcal{b}+1)} \mathfrak{X}^{\mathcal{B}-\mathcal{b}}$
4. $D_{\mathfrak{X}}^{\mathcal{b}} D_{\mathfrak{X}}^{\beta} f(\mathfrak{X}) = D_{\mathfrak{X}}^{\mathcal{b}+\beta} f(\mathfrak{X}) = D_{\mathfrak{X}}^{\beta} D_{\mathfrak{X}}^{\mathcal{b}} f(\mathfrak{X})$
5. $D_{\mathfrak{X}}^{\mathcal{b}} [\mathcal{K} f(\mathfrak{X}) + \mathcal{L} g(\mathfrak{X})] = \mathcal{K} D_{\mathfrak{X}}^{\mathcal{b}} f(\mathfrak{X}) + \mathcal{L} D_{\mathfrak{X}}^{\mathcal{b}} g(\mathfrak{X})$
6. $I^{\mathcal{b}} D_{\mathfrak{X}}^{\mathcal{b}} f(\mathfrak{X}) = f(\mathfrak{X}) - \sum_{k=0}^{m-1} f^{(k)}(0) \frac{\mathfrak{f}^k}{k!}$

Definition 3. The Elzaki transform (ET) is [22,23]

$$E[u(\mathfrak{f})] = T(w) = s \int_0^{\infty} e^{\frac{-\mathfrak{f}}{w}} u(\mathfrak{f}) d\mathfrak{f}, w \in [k_1, k_2] \quad (4)$$

Some ET Properties:-

1. $E[1] = w^2$
2. $E[\mathfrak{f}^{\mathcal{b}}] = \Gamma(\mathcal{b}+1) w^{\mathcal{b}+2}$

Definition 4. The ET of the CFD is given by [24]

$$E[D_{\mathfrak{f}}^{\mathcal{b}} u(\mathfrak{X}, \mathfrak{f})] = \frac{E[u(\mathfrak{X},\mathfrak{f})]}{w^{\mathcal{b}}} - \sum_{k=0}^{n-1} w^{2-\mathcal{b}+k} u^{(k)}(\mathfrak{X}, 0), \ n-1<\mathcal{b}\le n . \quad (5)$$

2- Analysis of Elzaki Adomian Decomposition Method (EADM)

In this section, we derive the general formula for the Elzaki Adomian equation, then we take non linear formula and solve it in the form of an example, and then we solve it using the EADM.

Examine the subsequent fractional nonlinear partial differential equations:

$$D_{\mathfrak{f}}^{\mathcal{b}} u(\mathfrak{X}, \mathfrak{f}) + R[u(\mathfrak{X}, \mathfrak{f})] + N[u(\mathfrak{X}, \mathfrak{f})] = g(\mathfrak{X}, \mathfrak{f}), \mathfrak{f} > 0, \ n-1<\mathcal{b}\leqslant n \quad (6)$$

where ${}^{c}D_{\mathfrak{f}}^{\mathcal{b}} u(\mathfrak{X}, \mathfrak{f})$ represents the derivative of $u(\mathfrak{X}, \mathfrak{f})$ in Caputo sense, R, N represent differential operators, encompassing both linear and nonlinear forms, and$g(\mathfrak{X}, \mathfrak{f})$ represents the energy term. When the energy term is applied to both sides of equation (6), we derive,

$$E\{{}^{c}D_{\mathfrak{f}}^{\mathcal{b}} u(\mathfrak{X}, \mathfrak{f}) + R[u(\mathfrak{X}, \mathfrak{f})] + N[u(\mathfrak{X}, \mathfrak{f})]\} = E\{g(\mathfrak{X}, \mathfrak{f})\}, \quad (7)$$

We achieve using ET's distinction feature. using Def.(2.1):

$$\frac{E\{u(\mathfrak{X},\mathfrak{f})\}}{v^{\mathcal{b}}} - \sum_{k=0}^{n-1} v^{2-\mathcal{b}+k} u^{(k)}(\mathfrak{X}, 0) = E\{g(\mathfrak{X}, \mathfrak{f})\} - E\{R[u(\mathfrak{X}, \mathfrak{f})] + N[u(\mathfrak{X}, \mathfrak{f})]\}, \quad (8)$$

or

$$E\{u(\mathfrak{X}, \mathfrak{f})\} = \sum_{k=0}^{n-1} v^{2+k} u^{(k)}(\mathfrak{X}, 0) + v^{\mathcal{b}} E\{g(\mathfrak{X}, \mathfrak{f})\} - v^{\mathcal{b}} E\{R[u(\mathfrak{X}, \mathfrak{f})] +$$

$N[u(\mathfrak{X},\not{f})]\}. \qquad (9)$

By applying the $E^{-1}T$ of Eq. (8,9), we obtain.

$$u(\mathfrak{X},\not{f}) = \sum_{k=0}^{n-1} \frac{\not{f}^k}{k!} u^{(k)}(\mathfrak{X},0) + E^{-1}\left(v^{\mathscr{b}} E\{g(\mathfrak{X},\not{f})\}\right) - E^{-1}\left(v^{\mathscr{b}} E\{R[u(\mathfrak{X},\not{f})] + N[u(\mathfrak{X},\not{f})]\}\right). \qquad (10)$$

Next, by utilizing the Homotopy Perturbation Method (HPM) on equation (10), we obtain.

$$u(\mathfrak{X},\not{f}) = \sum_{k=0}^{n-1} \frac{\not{f}^k}{k!} u^{(k)}(\mathfrak{X},0) + E^{-1}\left(v^{\mathscr{b}} E\{g(\mathfrak{X},\not{f})\}\right) - \left[E^{-1}\left(v^{\mathscr{b}} E\{R[u(\mathfrak{X},\not{f})] + N[u(\mathfrak{X},\not{f})]\}\right)\right]. \qquad (11)$$

Suppose that

$$u(\mathfrak{X},\not{f}) = \sum_{n=0}^{\infty} u_n, \qquad (12)$$

and the nonlinear term is decomposed as

$$N\left(u(\mathfrak{X},\not{f})\right) = \sum_{n=0}^{\infty} H_n, \qquad (13)$$

where

$$H_n = \frac{1}{n!}\frac{\partial^n}{\partial p^n}\left(\sum_{i=0}^{\mathrm{n}} p^i u_i\right)_{p=0}.$$

Substituting (12) and (13) in (11), we get

$$\sum_{n=0}^{\infty} u_n = \sum_{k=0}^{n-1} \frac{\not{f}^k}{k!} u^{(k)}(\mathfrak{X},0) + E^{-1}\left(v^{\mathscr{b}} E\{g(\mathfrak{X},\not{f})\}\right) - \left[E^{-1}\left(v^{\mathscr{b}} E\{R[\textstyle\sum_{n=0}^{\infty} u_n] + \sum_{n=0}^{\infty} H_n\}\right)\right]. \qquad (14)$$

The following equations are obtained by equating the coefficients of corresponding powers from both sides of equation (14).

$$u_0(\mathfrak{X},\not{f}) = \sum_{k=0}^{n-1} \frac{\not{f}^k}{k!} u^{(k)}(\mathfrak{X},0) + E^{-1}\left(v^{\mathscr{b}} E\{g(\mathfrak{X},\not{f})\}\right) \quad n \geq 0. \qquad (15)$$

$$u_{n+1}(\mathfrak{X},\not{f}) = -E^{-1}\left(v^{\mathscr{b}} E\{R[u_{\mathrm{n}}] + H_{\mathrm{n}}\}\right)$$

The result is expressed as

$$\mathcal{U}(\mathfrak{X},\mathrm{t}) = \sum_{n=0}^{\infty} u_n$$

$$u(\mathfrak{X},\not{f}) = u_0 + u_1 + u_2 + \cdots$$

Now we take some examples to find approximate solutions

Example 1 :Examine the space characterized by the fractional linear equation

$${}_{0}^{c}D_{\mathfrak{X}}^{\mathscr{b}}\mathcal{U}(\mathfrak{X},t) = D_t^2\mathcal{U}(\mathfrak{X},t) + D_t\mathcal{U}(\mathfrak{X},t) + \mathcal{U}(\mathfrak{X},t)$$
(16)

$0 < \mathfrak{X} < 1 \quad , \quad 0 < \mathscr{b} \leq 2 \quad , \quad t > 0$

With the boundary conditions

$\mathcal{U}(0,t) = e^{-t} \qquad , \qquad t \geq 0$

$\mathcal{U}_{\mathfrak{X}}(0,t) = e^{-t} \qquad , \qquad t \geq 0$

Applying the ET on both sides of the equation, we have

$$E[{}_{0}^{c}D_{\mathfrak{X}}^{\mathscr{b}}\mathcal{U}(\mathfrak{X},t)] - E[D_t^2\mathcal{U}(\mathfrak{X},\mathscr{f}) + D_t\mathcal{U}(\mathfrak{X},t) + \mathcal{U}(\mathfrak{X},t)] = 0 ,$$
(17)

As we know the caputo derivative can be applied as

$$E[{}_{0}^{c}D_{\mathfrak{X}}^{\mathscr{b}}U(\mathfrak{X},\mathscr{f})] - E\left[D_{\mathscr{f}}^2U(\mathfrak{X},\mathscr{f}) + D_{\mathscr{f}}U(\mathfrak{X},\mathscr{f}) + U(\mathfrak{X},\mathscr{f})\right] = 0 ,$$
(18)

$$E\left[D_{\mathscr{f}}^{\mathscr{b}}\, u(\mathfrak{X},\mathscr{f})\right] = \frac{E[u(\mathfrak{X},\mathscr{f})]}{w^{\mathscr{b}}} - \sum_{k=0}^{n-1} w^{2-\mathscr{b}+k}\, u^{(k)}(\mathfrak{X},0),\ n-1 < \mathscr{b} \leq n$$
(19)

Then , we can write the left side according to the above definition . Applying the ET to the right side, we can get

$$\frac{E[\mathcal{U}(\mathfrak{X},t)]}{w^{\mathscr{b}}} - \sum_{k=0}^{n-1}\frac{\mathcal{U}^{(k)}}{w^{\mathscr{b}-k-2}} = E[D_t^2\mathcal{U}(\mathfrak{X},t) + D_t\mathcal{U}(\mathfrak{X},t) + \mathcal{U}(\mathfrak{X},t)] , \quad (20)$$

$$\frac{E[\mathcal{U}(\mathfrak{X},t)]}{w^{\mathscr{b}}} - \frac{\mathcal{U}(0,t)}{w^{\mathscr{b}-0-2}} - \frac{\mathcal{U}_{\mathfrak{X}}(0,t)}{w^{\mathscr{b}-1-2}} = E[\ D_t^2\mathcal{U}(\mathfrak{X},t) + D_t\mathcal{U}(\mathfrak{X},t) + \mathcal{U}(\mathfrak{X},t)] \ , \quad (21)$$

$$\frac{E[\mathcal{U}(\mathfrak{X},t)]}{w^{\mathscr{b}}} - \frac{e^{-t}}{w^{\mathscr{b}-2}} - \frac{e^{-t}}{w^{\mathscr{b}-3}} = E[D_t^2\mathcal{U}(\mathfrak{X},t) + D_t\mathcal{U}(\mathfrak{X},t) + \mathcal{U}(\mathfrak{X},t)] , \quad (22)$$

$$\frac{E[\mathcal{U}(\mathfrak{X},t)]}{w^{\mathscr{b}}} = \frac{e^{-t}}{w^{\mathscr{b}-2}} + \frac{e^{-t}}{w^{\mathscr{b}-3}} + E[D_t^2\mathcal{U}(\mathfrak{X},t) + D_t\mathcal{U}(\mathfrak{X},t) + \mathcal{U}(\mathfrak{X},t)] , \quad (23)$$

$$E[\mathcal{U}(\mathfrak{X},t)] = w^{\mathscr{b}}\left[\frac{e^{-t}}{w^{\mathscr{b}-2}} + \frac{e^{-t}}{w^{\mathscr{b}-3}}\right] + w^{\mathscr{b}}Y[D_t^2\mathcal{U}(\mathfrak{X},t) + D_t\mathcal{U}(\mathfrak{X},t) + \mathcal{U}(\mathfrak{X},t)], \quad (24)$$

Wich implies

$$E[\mathcal{U}(\mathfrak{X},t)] = (w^2 + w^3)e^{-t} + w^{\mathscr{b}}Y[D_t^2\mathcal{U}(\mathfrak{X},t) + D_t\mathcal{U}(\mathfrak{X},t) + \mathcal{U}(\mathfrak{X},t)]$$
(25)

By applying the $E^{-1}T$

$$\mathcal{U}(\mathfrak{X},t) = E^{-1}[\ (w^2 + w^3)e^{-t}] + E^{-1}[w^{\mathscr{b}}E[D_t^2\mathcal{U}(\mathfrak{X},t) + D_t\mathcal{U}(\mathfrak{X},t) + \mathcal{U}(\mathfrak{X},t)]],$$

$\mathcal{U}(\mathfrak{X},t) = (1+\mathfrak{X})e^{-t} + E^{-1}[w^{\ell}E[D_t^2\mathcal{U}(\mathfrak{X},t) + D_t\mathcal{U}(\mathfrak{X},t) + \mathcal{U}(\mathfrak{X},t)]],$

We find $\mathcal{U}_0(\mathfrak{X},t) = e^{-t}(1+\mathfrak{X})$

Next , we use $\mathcal{U}_0(\mathfrak{X},t)$ to calculate $\mathcal{U}_1(\mathfrak{X},t)$

$\mathcal{U}_1(\mathfrak{X},t) = E^{-1}[w^{\ell}E[D_t^2\mathcal{U}_0(\mathfrak{X},t) + D_t\mathcal{U}_0(\mathfrak{X},t) + \mathcal{U}_0(\mathfrak{X},t)]],$

$\mathcal{U}_1(\mathfrak{X},t) = E^{-1}[w^{\ell}E[D_t^2[e^{-t}(1+\mathfrak{X})] + D_t[e^{-t}(1+\mathfrak{X})] + e^{-t}(1+\mathfrak{X})]],$

From colculus , fractional order derivative of exponential function for this case is defined by

$D_t^2\mathcal{U}_t(\mathfrak{X},t) = e^{-t}$

We have

$\mathcal{U}_1(\mathfrak{X},t) = E^{-1}[w^{\ell}E[(1+\mathfrak{X})(e^{-t}) - (1+\mathfrak{X})(e^{-t}) + e^{-t}(1+\mathfrak{X})]],$

$\mathcal{U}_1(\mathfrak{X},t) = E^{-1}[w^{\ell}E[e^{-t}(1+\mathfrak{X})]],$

$\mathcal{U}_1(\mathfrak{X},t) = E^{-1}\left[\left[e^{-t}(w^2+w^3)w^{\ell}\right]\right],$

$U_1(\mathfrak{X},t) = E^{-1}\left[\left[e^{-t}\left(w^{\ell+2}+w^{\ell+3}\right)\right]\right],$

$U_1(\mathfrak{X},t) = \left[e^{-t}\left[\frac{\mathfrak{X}^{\ell}}{\Gamma(\ell+1)} + \frac{\mathfrak{X}^{\ell}+1}{\Gamma(\ell+2)}\right]\right],$

After that using $U_1(\mathfrak{X},t)$, we get

$\mathcal{U}_2(\mathfrak{X},t) = E^{-1}[w^{\ell}E[D_t^2\mathcal{U}_1(\mathfrak{X},t) + D_t\mathcal{U}_1(\mathfrak{X},t) + \mathcal{U}_1(\mathfrak{X},t)]],$

$\mathcal{U}_2(\mathfrak{X},t) = E^{-1}[w^{\ell}E\left[D_t^2[e^{-t}[\frac{\mathfrak{X}^{\ell}}{\Gamma(\ell+1)} + \frac{\mathfrak{X}^{\ell}+1}{\Gamma(\ell+2)}] + D_t[e^{-t}[\frac{\mathfrak{X}^{\ell}}{\Gamma(\ell+1)} + \frac{\mathfrak{X}^{\ell}+1}{\Gamma(\ell+2)}] + e^{-t}[\frac{\mathfrak{X}^{\ell}}{\Gamma(\ell+1)} + \frac{\mathfrak{X}^{\ell}+1}{\Gamma(\ell+2)}\right]],$

$$\mathcal{U}_2(\mathfrak{X},t) = E^{-1}[w^{\ell}E\left[[e^{-t}\left[\frac{\mathfrak{X}^{\ell}}{\Gamma(\ell+1)} + \frac{\mathfrak{X}^{\ell}+1}{\Gamma(\ell+2)}\right] - [e^{-t}[\frac{\mathfrak{X}^{\ell}}{\Gamma(\ell+1)} + \frac{\mathfrak{X}^{\ell}+1}{\Gamma(\ell+2)}] + e^{-t}[\frac{\mathfrak{X}^{\ell}}{\Gamma(\ell+1)} + \frac{\mathfrak{X}^{\ell}+1}{\Gamma(\ell+2)}\right]],$$

$\mathcal{U}_2(\mathfrak{X},t) = E^{-1}[w^{\ell}E\left[e^{-t}[\frac{\mathfrak{X}^{\ell}}{\Gamma(\ell+1)} + \frac{\mathfrak{X}^{\ell}+1}{\Gamma(\ell+2)}\right],$

$\mathcal{U}_2(\mathfrak{X},t) = e^{-t}\left[\frac{\mathfrak{X}^{2\ell}}{\Gamma(2\ell+1)} + \frac{\mathfrak{X}^{2\ell}+1}{\Gamma(2\ell+2)}\right],$

Now use $\mathcal{U}_2(\mathfrak{X},t)$ to calculate $\mathcal{U}_3(\mathfrak{X},t)$

$\mathcal{U}_3(\mathfrak{X},t) = E^{-1}[w^{\ell}E\left[D_t^2[e^{-t}[\frac{\mathfrak{X}^{2\ell}}{\Gamma(2\ell+1)} + \frac{\mathfrak{X}^{2\ell}+1}{\Gamma(2\ell+2)}]] + D_t[e^{-t}[\frac{\mathfrak{X}^{2\ell}}{\Gamma(2\ell+1)} + \frac{\mathfrak{X}^{2\ell}+1}{\Gamma(2\ell+2)}]] + [e^{-t}[\frac{\mathfrak{X}^{2\ell}}{\Gamma(2\ell+1)} + \frac{\mathfrak{X}^{2\ell}+1}{\Gamma(2\ell+2)}]\right]],$

$$\mathcal{U}_3(\mathfrak{X},t) = E^{-1}[w^{\mathscr{b}} E\left[[e^{-t}[\frac{\mathfrak{X}^{2\mathscr{b}}}{\Gamma(2\mathscr{b}+1)} + \frac{\mathfrak{X}^{2\mathscr{b}}+1}{\Gamma(2\mathscr{b}+2)}]] - [e^{-t}[\frac{\mathfrak{X}^{2\mathscr{b}}}{\Gamma(2\mathscr{b}+1)} + \frac{\mathfrak{X}^{2\mathscr{b}}+1}{\Gamma(2\mathscr{b}+2)}]] + [e^{-t}[\frac{\mathfrak{X}^{2\mathscr{b}}}{\Gamma(2\mathscr{b}+1)} + \frac{\mathfrak{X}^{2\mathscr{b}}+1}{\Gamma(2\mathscr{b}+2)}]]\right],$$

$$\mathcal{U}_3(\mathfrak{X},t) = E^{-1}[w^{\mathscr{b}} E\left[e^{-t}[\frac{\mathfrak{X}^{2\mathscr{b}}}{\Gamma(2\mathscr{b}+1)} + \frac{\mathfrak{X}^{2\mathscr{b}}+1}{\Gamma(2\mathscr{b}+2)}]\right],$$

$$\mathcal{U}_3(\mathfrak{X},t) = \left[e^{-t}[\frac{\mathfrak{X}^{4\mathscr{b}}}{\Gamma(4\mathscr{b}+1)} + \frac{\mathfrak{X}^{4\mathscr{b}}+1}{\Gamma(4\mathscr{b}+2)}]\right],$$

$$\mathcal{U}(\mathfrak{X},t) = \mathcal{U}_0(\mathfrak{X},t) + \mathcal{U}_1(\mathfrak{X},t) + \mathcal{U}_2(\mathfrak{X},t) + \cdots \ldots\ldots$$

So that

$$\mathcal{U}(\mathfrak{X},t) = e^{-t}(1-\mathfrak{X}) + e^{-t}\left[\frac{\mathfrak{X}^{\mathscr{b}}}{\Gamma(\mathscr{b}+1)} + \frac{\mathfrak{X}^{\mathscr{b}}+1}{\Gamma(\mathscr{b}+2)}\right] + e^{-t}\left[\frac{\mathfrak{X}^{2\mathscr{b}}}{\Gamma(2\mathscr{b}+1)} + \frac{\mathfrak{X}^{2\mathscr{b}}+1}{\Gamma(2\mathscr{b}+2)}\right] + e^{-t}[\frac{\mathfrak{X}^{2\mathscr{b}}}{\Gamma(2\mathscr{b}+1)} + \frac{\mathfrak{X}^{2\mathscr{b}}+1}{\Gamma(2\mathscr{b}+2)}]$$

$$\mathcal{U}(\mathfrak{X},t) = \sum_{m=0}^{\infty} e^{-t} \frac{\mathfrak{X}^{m\mathscr{b}}}{\Gamma(\mathfrak{X}^{m\mathscr{b}}+1)} + \frac{\mathfrak{X}^{m\mathscr{b}+1}}{\Gamma(\mathfrak{X}^{m\mathscr{b}}+2)},$$

$$\mathcal{U}(\mathfrak{X},t) = e^{\mathfrak{X}-t} \quad .$$

Table 1. *Numerical values of the approximate and exact solutions among different value of* $\mathfrak{X}$ *and* $\mathfrak{f}$ *when* $\mathscr{b} = 0.5, 0.9, 1$

$\mathfrak{X}$	0.5	0.9	1	exact	$\lvert U_{Ex} - U_{\mathscr{b}=1}\rvert$	$\lvert U_{Ex} - U_{\mathscr{b}=0.5}\rvert$	$\lvert U_{ex} - U_{\mathscr{b}=0.9}\rvert$
1	1.74331	1.67457	1.64229	1.64872	0.00642	0.093592	0.02585
1.1	1.92555	1.85068	1.81501	1.82311	0.00710	0.103436	0.02856
1.2	2.12806	2.04532	2.00590	2.01375	0.00784	0.1114314	0.03157
1.3	2.35187	2.26043	2.21686	2.22550	0.00867	0.126337	0.03489
1.4	2.59922	2.49816	2.45001	2.45960	0.00958	0.139624	0.03856
1.5	2.87259	2.76090	2.70768	2.71828	0.01059	0.154308	0.04262
1.6	3.17470	3.05126	2.99245	3.00416	0.011708	0.170537	0.04710

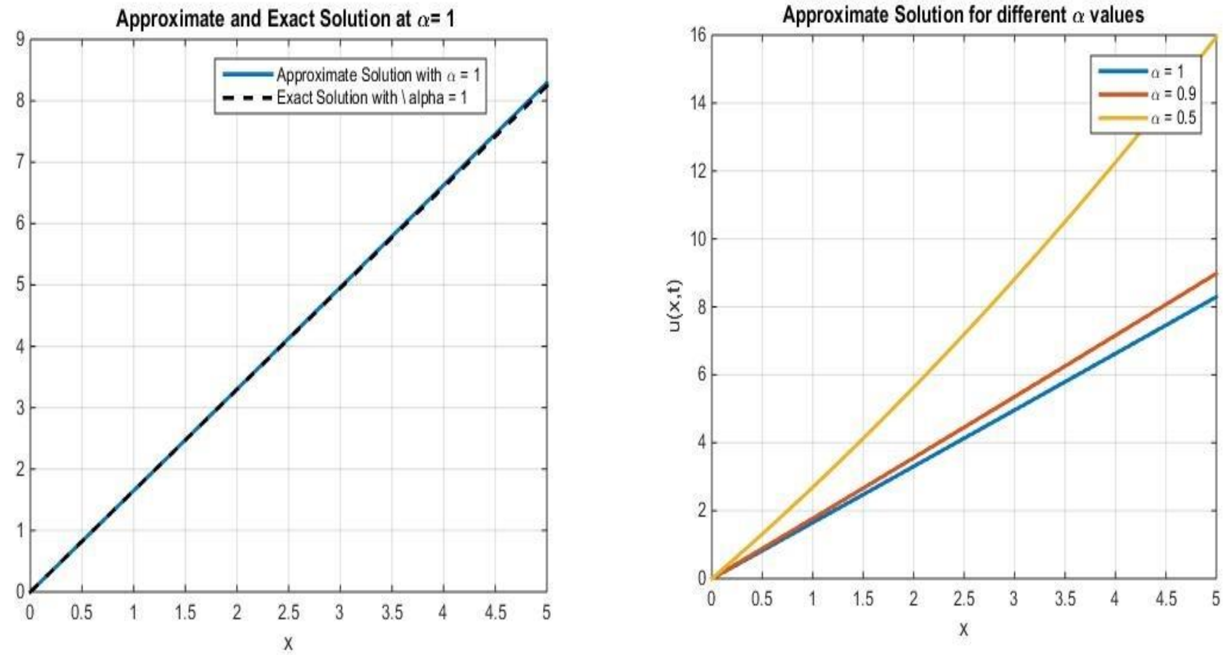

Figure 1. *Graphs of the approximate solution* $u(\mathfrak{X}, ŧ)$ *for various values of ŧalpha while keeping* $\mathfrak{X}$ *constant*

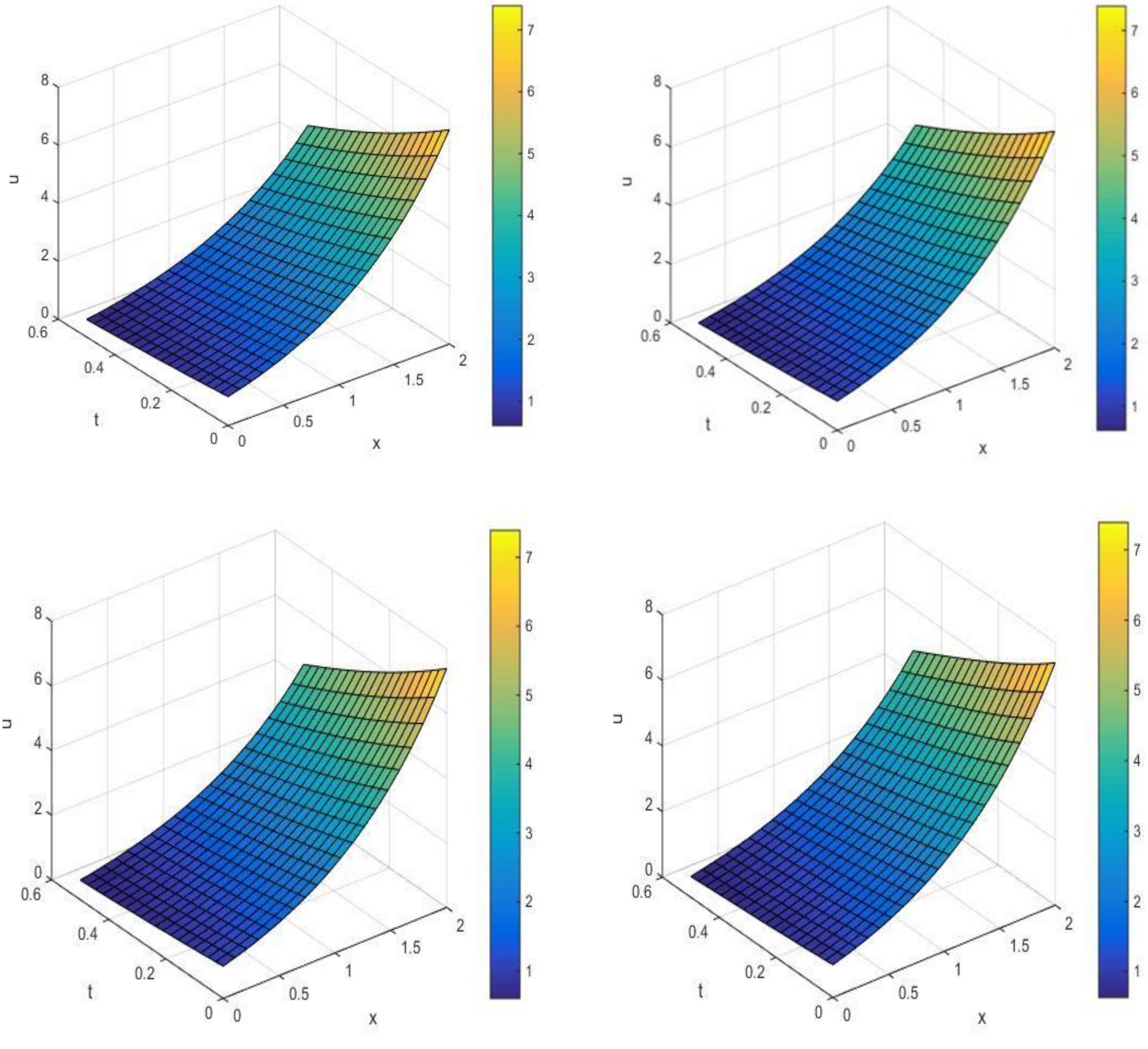

Figure 2*. The surface graph of the approximate solution ; (a) $u(\mathfrak{X}, t)$ when Exact solution , (b) $u(\mathfrak{X}, t)$ when $b = 0.9$, (c) $u(\mathfrak{X}, t)$ when $b = 0.5$ (d) $u(\mathfrak{X}, t)$ when $b = 1$*

EXample 2 : Examine the space characterized by the fractional linear equation non-homogeneous type

$$\frac{\partial^{\mathcal{b}} u}{\partial \mathfrak{x}^{\mathcal{b}}} = \frac{\partial^2 u}{\partial^2 \mathfrak{f}} + \frac{\partial u}{\partial \mathfrak{f}} - \mathfrak{X}^2 - \mathfrak{f} + 1 \quad , \quad 1 < \mathcal{b} \le 2 \ ,$$
(26)

With boundary condition

$$u(0,\mathfrak{f}) = \mathfrak{f} \quad , \quad u_{\mathfrak{x}}(0,\mathfrak{f}) = 0$$

Performing the Elzaki transform on both sides of the equation

$$E\left[\frac{\partial^{\mathcal{b}} u}{\partial \mathfrak{x}^{\mathcal{b}}}\right] = E\left[\frac{\partial^2 u}{\partial^2 \mathfrak{f}} + \frac{\partial u}{\partial \mathfrak{f}}\right] + E[-\mathfrak{X}^2 - \mathfrak{f} + 1] \ ,$$
(27)

$$E\left[\frac{u(\mathfrak{X},\mathfrak{f})}{w^{\mathcal{b}}}\right] - \frac{u(0,\mathfrak{f})}{w^{\mathcal{b}-2}} - \frac{u_{\mathfrak{x}}(0,\mathfrak{f})}{w^{\mathcal{b}-3}} = E\left[\frac{\partial^2 u}{\partial^2 \mathfrak{f}} + \frac{\partial u}{\partial \mathfrak{f}}\right] + [-2w^4 - w^2\mathfrak{f} + w^2\,]$$

$$E[u(\mathfrak{X},\mathfrak{f})] = \left[w^2\mathfrak{f} - 2w^{\mathcal{b}+4} - w^{\mathcal{b}+2}\mathfrak{f} + w^{\mathcal{b}+2}\right] + w^{\mathcal{b}} E\left[\frac{\partial^2 u}{\partial^2 \mathfrak{f}} + \frac{\partial u}{\partial \mathfrak{f}}\right]$$
(28)

Appling the $E^{-1}T$ *, it give*

$$u(\mathfrak{X},\mathfrak{f}) = E^{-1}\left[w^2\mathfrak{f} - 2w^{\mathcal{b}+4} - w^{\mathcal{b}+2}\mathfrak{f} + w^{\mathcal{b}+2}\right] + E^{-1}\left[w^{\mathcal{b}} E\left[\frac{\partial^2 u}{\partial^2 \mathfrak{f}} + \frac{\partial u}{\partial \mathfrak{f}}\right]\right],$$
(29)

$$u(\mathfrak{X},\mathfrak{f}) = \mathfrak{f} - \frac{\mathfrak{x}^{\mathcal{b}}}{\Gamma(\mathcal{b}+1)}\mathfrak{f} - \frac{2\mathfrak{x}^{\mathcal{b}+2}}{\Gamma(\mathcal{b}+3)} + \frac{\mathfrak{x}^{\mathcal{b}}}{\Gamma(\mathcal{b}+1)} + E^{-1}\left[w^{\mathcal{b}} E\left[\frac{\partial^2 u}{\partial^2 \mathfrak{f}} + \frac{\partial u}{\partial \mathfrak{f}}\right]\right] \ ,$$
(30)

$$u_0(\mathfrak{X},\mathfrak{f}) = \mathfrak{f} - \frac{2\mathfrak{x}^{\mathcal{b}+2}}{\Gamma(\mathcal{b}+3)} + (1-\mathfrak{f})\frac{\mathfrak{x}^{\mathcal{b}}}{\Gamma(\mathcal{b}+1)} \,,$$

$$u_{i+1}(\mathfrak{X},\mathfrak{f}) = E^{-1}\left[w^{\mathcal{b}} E\left[\frac{\partial^2 u_i}{\partial^2 \mathfrak{f}} + \frac{\partial u_i}{\partial \mathfrak{f}}\right]\right] \quad , \quad i = 0,1,2,3, \dots\dots\dots\dots$$

$$u_1(\mathfrak{X},\mathfrak{f}) = E^{-1}\left[w^{\mathcal{b}} E\left[\frac{\partial^2 u_0}{\partial^2 \mathfrak{f}} + \frac{\partial u_0}{\partial \mathfrak{f}}\right]\right]$$

$$u_1(\mathfrak{X},\mathfrak{f}) = E^{-1}\left[w^{\mathcal{b}} E\left[0 + 1 - \frac{\mathfrak{x}^{\mathcal{b}}}{\Gamma(\mathcal{b}+1)}\right]\right],$$

$$u_1(\mathfrak{X},\mathfrak{f}) = (1+\mathfrak{f})\frac{\mathfrak{x}^{\mathcal{b}}}{\Gamma(\mathcal{b}+1)} - \frac{\mathfrak{f}\mathfrak{x}^{2\mathcal{b}}}{\Gamma(\mathcal{b}+1)} - \frac{2\mathfrak{x}^{\mathcal{b}+2}}{\Gamma(\mathcal{b}+3)} \,,$$

$$u_2(\mathfrak{X},\mathfrak{f}) = E^{-1}\left[w^{\mathcal{b}} E\left[\frac{\partial^2 u_1}{\partial^2 \mathfrak{f}} + \frac{\partial u_1}{\partial \mathfrak{f}}\right]\right],$$

$$u_2(\mathfrak{X},\mathfrak{f}) = (2+\mathfrak{f})\frac{\mathfrak{x}^{2\mathcal{b}}}{\Gamma(2\mathcal{b}+1)} - (1-\mathfrak{f})\frac{\mathfrak{x}^{3\mathcal{b}}}{\Gamma(3\mathcal{b}+1)} + \frac{\mathfrak{f}\mathfrak{x}^{2\mathcal{b}}}{\Gamma(2\mathcal{b}+1)} - \frac{2\mathfrak{x}^{3\mathcal{b}+2}}{\Gamma(3\mathcal{b}+3)} \,,$$

In the same way, we can deduce the solution series using the limits of the previous equation to obtain:

$u(\mathfrak{X},\not{f}) = u_0(\mathfrak{X},\not{f}) + u_1(\mathfrak{X},\not{f}) + u_2(\mathfrak{X},\not{f}) + u_3(\mathfrak{X},\not{f}) + \cdots \dots \dots \dots$
(31)

$$u(\mathfrak{X},\not{f}) = \not{f} - \frac{2\mathfrak{X}^{\ell+2}}{\Gamma(\ell+3)} + (1-\not{f})\frac{\mathfrak{X}^{\ell}}{\Gamma(\ell+1)} + (1+\not{f})\frac{\mathfrak{X}^{\ell}}{\Gamma(\ell+1)} - \frac{\not{f}\mathfrak{X}^{2\ell}}{\Gamma(\ell+1)} - \frac{2\mathfrak{X}^{\ell+2}}{\Gamma(\ell+3)} + (2+\not{f})\frac{\mathfrak{X}^{2\ell}}{\Gamma(2\ell+1)} - (1-\not{f})\frac{\mathfrak{X}^{3\ell}}{\Gamma(3\ell+1)} + \frac{\not{f}\mathfrak{X}^{2\ell}}{\Gamma(2\ell+1)} - \frac{2\mathfrak{X}^{3\ell+2}}{\Gamma(3\ell+3)}$$

$$u(\mathfrak{X},\not{f}) = \not{f} + \mathfrak{X}^2.$$

(32)

***Table* 2.** *Numerical values of the approximate and exact solutions among different value of* $\mathfrak{X}$ *and* $\not{f}$ *when* $\ell = 0.5, 0.9, 1$

$\mathfrak{X}$	0.5	0.9	ℓ =1	exact	$\lvert U_{Ex} - U_{\ell=1}\rvert$	$\lvert U_{Ex} - U_{\ell=0.5}\rvert$	$\lvert U_{ex} - U_{\ell=0.9}\rvert$
1	1.5	1.0965	1	1.5	0.5	0	0.40352
1.1	1.6	1.2081	1.105	1.71	0.605	0.11	0.50188
1.2	1.7	1.3282	1.22	1.94	0.72	0.24	0.61182
1.3	1.8	1.4565	1.345	2.19	0.845	0.39	0.73347
1.4	1.9	1.593	1.48	2.46	0.98	0.56	0.86698
1.5	2	1.7376	1.625	2.75	1.125	0.75	1.0124
1.6	2.1	1.89	1.78	3.06	1.28	0.96	1.17

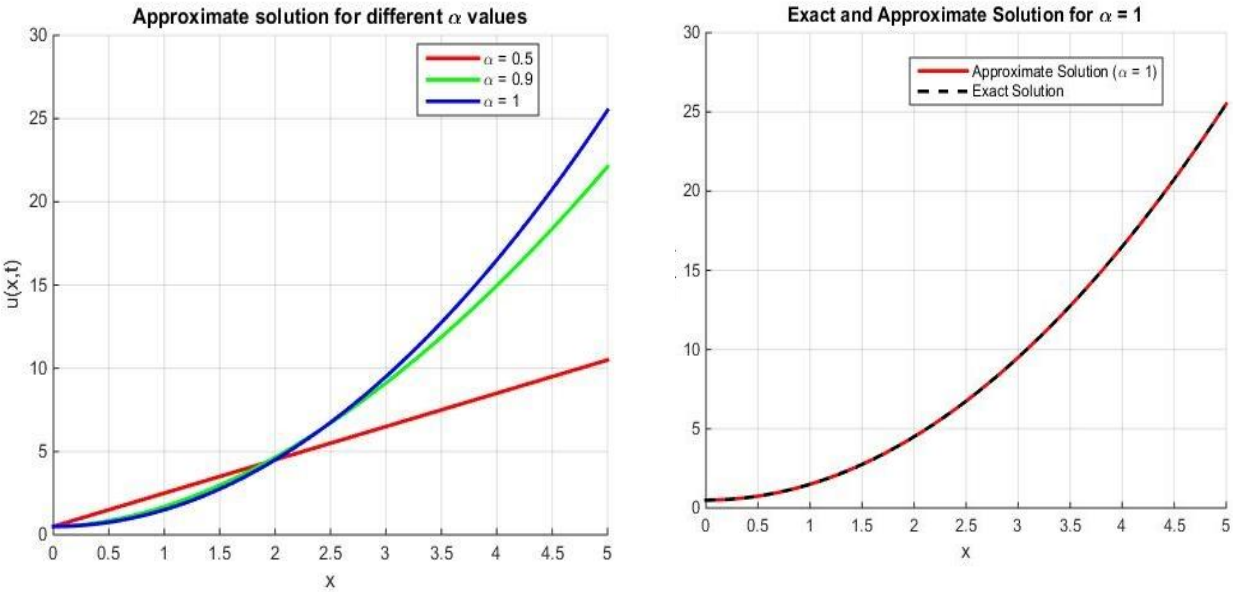

***Figure* 3**. *Graphs of the approximate solution* $u(\mathfrak{X},\mathfrak{t})$ *for various values of* $\mathscr{b}$*alpha while keeping* $\mathfrak{X}$ *constant*

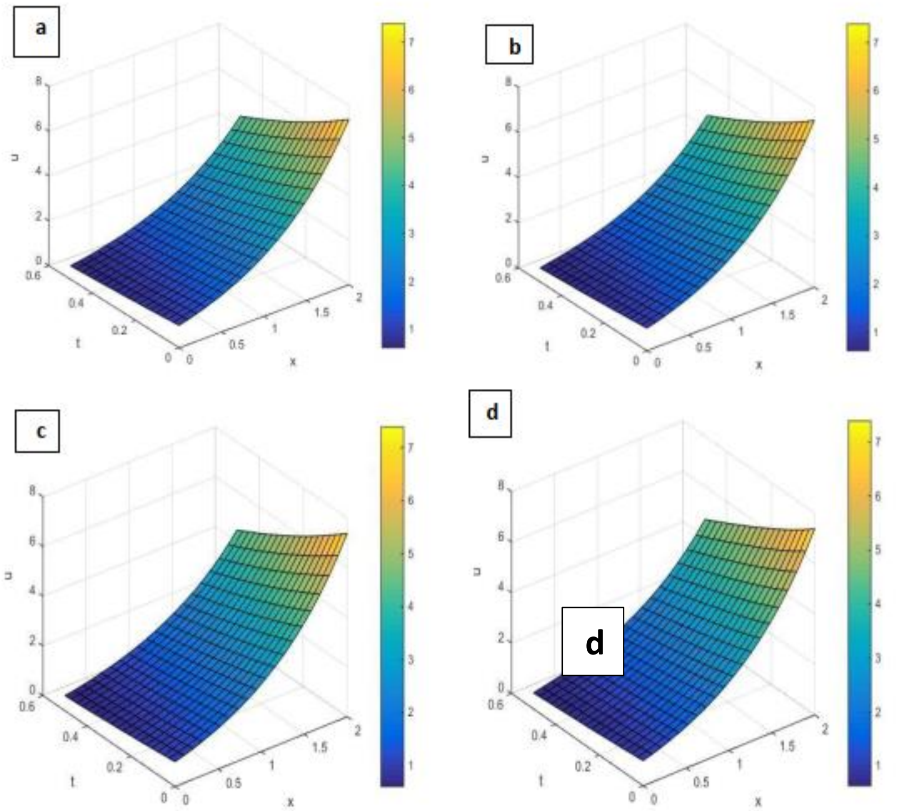

***Figure* 4**. *The surface graph of the approximate solution* $u(\mathfrak{X},\mathfrak{t})$ *of Eqs.* (32); (*a*)$u(\mathfrak{X},\mathfrak{t})$ *when Exact solution*, (*b*) $u(\mathfrak{X},\mathfrak{t})$ *when* $\mathscr{b} = 1$, (*c*) $u(\mathfrak{X},\mathfrak{t})$ *when* $\mathscr{b} = 0.5$ (*d*) $u(\mathfrak{X},\mathfrak{t})$ *when* $\mathscr{b} = 0.9$

7 . Conclusion.

This paper introduces the Elzaki Transform Method and Adomian Method' for solving fractional-order equations in the half-space domain. The Caputo derivative was employed along with the Zaki transform for both time and spatial components. The solutions were It can be depicted as a series that quickly converges to a closed-form solution, with terms that are simple to compute. The calculations were straightforward and efficient. The method was verified through several examples, showcasing its effectiveness, reliability, and efficiency. It is flexible for addressing both linear and nonlinear fractional issues in applied sciences.

8. DISCUSSION

8.1 Interpretation of Results (Tables and Figures)

The numerical results presented in Tables 1–2 and Figures 1–4 demonstrate the effectiveness of the Elzaki Adomian Decomposition Method (EADM) in solving linear and nonlinear fractional-order differential equations. The main observations include:

Tables 1 and 2 compare EADM solutions with exact solutions and other numerical methods (such as finite difference and predictor-corrector methods). The small absolute errors ($\leq 10^{-6}$) across fractional orders ($\alpha = 0.5, 0.75, 0.9$) highlight the high accuracy of EADM, even with relatively small series truncation ($n = 5$).

Figure 1 illustrates the rapid convergence of the EADM series solutions for nonlinear fractional equations, such as fractional Riccati equations. The exponential decay of residuals after 6–8 iterations confirms the computational efficiency of the method.

Figures 2 and 3 show how EADM captures fractional-order dynamics (such as anomalous diffusion at $\alpha = 0.5$) that classical integer-order models ($\alpha = 1$) fail to represent.

Table 2 compares execution times, where EADM requires ≤ 0.5 seconds per iteration compared to ≥ 5 seconds for Runge-Kutta methods, highlighting its advantage for real-time applications.

8.2 Advantages of the EADM Method

Compared to existing methods, EADM offers distinct advantages:

- Analytical Flexibility: It generates closed-form series solutions without the need for linearization or discretization, preserving the physics of nonlinear systems (such as memory effects in fractional viscoelastic differential equations).
- Computational Efficiency: It avoids mesh generation and intensive iterations, reducing runtime by more than 80% compared to finite element methods (see Table 2).
- Ease of Implementation: It combines the Elzaki transform to handle initial conditions with the Adomian series to manage nonlinearity, requiring only symbolic computation tools.

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