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Fixed Point for New Contraction Mapping in Fréchet Space via Fuzzy Structure with Application

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Abstract:

In this article, we introduce a new type of fuzzy contraction mapping in a fuzzy Fréchet space ($FF-$ space). This type is known as the $\vartheta-\theta-$fuzzy augmented contraction mapping, which is defined by $\vartheta-$acceptable mapping. We prove that this mapping possesses a fixed point by proving two results under specific conditions. To support our theoretical results, we studied an application that demonstrates the effectiveness of our approach for solving an integral equation and finding a unique solution.

Keywords: ϑ-acceptable mapping, ϑ-θ-fuzzy augmented contraction mapping, fuzzy Fréchet space($FF-$ space).

1. Introduction

In 1965, the renowned mathematician Zadeh introduced the concept of a fuzzy set [1], which led to significant advances in solving many mathematical problems. Since then, researchers have developed the concept of fuzziness and applied it to a wide range of fields. Fuzzy topology has played an important role in programming and design, particularly in algorithms, thereby advancing mathematical and computational applications. Through the fuzzy topological vector space [2], a combination of topology and functional analysis using fuzziness. In 2021, Al-Nafie and Ghanawi [3] introduced the concept of a fuzzy Fréchet space, a fuzzy topological vector space whose topology is generated by aa countable separating family of fuzzy semi-norms. In the same year, they defined the derivative in this space and used a fuzzy Fréchet space as a model space for the manifold space to study the derivative in fuzzy topological spaces [4]. This work is important in physics, in finding solutions to mathematical equations using these space, as well as in data science and robotics.

In 1922, Banach presented his important theorem about the fixed point of a contraction mapping in a complete metric space [5]. This theorem is a cornerstone of mathematical and functional analysis and has numerous applications, including solving mathematical equations, game theory, economics, operator theory, and more. Many mathematicians, especially those in mathematical analysis, functional analysis, and topology, have studied fixed point theorems with various contraction conditions in fuzzy spaces because of their great importance in mathematics, physics, and other fields [6,7, 8, 9, 10, 11].

Our aim in this study is to investigate a new type of fuzzy contraction mapping and to prove that it has fixed points in the fuzzy FF-space, for practical use in proving the existence of solutions to integral equations. Two key results are presented, ensuring the existence and uniqueness of fixed points under well-defined conditions. To validate the theoretical contributions, an illustrative example is provided. Moreover, we demonstrate the practical relevance of the results by applying them to prove the existence and uniqueness of a solution to a nonlinear integral equation, a type of application frequently encountered.

Through this work, we aim to contribute to the growing body of knowledge in fuzzy fixed-point theory, especially in fuzzy Fréchet settings, and to provide a theoretical framework for solving real-world problems modelled by integral or functional equations.

2. Preliminaries

This section presents essential definitions and fundamental concepts that will serve as the groundwork for the results developed throughout this paper.

Definition 1.1. [12] A continuous triangular norm (t-norm) refers to a binary function $*: [0,1]^2 \to [0,1]$ that meets the following conditions:

1. The operation is commutative, associative, and exhibits continuity over its domain.
2. It holds that $*$ has 1 as an identity element, i.e., $*(\ell, 1) = \ell$ for every $\ell \in [0,1]$.
3. Moreover, $*$ is monotonic in both arguments, meaning that for all $\ell, \hbar, g, f \in [0,1]$, if $\ell \leq g$ and $\hbar \leq f$, then $*(\ell, \hbar) \leq *(g, f)$.

Definition 1.2. [13] Consider a vector space $\mathcal{Q}$. A fuzzy subset $\mathcal{W}$ of $\mathcal{Q} \times \mathbb{R}$ is termed a fuzzy semi-norm on $\mathcal{Q}$, provided that it meets a specific set of axioms outlined below

1. $\mathcal{W}(\hbar, b) = 0, \forall b \leq 0$,
2. $\mathcal{W}(q\hbar, b) = \mathcal{W}\left(\hbar, \frac{b}{q}\right), \forall q \neq 0$,
3. $\mathcal{W}(\hbar, b) * \mathcal{W}(\ell, a) \leq \mathcal{W}(\ell + \hbar, b + a)$,
4. For every $\hbar \in \mathcal{Q}, \mathcal{W}(\hbar, b)$ is nondecreasing with respect to b, $\lim_{b \to 0} \mathcal{W}(\hbar, b) = 0$, $\lim_{b \to \infty} \mathcal{W}(\hbar, b) = 1$.

Definition 1.3. [3] A fuzzy Fréchet space (abbreviated as $FF - space$) is defined as a complete fuzzy topological vector space $\mathcal{Q}$, whose topology τ_{Ξ} arises from a countable and separating collection $\Xi = \{\mathcal{W}_k\}_{k \in I}$ of fuzzy semi-norms on $\mathcal{Q}$.

The foundational concepts of fuzzy continuity, fuzzy Cauchy sequence, and fuzzy convergence within the framework of fuzzy Fréchet spaces were thoroughly explored in [3], where the structural development of the $FF -$ space was also established.

Definition 1.4. [3] Let $\mathcal{S}: \mathcal{Q} \to \mathcal{O}$ be a transformation between two $FF -$ spaces. The map $\mathcal{S}$ is said to be continuous in the fuzzy Fréchet sense at a point $\ell \in \mathcal{Q}$ if, whenever a sequence $\{\ell_m\}_{m \in \mathbb{N}}$ in $\mathcal{Q}$ converges to ℓ, the sequence $\{\mathcal{S}(\ell_m)\}_{m \in \mathbb{N}}$ in $\mathcal{O}$ converges to $\mathcal{S}(\ell)$. When this property holds for every point $\ell \in \mathcal{Q}$, the function $\mathcal{S}$ is referred to as continuous throughout the $FF -$ space $\mathcal{Q}$.

3. Main Results

In this section, we develop the core contributions of the present work. As a preliminary step, we introduce two definitions that will play a crucial role in the forthcoming analysis.

Assume that $\mathcal{Q}$ is an $FF -$space, and its fuzzy topological structure is determined by the system of fuzzy semi-norms $\Xi = \{\mathcal{W}_k\}_{k \in I}$. Let us denote by Ψ the set of all mappings $\theta: [0, \infty) \to [0, \infty)$ satisfying the following conditions:

i. θ is nondecreasing and right-continuous,
ii. $\theta(b) < b$ for every $b > 0$,

iii. The set $\theta^{-1}(\{0\})$ consist solely of the point 0.

It is readily observed that, for each mapping $\theta \in \Psi$, the sequence defined by successive iterations $\theta^n(b)$ tends to zero as $n \to \infty$, for every $b > 0$.

Definition 3.1. Let $\mathcal{S}: \mathcal{Q} \to \mathcal{Q}$ be a self-map. When a mapping $\vartheta: \mathcal{Q} \times (0, \infty) \to [0, \infty)$ exists, we say that $\mathcal{S}$ is ϑ-acceptable if the following condition holds: $\forall b > 0, \ell, \ell\!\!k \in \mathcal{Q}$

$$\vartheta(\ell - k, b) \geq 1 \Longrightarrow \vartheta(\mathcal{S}(\ell) - \mathcal{S}(k), b) \geq 1. \tag{1}$$

Definition 3.2. A self-mapping $\mathcal{S}: \mathcal{Q} \to \mathcal{Q}$ is called ϑ-θ-fuzzy augmented contraction mapping if there exist two mappings $\vartheta: \mathcal{Q} \times (0, \infty) \to [0, \infty)$, and $\theta \in \Psi$ such that the following inequality is satisfied

$$\theta\left(\frac{1}{v(\ell, k, b)} - 1\right) \geq \vartheta(\ell - k, b)\left(\frac{1}{\mathcal{W}_k(\mathcal{S}(\ell) - \mathcal{S}(k), b)} - 1\right), \tag{2}$$

$\forall b > 0, \ell, k \in \mathcal{Q}$, and $\forall \mathcal{W} \in \Xi$, where

$$v(\ell, k, b) = \min\left\{\mathcal{W}_k(\ell - k, b), \mathcal{W}_k(\ell - \mathcal{S}(\ell), b), \mathcal{W}_k(k - \mathcal{S}(k), b), \frac{2\mathcal{W}_k(\ell - \mathcal{S}(k), b)\mathcal{W}_k(k - \mathcal{S}(\ell), b)}{\mathcal{W}_k(\ell - \mathcal{S}(k), b) + \mathcal{W}_k(k - \mathcal{S}(\ell), b)}\right\}. \tag{3}$$

At this stage, we proceed to formulate and rigorously validate the first principle theorem of the paper.

Theorem 3.3. Let $\mathcal{S}: \mathcal{Q} \to \mathcal{Q}$ be a ϑ-θ-fuzzy augmented contraction mapping, and the system of fuzzy semi-norms $\Xi = \{\mathcal{W}_k\}_{k \in I}$ is triangular. Assume that the following conditions are satisfied:

1. The mapping $\mathcal{S}$ is ϑ-acceptable;
2. There exists an initial point $\ell_0 \in \mathcal{Q}$ such that $\vartheta(\ell_0 - \mathcal{S}(\ell_0), b) \geq 1, \forall b > 0$.
3. The mapping $\mathcal{S}$ is continuous on $\mathcal{Q}$.

Under these assumptions, a fixed point of $\mathcal{S}$ is guaranteed to exist.

Proof. Consider the iterative sequence $\{\ell_m\}$ in $\mathcal{Q}$ given by the recurrence relation $\ell_m = \mathcal{S}(\ell_{m-1}), \forall m \in \mathbb{N}$. Suppose there exists an index $m \in \mathbb{N}$ for which the iteration stabilizes, i.e., $\ell_m = \ell_{m-1}$. Then the common value $\mathfrak{X} = \ell_{m-1}$ constitutes a fixed point of $\mathcal{S}$. Suppose that $\ell_m \neq \ell_{m-1}, \forall m \in \mathbb{N}$. Given that $\vartheta(\ell_0 - \mathcal{S}(\ell_0), b) = \vartheta(\ell_0 - \ell_1, b) \geq 1$ and since $\mathcal{S}$ is ϑ-acceptable, it follows that $\vartheta(\mathcal{S}(\ell_0) - \mathcal{S}(\ell_1), b) = \vartheta(\ell_1 - \ell_2, b) \geq 1, \forall b > 0$. By carrying on with this procedure, we obtain $\vartheta(\ell_{m-1} - \ell_m, b) \geq 1, \forall m \in \mathbb{N}$ and $\forall b > 0$. Moreover, by inequality (2), , $\forall m \in \mathbb{N}$, $\forall b > 0$ and $\forall \mathcal{W} \in \Xi$, we get

$$\begin{aligned}\frac{1}{\mathcal{W}_k(\ell_m - \ell_{m+1}, b)} - 1 = \frac{1}{\mathcal{W}_k(\mathcal{S}(\ell_{m-1}) - \mathcal{S}(\ell_m), b)} - 1 &\leq \vartheta(\ell_{m-1} - \ell_m, b)\left(\frac{1}{\mathcal{W}_k(\mathcal{S}(\ell_{m-1}) - \mathcal{S}(\ell_m), b)} - 1\right) \\ &\leq \theta\left(\frac{1}{v(\ell_{m-1}, \ell_m, b)} - 1\right),\end{aligned} \tag{4}$$

Where

$$v(\ell_{m-1},\ell_m,b) = \min\Big\{\mathcal{W}_k(\ell_{m-1}-\ell_m,b),\mathcal{W}_k(\ell_{m-1}-\mathcal{S}(\ell_{m-1}),b),\mathcal{W}_k(\ell_m - \mathcal{S}(\ell_m),b),\frac{2\mathcal{W}_k(\ell_{m-1}-\mathcal{S}(\ell_m),b)\mathcal{W}_k(\ell_m-\mathcal{S}(\ell_{m-1}),b)}{\mathcal{W}_k(\ell_m-\mathcal{S}(\ell_{m-1}),b)+\mathcal{W}_k(\ell_{m-1}-\mathcal{S}(\ell_m),b)}\Big\}$$
$$= \min\Big\{\mathcal{W}_k(\ell_{m-1}-\ell_m,b),\mathcal{W}_k(\ell_m-\ell_{m+1},b),\frac{2\mathcal{W}_k(\ell_{m-1}-\ell_{m+1},b)}{1+\mathcal{W}_k(\ell_{m-1}-\ell_{m+1},b)}\Big\}, \tag{5}$$

Quite the opposite,

$$\frac{2\mathcal{W}_k(\ell_{m-1}-\ell_{m+1},b)}{1+\mathcal{W}_k(\ell_{m-1}-\ell_{m+1},b)} = \frac{2}{\left(1/\mathcal{W}_k(\ell_{m-1}-\ell_{m+1},b)\right)+1}$$
$$\geq \frac{2}{\left(1/\mathcal{W}_k(\ell_m-\ell_{m+1},b)\right)+\left(1/\mathcal{W}_k(\ell_{m-1}-\ell_m,b)\right)}$$
$$\geq min\{\mathcal{W}_k(\ell_m-\ell_{m+1},b),\ \mathcal{W}_k(\ell_{m-1}-\ell_m,b)\}. \tag{6}$$

Inequalities (5) and (6) together imply that

$$v(\ell_{m-1},\ell_m,b) = min\{\mathcal{W}_k(\ell_m-\ell_{m+1},b),\ \mathcal{W}_k(\ell_{m-1}-\ell_m,b)\}.$$

Substituting the above result into (4) yields

$$\frac{1}{\mathcal{W}_k(\ell_m-\ell_{m+1},b)} - 1 \leq \theta\left(\frac{1}{min\{\mathcal{W}_k(\ell_m-\ell_{m+1},b),\mathcal{W}_k(\ell_{m-1}-\ell_m,b)\}} - 1\right). \tag{7}$$

In the event when $min\{\mathcal{W}_k(\ell_m-\ell_{m+1},b),\ \mathcal{W}_k(\ell_{m-1}-\ell_m,b)\} = \mathcal{W}_k(\ell_m-\ell_{m+1},b)$, according to the condition $\theta\ (b) < b$, we can infer that

$$\frac{1}{\mathcal{W}_k(\ell_m-\ell_{m+1},b)} - 1 \leq \theta\left(\frac{1}{\mathcal{W}_k(\ell_m-\ell_{m+1},b)} - 1\right) < \frac{1}{\mathcal{W}_k(\ell_m-\ell_{m+1},b)} - 1, \tag{8}$$

This leads to a contradiction. Therefore, it must be that $min\{\mathcal{W}_k(\ell_m-\ell_{m+1},b),\ \mathcal{W}_k(\ell_{m-1}-\ell_m,b)\} = \mathcal{W}_k(\ell_m-\ell_{m-1},b)$ and consequently

$$\frac{1}{\mathcal{W}_k(\ell_m-\ell_{m+1},b)} - 1 \leq \theta\left(\frac{1}{\mathcal{W}_k(\ell_m-\ell_{m-1},b)} - 1\right) < \frac{1}{\mathcal{W}_k(\ell_m-\ell_{m-1},b)} - 1. \tag{9}$$

It follows that $\mathcal{W}_k(\ell_m-\ell_{m-1},b) < \mathcal{W}_k(\ell_m-\ell_{m+1},b)$, which implies that the sequence $\{\mathcal{W}_k(\ell_m-\ell_{m+1},b)\}$ is strictly increasing on the interval [0,1], $\forall b > 0$ and $\forall \mathcal{W} \in \Xi$. Define $T(b) = \lim_{m\to\infty} \mathcal{W}_k(\ell_m-\ell_{m+1},b), \forall b > 0$ and $\forall \mathcal{W} \in \Xi$. We assert that $T(b) = 1$ holds $\forall b > 0$. Assume, for the sake of contradiction, that there exists some $b_0 > 0$ such that $T(b_0) < 1$. By taking the limit on both sides of inequality (9), we have

$$\frac{1}{T(b_0)}-1\leq\theta\left(\frac{1}{T(b_0)}-1\right)<\frac{1}{T(b_0)}-1,$$

(10)

This leads to a contradiction. Therefore, we conclude that $\lim_{m\to\infty}\mathcal{W}_k(\ell_m-\ell_{m+1},b)=1,\forall b>0$ and $\forall\mathcal{W}\in\Xi$. Next, fixing an arbitrary $r\in\mathbb{N}$, it follows that

$$\mathcal{W}_k(\ell_m-\ell_{m+r},b)\geq\mathcal{W}_k\left(\ell_m-\ell_{m+1},\frac{b}{r}\right)*\mathcal{W}_k\left(\ell_{m+1}-\ell_{m+2},\frac{b}{r}\right)*\ldots*\mathcal{W}_k\left(\ell_{m+r-1}-\ell_{m+r},\frac{b}{r}\right)$$
$$\to 1*1*\ldots*1=1, as\ m\to\infty$$

(11)

Therefore, the sequence $\{\ell_m\}$ is Cauchy. Given that the space $\mathcal{Q}$ is complete, there exist an element $\mathfrak{X}\in\mathcal{Q}$ such that $\ell_m\to\mathfrak{X}$ as $m\to\infty,\forall b>0$. Due to the continuity of $\mathcal{S}$, it follows that $\mathcal{S}(\ell_m)\to\mathcal{S}(\mathfrak{X})$ as $m\to\infty,\forall b>0$. Thus, we conclude

$$\lim_{m\to\infty}\mathcal{W}_k(\ell_{m+1}-\mathcal{S}(\mathfrak{X}),b)=\lim_{m\to\infty}\mathcal{W}_k(\mathcal{S}(\ell_m)-\mathcal{S}(\mathfrak{X}),b)=1,$$

$\forall b>0$ and $\forall\mathcal{W}\in\Xi$, in other words, $\ell_m\to\mathcal{S}(\mathfrak{X})$ as $m\to\infty$. By the uniqueness of limits, it follows that $\mathcal{S}(\mathfrak{X})=\mathfrak{X}$.■

In the preceding theorem, the uniqueness of the fixed point is ensured if $\vartheta(\ell-\hbar,b)\geq 1,\forall b>0$, $\ell,\hbar\in\mathcal{Q}$ such that $\mathcal{S}(\ell)=\ell$ and $\mathcal{S}(\hbar)=\hbar$.

The subsequent theorem is established without requiring the continuity of the mapping $\mathcal{S}$.

Theorem 3.4. . Let $\mathcal{S}:\mathcal{Q}\to\mathcal{Q}$ be a ϑ-θ-fuzzy augmented contraction mapping , and the system of fuzzy semi-norms $\Xi=\{\mathcal{W}_k\}_{k\in I}$ is triangular. Assume that the following conditions are satisfied:

1. The mapping $\mathcal{S}$ is ϑ-acceptable;
2. There exists an initial point $\ell_0\in\mathcal{Q}$ such that $\vartheta(\ell_0-\mathcal{S}(\ell_0),b)\geq 1,\forall b>0$.
3. For every sequence $\{\ell_m\}$ in $\mathcal{Q}$ satisfying $\vartheta(\ell_m-\ell_{m+1},b)\geq 1,\forall b>0$ and $\forall m>0$, and such that $\ell_m\to\ell$ as $m\to\infty$, it follows that $\vartheta(\ell_m-\ell,b)\geq 1,\forall b>0$ and $\forall m>0$.

Consequently, $\mathcal{S}$ possesses a fixed point under the stated hypotheses.

Proof. In accordance with the argument presented in the proof of Theorem 3.3, we construct a sequence $\{\ell_m\}$ in $\mathcal{Q}$ that is Cauchy and satisfies $\vartheta(\ell_m-\ell_{m+1},b)\geq 1,\forall b>0$ and $\forall m>0$. Furthermore, there exists an element $\mathfrak{X}\in\mathcal{Q}$ such that $\ell_m\to\mathfrak{X}$ as $m\to\infty$. Based on condition (3), it follows that $(\ell_m-\mathfrak{X},b)\geq 1,\forall b>0$ and $\forall m>0$. Consequently, $\forall b>0$ and $\forall\mathcal{W}\in\Xi$, we obtain

$$\frac{1}{\mathcal{W}_k(\mathcal{S}(\mathfrak{X})-\mathfrak{X},b)}-1\leq\left(\frac{1}{\mathcal{W}_k(\mathcal{S}(\mathfrak{X})-\mathcal{S}(\ell_m),b)}-1\right)+\left(\frac{1}{\mathcal{W}_k(\ell_{m+1}-\mathfrak{X},b)}-1\right)$$
$$\leq\vartheta(\ell_m-\mathfrak{X},b)\left(\frac{1}{\mathcal{W}_k(\mathcal{S}(\mathfrak{X})-\mathcal{S}(\ell_m),b)}-1\right)+\left(\frac{1}{\mathcal{W}_k(\ell_{m+1}-\mathfrak{X},b)}-1\right)$$
$$\leq\theta\left(\frac{1}{v(\ell_m,\mathfrak{X},b)}-1\right)+\left(\frac{1}{\mathcal{W}_k(\ell_{m+1}-\mathfrak{X},b)}-1\right).$$

(12)

Quite the opposite,

$$\begin{aligned}\mathcal{W}_k(\mathcal{S}(\mathfrak{X})-\mathfrak{X},b) &\geq v(\ell_m,\mathfrak{X},b)\\ &= \min\left\{\mathcal{W}_k(\ell_m-\mathfrak{X},b),\mathcal{W}_k(\ell_m-\mathcal{S}(\ell_m),b),\mathcal{W}_k(\mathfrak{X}\right.\\ &\left.-\mathcal{S}(\mathfrak{X}),b),\frac{2\mathcal{W}_k(\ell_m-\mathcal{S}(\mathfrak{X}),b)\mathcal{W}_k(\mathfrak{X}-\mathcal{S}(\ell_m),b)}{\mathcal{W}_k(\ell_m-\mathcal{S}(\mathfrak{X}),b)+\mathcal{W}_k(\mathfrak{X}-\mathcal{S}(\ell_m),b)}\right\}\\ &= \min\left\{\mathcal{W}_k(\ell_m-\mathfrak{X},b),\mathcal{W}_k(\ell_m-\ell_{m+1},b),\mathcal{W}_k(\mathfrak{X}\right.\\ &\left.-\mathcal{S}(\mathfrak{X}),b),\frac{2\mathcal{W}_k(\ell_m-\mathcal{S}(\mathfrak{X}),b)\mathcal{W}_k(\mathfrak{X}-\ell_{m+1},b)}{\mathcal{W}_k(\ell_m-\mathcal{S}(\mathfrak{X}),b)+\mathcal{W}_k(\mathfrak{X}-\ell_{m+1},b)}\right\}\end{aligned} \tag{13}$$

Where

$$\begin{aligned}\frac{2\mathcal{W}_k(\ell_m-\mathcal{S}(\mathfrak{X}),b)\mathcal{W}_k(\mathfrak{X}-\ell_{m+1},b)}{\mathcal{W}_k(\ell_m-\mathcal{S}(\mathfrak{X}),b)+\mathcal{W}_k(\mathfrak{X}-\ell_{m+1},b)} &= \frac{2}{\left({}^{1}\!/\!_{\mathcal{W}_k(\mathfrak{X}-\ell_{m+1},b)}\right)+\left({}^{1}\!/\!_{\mathcal{W}_k(\ell_m-\mathcal{S}(\mathfrak{X}),b)}\right)}\\ &\geq \frac{2}{\left({}^{1}\!/\!_{\mathcal{W}_k(\mathfrak{X}-\ell_{m+1},b)}\right)+\left({}^{1}\!/\!_{\mathcal{W}_k(\ell_m-\mathfrak{X},b)}\right)+\left({}^{1}\!/\!_{\mathcal{W}_k(\mathfrak{X}-\mathcal{S}(\mathfrak{X}),b)}\right)-1}\\ &\to \frac{2}{1+\left({}^{1}\!/\!_{\mathcal{W}_k(\mathfrak{X}-\mathcal{S}(\mathfrak{X}),b)}\right)} \geq \frac{2}{\left({}^{1}\!/\!_{\mathcal{W}_k(\mathfrak{X}-\mathcal{S}(\mathfrak{X}),b)}\right)+\left({}^{1}\!/\!_{\mathcal{W}_k(\mathfrak{X}-\mathcal{S}(\mathfrak{X}),b)}\right)}\\ &= \mathcal{W}_k(\mathfrak{X}-\mathcal{S}(\mathfrak{X}),b).\end{aligned} \tag{14}$$

Letting m tend to infinity in both sides of inequality (13), we conclude that the limit satisfies $\lim_{m\to\infty} v(\ell_m,\mathfrak{X},b) = \mathcal{W}_k(\mathfrak{X}-\mathcal{S}(\mathfrak{X}),b)$. Using the right-continuity condition of θ, and as m approaches infinity in inequality (12), the resulting inequality is given by

$$\left({}^{1}\!/\!_{\mathcal{W}_k(\mathfrak{X}-\mathcal{S}(\mathfrak{X}),b)}\right)-1 \leq \theta\left(\left({}^{1}\!/\!_{\mathcal{W}_k(\mathfrak{X}-\mathcal{S}(\mathfrak{X}),b)}\right)-1\right).$$

It follows that $\left({}^{1}\!/\!_{\mathcal{W}_k(\mathfrak{X}-\mathcal{S}(\mathfrak{X}),b)}\right)-1=0$. Hence, $\mathcal{W}_k(\mathfrak{X}-\mathcal{S}(\mathfrak{X}),b)=1$ consequently $\mathfrak{X}=\mathcal{S}(\mathfrak{X})$. ■

Example 3.5. Let $\mathcal{Q}=F\cup G\cup H$ be $FF-$space such that $F=\left\{\frac{v}{u},u=1,4,\dots,3r+1,v=0,1,3,9,\dots\right\}, G=\left\{\frac{v}{u},u=2,5,\dots,3r+2,v=1,3,9,\dots\right\}$, and $H=\{2r: r\in\mathbb{N}\}$. Let $\mathcal{W}_k(\ell,b)=\frac{b}{b+|\ell|}, \forall\ell\in\mathcal{Q}$ and $b>0$, and let $d*c=dc, \forall d,c\in[0,1]$. Construct $\mathcal{S}:\mathcal{Q}\to\mathcal{Q}$ by

$$\mathcal{S}(\ell)=\begin{cases}\frac{3\ell}{11}, & \ell\in F,\\ \frac{\ell}{8}, & \ell\in G,\\ 2\ell, & \ell\in H.\end{cases} \tag{15}$$

and $\vartheta:\mathcal{Q}\times(0,\infty)\to[0,\infty)$ by

$$\vartheta(\ell-k,b)=\begin{cases}1, & \ell,k\in F\cup G,\\ 0, & \text{otherwise.}\end{cases}$$

(16)

Now, if $\ell,k\in F$, we have

$$\frac{1}{\mathcal{W}_k(\mathcal{S}(\ell)-\mathcal{S}(k),b)}-1=\frac{\left|\frac{3\ell}{11}-\frac{3k}{11}\right|}{b}=\frac{3}{11}\frac{|\ell-k|}{b}=\frac{3}{11}\left(\frac{1}{\mathcal{W}_k(\ell-k,b)}-1\right)\le\frac{6}{11}\left(\frac{1}{v(\ell,k,b)}-1\right).$$

(17)

if $\ell,k\in G$, we have

$$\frac{1}{\mathcal{W}_k(\mathcal{S}(\ell)-\mathcal{S}(k),b)}-1=\frac{\left|\frac{\ell}{8}-\frac{k}{8}\right|}{b}=\frac{1}{8}\frac{|\ell-k|}{b}=\frac{1}{8}\left(\frac{1}{\mathcal{W}_k(\ell-k,b)}-1\right)\le\frac{6}{11}\left(\frac{1}{v(\ell,k,b)}-1\right).$$

(18)

if $\ell,k\in G$, we get $\vartheta(\ell-k,b)=0$, moreover, it is clear that inequality (5) remains valid under the given assumptions.

If $\ell\in F$ and $k\in G$, we obtain

$$\frac{1}{\mathcal{W}_k(\mathcal{S}(\ell)-\mathcal{S}(k),b)}-1=\frac{\left|\frac{3\ell}{11}-\frac{k}{8}\right|}{b}=\frac{3}{11}\frac{\left|\ell-\left(\frac{11}{24}\right)k\right|}{b}.$$

(19)

Therefore, in this case where $\ell>\left(\frac{11}{24}\right)k$, the following conclusion can be drawn:

$$\begin{aligned}\frac{1}{\mathcal{W}_k(\mathcal{S}(\ell)-\mathcal{S}(k),b)}-1&=\frac{3}{11}\frac{\left(\ell-\left(\frac{11}{24}\right)k\right)}{b}\le\frac{3}{11}\frac{\left(\ell-\left(\frac{1}{8}\right)k\right)}{b}=\frac{6}{11}\left(\frac{1}{2}\left(\frac{1}{\mathcal{W}_k(\ell-\mathcal{S}(k),b)}-1\right)\right)\\&=\frac{6}{11}\left(\frac{1}{2}\left(\frac{1}{\mathcal{W}_k(\ell-\mathcal{S}(k),b)}+1\right)-1\right)\le\frac{6}{11}\left(\frac{1}{2}\left(\frac{1}{\mathcal{W}_k(\ell-\mathcal{S}(k),b)}+\frac{1}{\mathcal{W}_k(k-\mathcal{S}(\ell),b)}\right)-1\right)\\&=\frac{6}{11}\left(\frac{1}{((2\mathcal{W}_k(\ell-\mathcal{S}(k),b)\mathcal{W}_k(k-\mathcal{S}(\ell),b)/(\mathcal{W}_k(k-\mathcal{S}(\ell),b)+\mathcal{W}_k(\ell-\mathcal{S}(k),b)))}-1\right)\\&\le\frac{6}{11}\left(\frac{1}{v(\ell,k,b)}-1\right).\end{aligned}$$

(20)

and in this case where $\ell>\left(\frac{11}{24}\right)k$, the following conclusion can be drawn:

$$\frac{1}{\mathcal{W}_k(\mathcal{S}(\ell)-\mathcal{S}(k),b)}-1=\frac{3}{11}\frac{\left(\left(\frac{11}{24}\right)k-\ell\right)}{b}\le\frac{3}{11}\frac{(k-\ell)}{b}\le\frac{3}{11}\left(\frac{1}{\mathcal{W}_k(\ell-k,b)}-1\right)\le\frac{6}{11}\left(\frac{1}{v(\ell,k,b)}-1\right).$$

(21)

Accordingly, we find that $\left(\frac{1}{\mathcal{W}_k(\mathcal{S}(\ell)-\mathcal{S}(\mathscr{k}),b)}-1\right) \leq \frac{6}{11}\left(\frac{1}{v(\ell,\mathscr{k},b)}-1\right)$, $\forall \ell,\mathscr{k} \in F \cup G$. Therefore, based on the definition of the mapping ϑ, we obtain the following:

$$\vartheta(\ell-\mathscr{k},b)\left(\frac{1}{\mathcal{W}_k(\mathcal{S}(\ell)-\mathcal{S}(\mathscr{k}),b)}-1\right) \leq \frac{6}{11}\left(\frac{1}{v(\ell,\mathscr{k},b)}-1\right),$$

$\forall \ell,\mathscr{k} \in Q$. Hence, $\mathcal{S}$ is a ϑ-θ-fuzzy augmented contraction mapping with $\theta(b) = \left(\frac{6}{11}\right)b$. In addition, when $\ell_0 = 1$, it follows that

$$\vartheta(\ell_0 - \mathcal{S}(\ell_0), b) = \vartheta\left(1-\frac{3}{11}, b\right) = 1.$$

It can be readily verified that mapping $\mathcal{S}$ is ϑ-acceptable and the condition (3) in Theorem 3.4 is satisfied. Consequently, by invoking Theorem 3.4, we conclude that $\mathcal{S}$ has a fixed point. In this context, we observe that $\mathcal{S}(0) = 0$.

4. Application

In this section, we explore an application that demonstrates the uniqueness of the solution to a certain class of integral equations. To this end, we consider an integral equation of the following form, which will serve as a framework for applying the established fixed-point results,

$$\ell(\upsilon) = \mathcal{P}(\upsilon) + \int_0^{\upsilon} \mathcal{G}(\upsilon,u,\ell(u))du, \forall \upsilon \in [0,k], k > 0, \tag{22}$$

and Fréchet space $C([0,k],\mathbb{R})$, consisting of all real-valued continuous functions defined on the interval $[0,k]$, endowed with the topology induced by the supremum semi-norm,

$$w_k(\ell) = sup_{\upsilon\in[0,k]}|\ell(\upsilon)|, \ell \in C([0,k],\mathbb{R}), \tag{23}$$

We proceed by considering a fuzzy Fréchet space $(FF - space)$ structured via the product t-norm, with fuzzy semi-norms specified in the following manner:

$$\mathcal{W}_k(\ell,b) = \frac{b}{b+w_k(\ell)}, \forall \ell \in C([0,k],\mathbb{R}) \text{ and } b > 0. \tag{24}$$

Now, consider the integral operator $\mathcal{S}$ defined on a $FF - space$

$C([0,k],\mathbb{R})$ as follows

$$\mathcal{S}\ell(\upsilon) = \mathcal{P}(\upsilon) + \int_0^{\upsilon} \mathcal{G}(\upsilon,u,\ell(u))du, \tag{25}$$

where $\mathcal{G}$ satisfies the following assumptions: there exists a function $\mathcal{T}: [0,k]^2 \to [0,\infty)$ such that $\mathcal{T} \in L^1([0,k],\mathbb{R})$, and $\forall \ell,\mathscr{k} \in C([0,k],\mathbb{R})$, $\upsilon,u \in [0,k]$, the following condition holds:

$$\left|\mathcal{G}(\mathcal{v},u,\ell(u))-\mathcal{G}(\mathcal{v},u,\mathscr{k}(u))\right| \leq \mathcal{T}(\mathcal{v},u)\max\left\{|\ell(u)-\mathscr{k}(u)|,|\ell(u)-\mathcal{S}\mathscr{k}(u)|,|\mathscr{k}(u)-\mathcal{S}\mathscr{k}(u)|,\frac{2|\ell(u)-\mathcal{S}\mathscr{k}(u)||\mathscr{k}(u)-\mathcal{S}\ell(u)|}{|\ell(u)-\mathcal{S}\mathscr{k}(u)|+|\ell(u)-\mathscr{k}(u)|}\right\}, \quad (26)$$

where

$$sup_{\mathcal{v}\in[0,k]}\int_0^{\mathcal{v}}\mathcal{T}(\mathcal{v},u)d\,u \leq \beta < 1. \quad (27)$$

In what follows, we aim to establish that the integral equation (22) admits a unique solution.

Assume that ℓ and $\mathscr{k}$ are elements of the space $C([0,k],\mathbb{R})$, and let us examine the following expression:

$$|\mathcal{S}\ell(\mathcal{v})-\mathcal{S}\mathscr{k}(\mathcal{v})| \leq \int_0^{\mathcal{v}}\left|\mathcal{G}(\mathcal{v},u,\ell(u))-\mathcal{G}(\mathcal{v},u,\mathscr{k}(u))\right|d\,u$$

$$\leq \int_0^{\mathcal{v}}\mathcal{T}(\mathcal{v},u)\max\left\{|\ell(u)-\mathscr{k}(u)|,|\ell(u)-\mathcal{S}\mathscr{k}(u)|,|\mathscr{k}(u)-\mathcal{S}\mathscr{k}(u)|,\frac{2|\ell(u)-\mathcal{S}\mathscr{k}(u)||\mathscr{k}(u)-\mathcal{S}\ell(u)|}{|\ell(u)-\mathcal{S}\mathscr{k}(u)|+|\ell(u)-\mathscr{k}(u)|}\right\}d\,u$$

$$\leq \int_0^{\mathcal{v}}\mathcal{T}(\mathcal{v},u)\max\left\{sup_{\mathcal{v}\in[0,k]}|\ell(u)-\mathscr{k}(u)|,sup_{\mathcal{v}\in[0,k]}|\ell(u)-\mathcal{S}\mathscr{k}(u)|,sup_{\mathcal{v}\in[0,k]}|\mathscr{k}(u)-\mathcal{S}\mathscr{k}(u)|,\frac{2sup_{\mathcal{v}\in[0,k]}|\ell(u)-\mathcal{S}\mathscr{k}(u)|sup_{\mathcal{v}\in[0,k]}|\mathscr{k}(u)-\mathcal{S}\ell(u)|}{sup_{\mathcal{v}\in[0,k]}|\ell(u)-\mathcal{S}\mathscr{k}(u)|+sup_{\mathcal{v}\in[0,k]}|\ell(u)-\mathscr{k}(u)|}\right\}d\,u$$

$$\leq \max\left\{w_k(\ell-\mathscr{k}),w_k(\ell-\mathcal{S}\mathscr{k}),w_k(\mathscr{k}-\mathcal{S}\mathscr{k}),\frac{2w_k(\ell-\mathcal{S}\mathscr{k})w_k(\mathscr{k}-\mathcal{S}\ell)}{w_k(\ell-\mathcal{S}\mathscr{k})+w_k(\ell-\mathscr{k})}\right\}.$$

Accordingly,

$$w_k(\mathcal{S}\ell-\mathcal{S}\mathscr{k}) \leq \beta\max\left\{w_k(\ell-\mathscr{k}),w_k(\ell-\mathcal{S}\mathscr{k}),w_k(\mathscr{k}-\mathcal{S}\mathscr{k}),\frac{2w_k(\ell-\mathcal{S}\mathscr{k})w_k(\mathscr{k}-\mathcal{S}\ell)}{w_k(\ell-\mathcal{S}\mathscr{k})+w_k(\ell-\mathscr{k})}\right\}. \quad (28)$$

By applying (24), we obtain the following expression

$$v(\ell-\mathscr{k},b)=\min\left\{\frac{b}{b+w_k(\ell-\mathscr{k})},\frac{b}{b+w_k(\ell-\mathcal{S}\mathscr{k})},\frac{b}{b+w_k(\mathscr{k}-\mathcal{S}\mathscr{k})},\frac{b}{b+\frac{2w_k(\ell-\mathcal{S}\mathscr{k})w_k(\mathscr{k}-\mathcal{S}\ell)}{w_k(\ell-\mathcal{S}\mathscr{k})+w_k(\ell-\mathscr{k})}}\right\}$$

$$=\frac{b}{b+\max\left\{w_k(\ell-\mathscr{k}),w_k(\ell-\mathcal{S}\mathscr{k}),w_k(\mathscr{k}-\mathcal{S}\mathscr{k}),\frac{2w_k(\ell-\mathcal{S}\mathscr{k})w_k(\mathscr{k}-\mathcal{S}\ell)}{w_k(\ell-\mathcal{S}\mathscr{k})+w_k(\ell-\mathscr{k})}\right\}}.$$

Then,

$$\frac{1}{v(\ell-\mathscr{k},b)}-1=\frac{b+\max\left\{w_k(\ell-\mathscr{k}),w_k(\ell-\mathcal{S}\mathscr{k}),w_k(\mathscr{k}-\mathcal{S}\mathscr{k}),\frac{2w_k(\ell-\mathcal{S}\mathscr{k})w_k(\mathscr{k}-\mathcal{S}\ell)}{w_k(\ell-\mathcal{S}\mathscr{k})+w_k(\ell-\mathscr{k})}\right\}}{b}-1$$
$$=\frac{\max\left\{w_k(\ell-\mathscr{k}),w_k(\ell-\mathcal{S}\mathscr{k}),w_k(\mathscr{k}-\mathcal{S}\mathscr{k}),\frac{2w_k(\ell-\mathcal{S}\mathscr{k})w_k(\mathscr{k}-\mathcal{S}\ell)}{w_k(\ell-\mathcal{S}\mathscr{k})+w_k(\ell-\mathscr{k})}\right\}}{b}.$$

Hence,

$$\frac{1}{\mathcal{W}_k(\mathcal{S}(\ell)-\mathcal{S}(\mathscr{k}),b)}-1=\frac{w_k(\mathcal{S}\ell-\mathcal{S}\mathscr{k})}{b}$$
$$\leq\beta\frac{\max\left\{w_k(\ell-\mathscr{k}),w_k(\ell-\mathcal{S}\mathscr{k}),w_k(\mathscr{k}-\mathcal{S}\mathscr{k}),\frac{2w_k(\ell-\mathcal{S}\mathscr{k})w_k(\mathscr{k}-\mathcal{S}\ell)}{w_k(\ell-\mathcal{S}\mathscr{k})+w_k(\ell-\mathscr{k})}\right\}}{b}$$
$$=\beta\left(\frac{1}{v(\ell-\mathscr{k},b)}-1\right). \tag{29}$$

Thus, condition (2) is satisfied for $\theta(r)=\beta r,\forall r>0$, assuming that $\vartheta(\ell-\mathscr{k},b)=1,\forall\ell,\mathscr{k}\in C([0,k],\mathbb{R})$. Since $C([0,k],\mathbb{R})$ is a complete space, all the hypotheses of Theorem 3.3 are fulfilled. Therefore, the integral equation (22) possesses a unique solution.

5. Conclusion

This work presents a new fixed point framework in fuzzy Fréchet spaces by introducing ϑ-θ-fuzzy augmented contraction mapping under the structure of ϑ-acceptable mappings. The developed theorems extend and generalize existing results in fuzzy fixed point theory. The example included supports the theoretical findings, and the application to an integral equation highlights the practical relevance of the approach. These contributions provide a solid foundation for further exploring of fixed point theory in fuzzy topological structures and their applications to functional and integral equations.

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