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SOME PROPERTIES OF HIGHER ORDER (n, m)-DRAZIN INTUTIONISTIC FUZZY NORMAL OPERATORS

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Abstract:

This paper introduces the concept of the $\wp$-tuple of (n, m)-$\mathcal{D}$-intuitionistic fuzzy normal operators in Intuitionistic Fuzzy Hilbert Spaces ($\mathcal{IFH}$-spaces). New definitions and theorems are established, and key algebraic and structural properties are investigated. The results provide a rigorous foundation for extending intuitionistic fuzzy operator theory and highlight the significance of these operators in the broader context of $\mathcal{IFH}$-spaces.

Keywords: ($\mathcal{IFH}$ -Space), $\mathcal{IF}$- $\mathcal{NB}_d[\mathcal{X}]$, $\mathcal{IF}(n)$- $\mathcal{NB}_d[\mathcal{X}]$, $\mathcal{IF}(n, m)$- $\mathcal{NB}_d[\mathcal{X}]$, $\mathcal{IF}(n, m)$- $\mathcal{NB}_d[\mathcal{X}]^{\wp}$.

1-Introduction

In recent years, intuitionistic fuzzy spaces have attracted significant attention due to their diverse applications in fields such as artificial intelligence, approximation theory, and the analysis of imprecise data. Despite these developments, the theory of (n, m)-$\mathcal{D}$-intuitionistic fuzzy normal operators, particularly in the context of $\wp$-tuples, remains insufficiently explored. This study aims to address this gap by introducing the $\wp$-tuple of (n, m)-$\mathcal{D}$-intuitionistic fuzzy normal operators within Intuitionistic Fuzzy Hilbert Spaces ($\mathcal{IFH}$ -spaces), establishing fundamental definitions, and proving theorems that describe their algebraic and structural properties. Specifically, the Intuitionistic Fuzzy Hilbert Space ($\mathcal{IFH}$-space) serves as a natural extension of the classical Hilbert space in the intuitionistic fuzzy framework, playing a central role in the analysis of both algebraic and structural aspects of these operators. This contribution extends the existing framework of intuitionistic fuzzy operator theory and provides a solid foundation for future research in this area.

Park [25] was among the first to explore the concept of Intuitionistic Fuzzy Metric Spaces, providing a theoretical foundation for measuring distances under uncertainty and fuzziness. Later, Saadati [20] extended this concept by introducing the intuitionistic fuzzy metric and norm, paving the way for more sophisticated analytical constructions.

In 2009, Goudarzi et al. [22] intr0duced the notion ($\mathcal{IFIP}$-spaces), marking an essential step toward a comprehensive theory of intuitionistic fuzzy functional analysis. Subsequently, Samanta and Bera [21] redefined and refined this concept in 2019.

A major development came in 2020, when Radharamani and her collaborators [18, 19] provided a formal definition of ($\mathcal{IFH}$-Space), highlighting its structural and foundational properties. This advancement enabled the study of linear operators within an intuitionistic fuzzy environment, analogous to classical Hilbert space theory, but incorporating the degrees membership and non-membership each element

In this context, the concept of the Drazin inverse has emerged as an important analytical tool for studying linear operators. It was first introduced for bounded operators on complex Banach spaces by Sheibani, Rashidi, and Rezaei [17] and King [9]. Since then, numerous detailed studies and applications of Drazin invertibility have been conducted, which can be explored further in reference [5].

Additionally, significant contributions from functional analysis provide theoretical support for the fuzzy operator theory presented in this work, particularly those outlined in [1, 8, 11, 12, 13, 14, 15, 23], which address operator properties and spectral theory in the context of generalized and fuzzy structures

The paper regulations are as follows

Section two includes several preliminary results. In Section three, we introduce the concept the intuitionistic fuzzy Drazin inverse, present several theorems, and discuss some properties. We also cover $\mathcal{D}$ - intuitionistic fuzzy normal, n-power $\mathcal{D}$-intuitionistic fuzzy normal, (n, m)-power $\mathcal{D}$-intuitionistic fuzzy normal, jointly intuitionistic fuzzy normal, jointly intuitionistic fuzzy n- normal, jointly intuitionistic fuzzy (n, m)- normal, p-tuple of $\mathcal{D}$-intuitionistic Fuzzy normal operator and p-tuple of n- power $\mathcal{D}$- intuitionistic Fuzzy normel operator which are utilized in this paper. In Section four we introduce the concept of a p-tuple of (n, m) - $\mathcal{D}$- intuitionistic Fuzzy normal operator $\mathcal{IF}(n, m)$- $N\mathcal{B}_d[\mathcal{X}]^{p}$, present several theorems, and discusses some properties.

2- Preliminaries

Definition 2.1: [22]

A continuous t-norm $\mathcal{T}$ is said to be continuous t-representable if and only if there exist a continuous t -norm * , a continuous t – conorm ⋄ defined on the interval [0,1], ∀ elements $x = (x_1, x_2)$, $y = (y_1, y_2) \in \mathcal{L}^*$, the following hold:

$$\mathcal{T}(x, y) = (x_1 * y_1, x_2 \diamond y_2).$$

Definition 2. 2: [22]

Let

$$\mu : \mathcal{X}^2 \times (0, +\infty) \to [0,1], \ \vartheta: \mathcal{X}^2 \times (0, +\infty) \to [0,1],$$

$$\mu(x, y, t) + \vartheta(x, y, t) \leq 1, \forall x, y \in \mathcal{X} \ \& \ t > 0.$$

An Intuitionistic Fuzzy Inner Product Space ($\mathcal{IFIP}$-Space) is defined as a triplet $(\mathcal{X}, \mathcal{F}_{\mu,v}, \mathcal{J})$, where $\mathcal{X}$ is real Vector Space, $\mathcal{F}_{\mu,v}$ is an Intuitionistic Fuzzy set on $\mathcal{X}^2 \times \mathbb{R}$ and $\mathcal{J}$ is a continuous t –representable , the following conditions are satisfied for all $x, y, z \in \mathcal{X}$ and $s, r, t \in \mathbb{R}$:

(IFI-1) $\mathcal{F}_{\mu,v}(x, y, 0) = 0$ and $\mathcal{F}_{\mu,v}(x, x, t) > 0$, $\forall t > 0$.

(IFI-2) $\mathcal{F}_{\mu,v}(x, y, t) = \mathcal{F}_{\mu,v}(y, x, t)$.

(IFI-3) $\mathcal{F}_{\mu,v}(x, x, t) \neq \mathrm{H}(t)$

for some $t \in \mathbb{R}$ iff $x \neq 0$

Where $H(t) = \begin{cases} 1, & \text{if } t > 0 \\ 0, & \text{if } t \leq 0 \end{cases}$

(IFI-4) For all scalars $\alpha \in \mathbb{R}$

$$\mathcal{F}_{\mu,v}(x\alpha, y, t) = \begin{cases} \mathcal{F}_{\mu,v}\left(x, y, \frac{t}{\alpha}\right), & \text{if } \alpha > 0 \\ H(t), & \text{if } \alpha = 0 \\ \mathcal{N}_s\left(\mathcal{F}_{\mu,v}\left(x, y, \frac{t}{\alpha}\right)\right), & \text{if } \alpha < 0 \end{cases}$$

(IFI-5) $\sup\left\{\mathcal{T}\left(\mathcal{F}_{\mu,v}(x,z,s), \mathcal{F}_{\mu,v}(y,z,r)\right)\right\} = \mathcal{F}_{\mu,v}(x+y, y, t)$.

(IFI-6) $\mathcal{F}_{\mu,v}(x, y, .): \mathbb{R} \to [0,1]$ is continuous on $\mathbb{R} \setminus \{0\}$.

(IFI-7) $\lim_{t\to 0} \mathcal{F}_{\mu,v}(x, y, t) = 1$.

Note 2.3: [22]

(i) The standard negator is defined as $\mathcal{N}_s(x) = 1 - x$, $\forall x \in [0,1]$

(ii) By defining the inner product as $\langle x, y\rangle = \mathcal{F}_{\mu,v}(x, y, .)$, it becomes evident that the Intuitionistic Fuzzy Inner Product behaves in a manner analogous to the classical inner product.

(iii) Schwarz inequality:

for $x, y \in \mathcal{X}$ and for any $s, t > 0$ the following inequality holds

$$\mathcal{F}_{\mu,v}(x, y, ts) \geq \mathcal{T}\left(\mathcal{F}_{\mu,v}(x, x, t^2)\mathcal{F}_{\mu,v}(y, y, s^2)\right),$$

(iv) A sequence $\{x_n\} \subseteq \mathcal{X}$ is said to be $t-$convergent to $x \in \mathcal{X}$, if for any given $\epsilon > 0$, $\lambda > 0$, there exists anatural number, $N^0 = N^0(\epsilon, \lambda)$, such that $P(x_n - x, \epsilon) > \mathcal{N}_s(\lambda)$, whenever $n > N^0$.

(v) Enable $f(x)$ be a continuous linear functional on $\mathcal{X}$. Then its said to be $\mathcal{T}_{\mathcal{F}_{\mu,v}}$ - continuous if for any sequence x_n in $\mathcal{X}$ $x_n \xrightarrow{\mathcal{T}_{\mathcal{F}_{\mu,v}}} x \Rightarrow f(x_n) \xrightarrow{\mathcal{T}_{\mathcal{F}_{\mu,v}}} f(x)$,

Theorem 2.4: [18]

Let $(\mathcal{X}, \mathcal{F}_{\mu,v}, \mathcal{I})$ exist as any $IFIP-$Space , where $\mathcal{T}$ is a continuous t - representable such that $\forall x, y \in \mathcal{X}$, $\sup\{t \in \mathbb{R}\ \mathcal{F}_{\mu,v}(x, y, t) < 1\} < \infty$. Define $\langle .,. \rangle : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ by $\langle x, y\rangle = \sup\{t \in \mathbb{R} : \mathcal{F}_{\mu,v}(x, y, t) < 1$. Then $(\mathcal{X}\langle .,. \rangle)$ is an $IFIP$-space, that $(\mathcal{X}, \mathcal{P}_{\mu,v})$ is normed space, where $\mathcal{P}_{\mu,v}(x, z) = \langle x, x\rangle^{1/2}\ \forall x \in \mathcal{X}$.

Definition2.5:[18]

Let $(\mathcal{X}, \mathcal{F}_{\mu,v}, \mathcal{I})$ a denote an $IFIP-$Space with $IP: \langle x, y\rangle = \sup\{t \in \mathbb{R}: \mathcal{F}_{\mu,v}(x, y, t) < 1\}, \forall x, y \in \mathcal{X}$. If $(\mathcal{X}, \mathcal{F}_{\mu,v}, \mathcal{I})$ is complete in the norm $\mathcal{P}_{\mu,v}$ then $\mathcal{X}$ is an intuitionistic Fuzzy Hilbert space ($IF\mathcal{H}-$Space).

Theorem 2.6: [18]

Enable $(\mathcal{X}, \mathcal{F}_{\mu,v}, \mathcal{I})$ be a ($IF\mathcal{H}$- space) with $IP\langle x, y\rangle = Sup\ \{t \in \mathbb{R} : \mathcal{F}(x, y, t) < 1\}, \forall x, y \in \mathcal{X}$. A sequence $\{x_n\}$ on $\mathcal{X}$ is $\mathcal{T}_{\mathcal{F}_{\mu,v}}$- convergent (i.e. $x_n \xrightarrow{\mathcal{T}_{\mathcal{F}_{\mu,v}}} x$) if $x_n \xrightarrow{\mathcal{P}_{\mu,v}} x$

Theorem 2.7: [18](Riesz Theorem)

Let $(\mathcal{X}, \mathcal{F}_{\mu,v}, \mathcal{I})$ be an $IF\mathcal{H}-$Space.

For every $\mathcal{T}_{\mathcal{F}_{\mu,v}}$- continuous linear functional f, there exists a unique vector $y \in \mathcal{X}$, $\forall x \in \mathcal{X}$, the following $f(x) = \sup\{t \in \mathbb{R}: \mathcal{F}_{\mu,v}(x, y, t) < 1\}$.

Theorem 2.8:[18]

Let $(\mathcal{X}, \mathcal{F}_{\mu,v}, \mathcal{I})$ exist as any $IFIP$-Space, where $\mathcal{I}$ isa continuous $t-$representable and suppose satisfying that for every

$$x, y \in \mathcal{X}, \qquad \sup\{ t \in \mathbb{R}: \mathcal{F}_{\mu,v}(x, y, t) < 1 \} < \infty ,\}$$

$$\text{supt} \in \mathbb{R}: \mathcal{F}_{\mu,v}(x + y, z, t < 1\} = \sup\{ t \in \mathbb{R} <: \mathcal{F}_{\mu,v}(x, z, t) < 1$$

$$+ \sup\{ t \in \mathbb{R}: \mathcal{F}_{\mu,v}(y, z, t) < 1 \} \forall\, x, y \in \mathcal{X}\ .$$

Remark 2.9:[18]
Let $I\mathcal{FB}[\mathcal{X}]$ denote the collection of all intuitionistic fuzzy bounded linear defined on the $I\mathcal{FH}$ -Space

Theorem 2.10:_($I\mathcal{F}$A − operator in $I\mathcal{FH}$ − Space) [18]
Let $(\mathcal{X}, \mathcal{F}_{\mu,v}, \mathcal{I})$ exist as any $I\mathcal{FH}$-Space, let $\mathcal{S} \in I\mathcal{FB}[\mathcal{X}]$. Then a unique exists

$$\mathcal{S}^* \in I\mathcal{FB}[\mathcal{X}] \ni \langle \mathcal{S}x, y \rangle = \langle x, \mathcal{S}^* y \rangle \,\forall\, x, y \in \mathcal{X}\ .$$

Definition 2.11: ($I\mathcal{FSA}$ − operator) [18]
Let $(\mathcal{X}, \mathcal{F}_{\mu,v}, \mathcal{I})$ a denote an $I\mathcal{FH}$-Space with

$$IP: \langle x, y \rangle = \sup\{ t \in \mathbb{R}: \mathcal{F}_{\mu,v}(x, y, t) < 1 \}, \forall\, x, y \in \mathcal{X}$$

Let $\mathcal{T} \in I\mathcal{FB}[\mathcal{X}]$. Next $\mathcal{T}$ is Intuitionistic Fuzzy Self- Adjoint operat0r, if $\mathcal{T} = \mathcal{T}^*$,where of $\mathcal{T}^* \in I\mathcal{FSA}$ of $\mathcal{T}$.

Theorem 2.12:[18]
Let $(\mathcal{X}, \mathcal{F}_{\mu,v}, \mathcal{I})$ exist as any $I\mathcal{FH}$ -space with

$$IP: < x, y > = \sup\{ t \in \mathbb{R}: \mathcal{F}_{\mu,v}(x, y, z) < 1 \}, \forall\, x, y \in \mathcal{X}\ , \text{let } \mathcal{T} \in I\mathcal{FB}[\mathcal{X}].$$

$\mathcal{T}$ is $I\mathcal{F}$SA operator.

Theorem 2.13:[18]
Let $(\mathcal{X}, \mathcal{F}_{\mu,v}, \mathcal{I})$ a denote an $I\mathcal{F}IP$ -space with $IP: < x, y > = \sup\{ t \in \mathbb{R}: \mathcal{F}_{\mu,v}(x, y, t) < 1 \}$, $\forall\, x, y \in \mathcal{X}$ and let $\mathcal{T}^*$ a denote the $I\mathcal{F}$A − operator of $\mathcal{T}$. Next the following properties hold:

1. $(\mathcal{T}^*)^* = \mathcal{T}$
2. $(\delta\mathcal{T})^* = \delta\mathcal{T}^*$
3. $(\delta\mathcal{T}_1 + \gamma\mathcal{T}_2)^* = \delta\mathcal{T}_1^* + \gamma\mathcal{T}_2^*$ where δ, γ are scalars.
4. $(\mathcal{T}_1\mathcal{T}_2)^* = \mathcal{T}_2^*\mathcal{T}_1^*$.

3. MAIN RESULTS

In the following section, we introduce the concept of the intuitionistic fuzzy Drazin inverse ($I\mathcal{FH}$-space) and examine some of its fundamental properties. Furthermore, we explore the structure and behavior of (n, m) - power $\mathcal{D}$-intuitionistic fuzzy normal.

Definition. (3.1): (Intuitionistic fuzzy Drazin inverse)
Enable $(\mathcal{X}, \mathcal{F}_{\mu,v}, \mathcal{I})$ be any $I\mathcal{FH}$-space using

$$IP: \langle u, v \rangle = sup\{ x \in \mathbb{R}: \mathcal{F}(u, v, x) < 1 \} \,\forall\, u, v \in \mathcal{X}$$

as well as let $\mathcal{T} \in \mathcal{IFB}[\mathcal{X}]$, thereafter intuitionistic fuzzy Drazin inverse of $\mathcal{T}$ is that unique operator $\mathcal{T}^{\mathcal{D}} \in \mathcal{IFB}[\mathcal{X}]$ in case present as well as meets the following requirements

$$p_{\mu,\nu}\left((\mathcal{T}^{\mathcal{D}}\mathcal{T} - \mathcal{T}\mathcal{T}^{\mathcal{D}})u, t\right) = 0$$

$$p_{\mu,\nu}\left(\left((\mathcal{T}^{\mathcal{D}})^2\mathcal{T} - \mathcal{T}^{\mathcal{D}}\right)u, t\right) = 0$$

$$p_{\mu,\nu}\left((\mathcal{T}^{v+1}\mathcal{T}^{D} - \mathcal{T}^{v})u, t\right) = 0$$

Given an integer $v \geq 0$

Denoted by $\mathcal{IFB}_d[\mathcal{X}]$ all elements of intuitionistic fazzy Drazin invertible elements of $\mathcal{IFB}[\mathcal{X}]$ for $\mathcal{T} \in \mathcal{IFB}_d[\mathcal{X}]$ It became evident that intuitionistic fazzy Drazin inverse $\mathcal{T}^{\mathcal{D}}$ of $\mathcal{T}$ conforms to the following rules

$$\begin{cases} p_{\mu,\nu}\left(((\mathcal{T}^*)^{\mathcal{D}} - (\mathcal{T}^{\mathcal{D}})^*)u, t\right) = 0 \\ p_{\mu,\nu}\left(\left((\mathcal{T}^k)^{\mathcal{D}} - (\mathcal{T}^{\mathcal{D}})^k\right)u, t\right) = 0 \quad \forall k \in \mathbb{N} \end{cases}$$

Furthermore, it was noted that in the case where $\mathcal{T} \in \mathcal{IFB}_d[\mathcal{X}]$ and $\mathcal{N} \in \mathcal{IFB}[\mathcal{X}]$ is an invertible operator then

$$\mathcal{N}^{-1}\mathcal{T}\mathcal{N} \in \mathcal{IFB}_d[\mathcal{X}] \text{ and } p_{\mu,\nu}\left(((\mathcal{N}^{-1}\mathcal{T}\mathcal{N})^{D} - \mathcal{N}^{-1}\mathcal{T}^{D}\mathcal{N})u, t\right) = 0$$

Definition (3.2):

Let $(\mathcal{X}, \mathcal{F}_{\mu,\nu}, \mathcal{I})$ *be an* IFH-space with

$$IP: \langle u, v\rangle = sup\{t \in \mathbb{R}: \mathcal{F}_{\mu,\nu}(u, v, t) < 1\} \ \forall\, u, v \in \mathcal{X} \text{ and let } \mathcal{T} \in \mathcal{IFB}[\mathcal{X}],$$

then $\mathcal{T}$ is called

(i) intuitionistic Fuzzy normal if $p_{\mu,\nu}\left((\mathcal{T}\mathcal{T}^* - \mathcal{T}^*\mathcal{T})u, t\right) = 0$ denoted by $\mathcal{IF}$- $\mathcal{NB}[\mathcal{X}]$

(ii) intuitionistic Fuzzy n- normal operator if $p_{\mu,\nu}\left((\mathcal{T}^n\mathcal{T}^* - \mathcal{T}^*\mathcal{T}^n)u, t\right) = 0$ denoted by $\mathcal{IF}(n)$-$\mathcal{NB}[\mathcal{X}]$

(iii) intuitionistic Fuzzy (n, m)- normal operator if $p_{\mu,\nu}\left((\mathcal{T}^n\mathcal{T}^{*m} - \mathcal{T}^{*m}\mathcal{T}^n)u, t\right) = 0$ denoted by $\mathcal{IF}(n, m)$-$\mathcal{NB}[\mathcal{X}]$ for some positive integers n, m

Definition (3.3):

Let $(\mathcal{X}, \mathcal{F}_{\mu,\nu}, \mathcal{I})$ *be a* $\mathcal{IFH}$-space with

$$IP: \langle u, v\rangle = sup\{t \in \mathbb{R}: \mathcal{F}_{\mu,\nu}(u, v, t) < 1\} \forall\, u, v \in \mathcal{X} \text{ and let } \mathcal{T} \in \mathcal{IFB}_d[\mathcal{X}].$$

then $\mathcal{T}$ is called

(i) $\mathcal{D}$ - intuitionistic Fuzzy normal If $p_{\mu,\nu}\left((\mathcal{T}^{\mathcal{D}}\mathcal{T}^* - \mathcal{T}^*\mathcal{T}^{\mathcal{D}})u, t\right) = 0$ This is denoted by $\mathcal{IF}$- $\mathcal{NB}_d[\mathcal{X}]$

(ii) n- power $\mathcal{D}$- intuitionistic Fuzzy normal if $p_{\mu,\nu}\left(\left((\mathcal{T}^{\mathcal{D}})^n\mathcal{T}^* - \mathcal{T}^*(\mathcal{T}^{\mathcal{D}})^n\right)u, t\right) = 0$ This is denoted by $\mathcal{IF}(n)$- $\mathcal{NB}_d[\mathcal{X}]$

(iii) (n, m)-power $\mathcal{D}$-intuitionistic Fuzzy normal if $p_{\mu,\nu}\left(\left((\mathcal{T}^{\mathcal{D}})^n\mathcal{T}^{*m} - \mathcal{T}^{*m}(\mathcal{T}^{\mathcal{D}})^n\right)u, t\right) = 0$ This is denoted by $\mathcal{IF}(n, m)$- $\mathcal{NB}_d[\mathcal{X}]$ for certain positive whole numbers n , m

The study p-tuples (collections of p operators) has garnered significant attention from several researchers In recent years. Notable advancements have occurred this area discussed [2,3,4,6,7,10,16,24]and additional information can be found within those sources.

Given a p-tuple

$$\mathcal{T} := (\mathcal{T}_1, \dots\dots\dots, \mathcal{T}_p) \in \mathcal{IFB}[\mathcal{X}]^p$$

we define

$$p_{\mu,\nu}\left((\mathcal{T}^*\mathcal{T} - \mathcal{T}\mathcal{T}^*)u, t\right) = 0 \in \mathcal{IFB}[\mathcal{X} \oplus \dots\dots \oplus \mathcal{X}]$$

as the self- commutator of $\mathcal{T}$, defined by

$$p_{\mu,v}\left((\mathcal{T}^*\mathcal{T}-\mathcal{T}\mathcal{T}^*)_{i,j}u,t\right) := p_{\mu,v}\left(\left(\mathcal{T}_j^*\mathcal{T}_i-\mathcal{T}_i\mathcal{T}_j^*\right)u,t\right) \quad \forall (i,j)\in\{1\ldots\ldots p\}^2$$

Note thet this definition of self- commutator for p-tuples of operators on a ($\mathcal{IFH}$- Space)
Where $\mathcal{T}^* := \left(\mathcal{T}_1^{|*},\ldots,\mathcal{T}_p^*\right)$ that $\mathcal{T}$ is jointly intuitionistic fuzzy hyponormal if

$$p_{\mu,v}\left((\mathcal{T}^*\mathcal{T}-\mathcal{T}\mathcal{T}^*)u,t\right) = \begin{pmatrix} p_{\mu,v}\left((\mathcal{T}_1^*\mathcal{T}_1-\mathcal{T}_1\mathcal{T}_1^*)u,t\right) & p_{\mu,v}\left((\mathcal{T}_2^*\mathcal{T}_1-\mathcal{T}_1\mathcal{T}_2^*)u,t\right) & \cdots & p_{\mu,v}\left(\left(\mathcal{T}_p^*\mathcal{T}_1-\mathcal{T}_1\mathcal{T}_p^*\right)u,t\right) \\ p_{\mu,v}\left((\mathcal{T}_1^*\mathcal{T}_2-\mathcal{T}_2\mathcal{T}_1^*)u,t\right) & p_{\mu,v}\left((\mathcal{T}_2^*\mathcal{T}_2-\mathcal{T}_2\mathcal{T}_2^*)u,t\right) & \cdots & p_{\mu,v}\left(\left(\mathcal{T}_p^*\mathcal{T}_2-\mathcal{T}_2\mathcal{T}_p^*\right)u,t\right) \\ \vdots & \vdots & \vdots & \vdots \\ p_{\mu,v}\left(\left(\mathcal{T}_1^*\mathcal{T}_p-\mathcal{T}_p\mathcal{T}_1^*\right)u,t\right) & p_{\mu,v}\left(\left(\mathcal{T}_2^*\mathcal{T}_p-\mathcal{T}_p\mathcal{T}_2^*\right)u,t\right) & \cdots & p_{\mu,v}\left(\left(\mathcal{T}_p^*\mathcal{T}_p-\mathcal{T}_p\mathcal{T}_p^*\right)u,t\right) \end{pmatrix}$$

is a non-negative operator on $\mathcal{X}\oplus\ldots\oplus\mathcal{X}$, or equivalently

$$\sum_{1\le i,j\le p} p_{\mu,v}\left(\left(\mathcal{T}_i^*\mathcal{T}_j-\mathcal{T}_j\mathcal{T}_i^*\right)u,t\right)\ge 0 \quad \forall x\in\mathcal{X}$$

$\boldsymbol{\mathcal{T}}$ is called jointly intuitionistic fuzzy normal if $\boldsymbol{\mathcal{T}}$ satisfying

$$\begin{cases} p_{\mu,v}\left(\left(\mathcal{T}_i\mathcal{T}_j-\mathcal{T}_j\mathcal{T}_i\right)u,t\right)=0 & i,j\in\{1,\ldots,p\} \\ p_{\mu,v}\left((\mathcal{T}_i^{\,*}\mathcal{T}_i-\mathcal{T}_i\mathcal{T}_i^{\,*})u,t\right)=0 & i=1,\ldots,p \end{cases}$$

This is denoted by $I\mathcal{F}$- $N\mathcal{B}[\mathcal{X}]^{p}$

$\boldsymbol{\mathcal{T}}$ is called jointly intuitionistic fuzzy n- normal if $\boldsymbol{\mathcal{T}}$ satisfying

$$\begin{cases} p_{\mu,v}\left(\left(\mathcal{T}_i\mathcal{T}_j-\mathcal{T}_j\mathcal{T}_i\right)u,t\right)=0 & i,j\in\{1,\ldots,p\} \\ p_{\mu,v}\left((\mathcal{T}_i^{\,n}\mathcal{T}_i^{\,*}-\mathcal{T}_i^{\,*}\mathcal{T}_i^{\,n})u,t\right)=0 & i=1,\ldots,p \end{cases}$$

This is denoted by $I\mathcal{F}(n)$- $N\mathcal{B}[\mathcal{X}]^{p}$

$\boldsymbol{\mathcal{T}}$ is called jointly intuitionistic fuzzy (n,m)- normal if $\boldsymbol{\mathcal{T}}$ satisfying

$$\begin{cases} p_{\mu,v}\left(\left(\mathcal{T}_i\mathcal{T}_j-\mathcal{T}_j\mathcal{T}_i\right)u,t\right)=0 & i,j\in\{1,\ldots,p\} \\ p_{\mu,v}\left((\mathcal{T}_i^{\,n}\mathcal{T}_i^{\,*m}-\mathcal{T}_i^{\,*m}\mathcal{T}_i^{\,n})u,t\right)=o & i=1,\ldots,p \end{cases}$$

This is denoted by $I\mathcal{F}(n,m)$- $N\mathcal{B}[\mathcal{X}]^{p}$ for certain positive integers n and m

Definition (3.4):
Let $\boldsymbol{\mathcal{T}} := \left(\mathcal{T}_1,\ldots\ldots\ldots\ldots,\mathcal{T}_p\right)\in I\mathcal{F}\text{-}N\mathcal{B}_d[\mathcal{X}]^{p}$ we say that $\boldsymbol{\mathcal{T}}$ is p-tuple of $\mathcal{D}$-intuitionistic Fuzzy normal operator if $\boldsymbol{\mathcal{T}}$ satisfying

$$\begin{cases} p_{\mu,v}\left(\left(\mathcal{T}_i\mathcal{T}_j-\mathcal{T}_j\mathcal{T}_i\right)u,t\right)=0 & i,j\in\{1,\ldots,p\}^2 \\ p_{\mu,v}\left(\left(\mathcal{T}_i^{\,D}\mathcal{T}_i^{\,*}-\mathcal{T}_i^{\,*}\mathcal{T}_i^{\,D}\right)u,t\right)=0 & i=1,\ldots,p \end{cases}$$

This is denoted by $\mathcal{IF}$-$N\mathcal{B}_d[\mathcal{X}]^{\wp}$

Definition (3.5):
Let $\boldsymbol{\mathcal{T}} := (\mathcal{T}_1, \ldots \ldots \ldots, \mathcal{T}_{\wp}) \in \mathcal{IF}(n)$-$N\mathcal{B}_d[\mathcal{X}]^{\wp}$ we say that $\boldsymbol{\mathcal{T}}$ is $\wp$-tuple of n- power $\mathcal{D}$- intuitionistic Fuzzy normel operator if $\boldsymbol{\mathcal{T}}$ satisfying

$$\begin{cases} p_{\mu,v}\left((\mathcal{T}_i\mathcal{T}_j - \mathcal{T}_j\mathcal{T}_i)u, t\right) = 0 & i,j \in \{1, \ldots, \wp\}^2 \\ p_{\mu,v}\left(\left(\left(\mathcal{T}_i^{\mathcal{D}}\right)^n \mathcal{T}_i^* - \mathcal{T}_i^*\left(\mathcal{T}_i^{\mathcal{D}}\right)^n\right)u, t\right) = 0 & i = 1, \ldots, \wp \end{cases}$$

This is denoted by $\mathcal{IF}(n)$-$N\mathcal{B}_d[\mathcal{X}]^{\wp}$ for certain positive integers n

4. $\wp$-tuple of (n, m) -Drazin intuitionistic fuzzy normal operators

" This portion is concerned with investigating the behavior $\mathcal{IF}(n, m)$- $N\mathcal{B}_d[\mathcal{X}]^{\wp}$ in $\mathcal{IFH}$-spaces, including their transformations under powers, adjoints, and unitary equivalence."

Definition (4.1):
Let $\mathcal{T} := (\mathcal{T}_1, \ldots \ldots \ldots, \mathcal{T}_{\wp}) \in \mathcal{IF}(n, m)$-$N\mathcal{B}_d[\mathcal{X}]^{\wp}$ we say that $\mathcal{T}$ is $\wp$-tuple of (n, m) - $\mathcal{D}$- intuitionistic Fuzzy normal operator if $\mathcal{T}$ satisfying

$$\begin{cases} p_{\mu,v}\left((\mathcal{T}_i\mathcal{T}_j - \mathcal{T}_j\mathcal{T}_i)u, t\right) = 0 & i,j \in \{1, \ldots, \wp\}^2 \\ p_{\mu,v}\left(\left(\left(\mathcal{T}_i^{\mathcal{D}}\right)^n \mathcal{T}_i^{*m} - \mathcal{T}_i^{*m}\left(\mathcal{T}_i^{\mathcal{D}}\right)^n\right)u, t\right) = 0 & i = 1, \ldots, \wp \end{cases}$$

This is denoted by $\mathcal{IF}(n, m)$- $N\mathcal{B}_d[\mathcal{X}]^{\wp}$ for certain positive integers n and m

Example (4.2):
Let $\mathcal{T} \in \mathcal{IF}(n, m)$- $N\mathcal{B}_d[\mathcal{X}]$ be an (n, m)-$\mathcal{D}$- intuitionistic Fuzzy normal operator, then $\mathcal{T} = (\mathcal{T}, \ldots, \mathcal{T}) \in \mathcal{IF}(n, m)$- $N\mathcal{B}_d[\mathcal{X}]^{\wp}$

Lemma (4.3):
Let $\mathcal{T}_1, \mathcal{T}_2 \in \mathcal{IFB}[\mathcal{X}]$ *and* $r, s \in \mathbb{N}^*$ then

$$p_{\mu,v}\left((\mathcal{T}_1^r\mathcal{T}_2^s - \mathcal{T}_2^s\mathcal{T}_1^r)u, t\right) = \sum_{i=0}^{r-1}\sum_{j=0}^{s-1} \mathcal{T}_1^{r-i-1}\mathcal{T}_2^{s-j-1}\, p_{\mu,v}\left((\mathcal{T}_1\mathcal{T}_2 - \mathcal{T}_2\mathcal{T}_1)u, t\right)\mathcal{T}_2^j\mathcal{T}_1^i$$

Proof: Observe that

$$p_{\mu,v}\left((\mathcal{T}_1^r\mathcal{T}_2 - \mathcal{T}_2\mathcal{T}_1^r)u, t\right) = \sum_{i=0}^{r-1}\left(\mathcal{T}_1^{r-i}\mathcal{T}_2\mathcal{T}_1^r - \mathcal{T}_1^{r-i-1}\mathcal{T}_2\mathcal{T}_1^{r+1}\right)$$

$$= \sum_{i=0}^{r-1} \mathcal{T}_1^{r-i-1}\, p_{\mu,v}\left((\mathcal{T}_1\mathcal{T}_2 - \mathcal{T}_2\mathcal{T}_1)u, t\right)\mathcal{T}_1^i \quad \ldots\ldots \quad (1)$$

Since $p_{\mu,v}\left((\mathcal{T}_1\mathcal{T}_2^s - \mathcal{T}_2^s\mathcal{T}_{1,})u, t\right) = -p_{\mu,v}\left((\mathcal{T}_2^s\mathcal{T}_1 - \mathcal{T}_1\mathcal{T}_2^s)u, t\right),$

We immediately get

$$p_{\mu,\nu}((\mathcal{T}_1\mathcal{T}_2^{s} - \mathcal{T}_2^{s}\mathcal{T}_1)u,t) = \sum_{j=0}^{s-1}\mathcal{T}_2^{s-j-1}p_{\mu,\nu}\left((\mathcal{T}_1\mathcal{T}_2 - \mathcal{T}_2\mathcal{T}_{1,})u,t\right)\mathcal{T}_2^{j} \cdots\cdots \quad (2)$$

Substituting $\mathcal{T}_2^{s}$ for $\mathcal{T}_2$ in (1) and (2)→

$$p_{\mu,\nu}((\mathcal{T}_1^{r}\mathcal{T}_2^{s} - \mathcal{T}_2^{s}\mathcal{T}_1^{r})u,t) = \sum_{i=0}^{r-1}\sum_{j=0}^{s-1}\mathcal{T}_1^{r-i-1}\mathcal{T}_2^{s-j-1}p_{\mu,\nu}((\mathcal{T}_1\mathcal{T}_2 - \mathcal{T}_2\mathcal{T}_1)u,t)\mathcal{T}_2^{j}\mathcal{T}_1^{i}$$

Proposition (4.4):

Let $\mathcal{T} \in I\mathcal{F}(n,m)$-$N\mathcal{B}_d[\mathcal{X}]$. If $\mathcal{T} \in I\mathcal{F}(n,m)$-$N\mathcal{B}_d[\mathcal{X}]$, then so is $\mathcal{T}^k$ For all positive integers k.

Proof:

To show that $\mathcal{T}^k \in I\mathcal{F}(n,m) - N\mathcal{B}_d[\mathcal{X}]$, we have to prove that

$$p_{\mu,\nu}\left(\left(\left((\mathcal{T}^k)^{\mathcal{D}}\right)^n (\mathcal{T}^k)^{*m} - (\mathcal{T}^k)^{*m}\left((\mathcal{T}^k)^{\mathcal{D}}\right)^n\right)u,t\right), \quad k = 1,2,3,\ldots$$

We establish the validity of the statement through mathematical induction on k . Given that $\mathcal{T} \in I\mathcal{F}(n,m)$-$N\mathcal{B}_d[\mathcal{X}]$, the statement holds for the base case $k = 1$. Now, we assume as the induction hypothesis that the statement holds true for some arbitrary k , that is

$$p_{\mu,\nu}\left(\left(\left((\mathcal{T}^k)^{\mathcal{D}}\right)^n (\mathcal{T}^k)^{*m} - (\mathcal{T}^k)^{*m}\left((\mathcal{T}^k)^{\mathcal{D}}\right)^n\right)u,t\right),$$

And prove it for $k + 1$

$$p_{\mu,\nu}\left(\left(\left((\mathcal{T}^{k+1})^{\mathcal{D}}\right)^n (\mathcal{T}^{k+1})^{*m} - (\mathcal{T}^{k+1})^{*m}\left((\mathcal{T}^{k+1})^{\mathcal{D}}\right)^n\right)u,t\right)$$

$$= p_{\mu,\nu}\left(\left((\mathcal{T}^{\mathcal{D}})^n(\mathcal{T}^k)^{*m}\left(\mathcal{T}^{k\mathcal{D}}\right)^n\mathcal{T}^{*m} - (\mathcal{T}^{k+1})^{*m}\left((\mathcal{T}^{k+1})^{\mathcal{D}}\right)^n\right)u,t\right)$$

$$= p_{\mu,\nu}\left(\left((\mathcal{T}^{k+1})^{*m}\left((\mathcal{T}^{k+1})^{\mathcal{D}}\right)^n - (\mathcal{T}^{k+1})^{*m}\left((\mathcal{T}^{k+1})^{\mathcal{D}}\right)^n\right)u,t\right) = 0$$

Hence, $\mathcal{T}^{k+1}$ is (n,m)- power $\mathcal{D}$- intuitionistic fuzzy normal operators. Hence, we verify the truth of the proposition.

Theorem (4.5):

Let $\mathcal{T} = (\mathcal{T}_1,\ldots,\mathcal{T}_p) \in I\mathcal{F}(n,m)$- $N\mathcal{B}_d[\mathcal{X}]^p$ then

(i) $\mathcal{T} \in I\mathcal{F}(rn,sm)$- $N\mathcal{B}_d[\mathcal{X}]^p$ for some positive integers r,s

(ii) $\mathcal{T}^q := (\mathcal{T}_1^{q_1},\ldots,\mathcal{T}_p^{q_p}) \in I\mathcal{F}(n,m)$- $N\mathcal{B}_d[\mathcal{X}]^p$ for $q = (q_1,\ldots,q_p) \in \mathbb{N}^p$

(iii) $\mathcal{T}^* = (\mathcal{T}_1^*,\ldots,\mathcal{T}_p^*) \in I\mathcal{F}(n,m)$- $N\mathcal{B}_d[\mathcal{X}]^p$

(iv) If $\mathcal{U}$ is an unitary operator, then $\mathcal{U}^*\mathcal{T}\mathcal{U} := (\mathcal{U}^*\mathcal{T}_1\mathcal{U},\ldots,\mathcal{U}^*\mathcal{T}_p\mathcal{U}) \in I\mathcal{F}(n,m)$- $N\mathcal{B}_d[\mathcal{X}]^p$

Proof (i)

Since $\boldsymbol{\mathcal{T}} \in I\mathcal{F}(n,m)$- $N\mathcal{B}_d[\mathcal{X}]^p$, This implies that

$$p_{\mu,\nu}\left((\mathcal{T}_i\mathcal{T}_j - \mathcal{T}_j\mathcal{T}_i)u,t\right) = 0 \quad \text{for } i,j = 1,\ldots,p.$$

$$p_{\mu,v}\left(\left(\left(\mathcal{T}_i^{\mathcal{D}}\right)^{rn}\mathcal{T}_i^{*(sm)}-\mathcal{T}_i^{*(sm)}\left(\mathcal{T}_i^{\mathcal{D}}\right)^{sn}\right)u,t\right)$$

$$=p_{\mu,v}\left(\left(\underbrace{\left(\mathcal{T}_i^{\mathcal{D}}\right)^n\ldots\left(\mathcal{T}_i^{\mathcal{D}}\right)^n}_{r-times}.\underbrace{\mathcal{T}_i^{*m}\ldots T_i^{*m}}_{s-times}-\underbrace{\mathcal{T}_i^{*m}\ldots\mathcal{T}_i^{*m}}_{s-times}.\underbrace{\left(\mathcal{T}_i^{\mathcal{D}}\right)^n}_{r-times}\ldots\left(\mathcal{T}_i^{\mathcal{D}}\right)^n\right)u,t\right)$$

$$\Longrightarrow p_{\mu,v}\left(\left(\underbrace{\mathcal{T}_i^{*m}\ldots\mathcal{T}_i^{*m}}_{s-times}\underbrace{\left(\mathcal{T}_i^{\mathcal{D}}\right)^n\ldots\left(\mathcal{T}_i^{\mathcal{D}}\right)^n}_{r-times}-\underbrace{\mathcal{T}_i^{*m}\ldots\mathcal{T}_i^{*m}}_{s-times}.\underbrace{\left(\mathcal{T}_i^{\mathcal{D}}\right)^n}_{r-times}\ldots\left(\mathcal{T}_i^{\mathcal{D}}\right)^n\right)u,t\right)=0$$

(ii) $If\ q_i=1\ \forall\ i\in\{1,\ldots,p\}$, then $p_{\mu,v}\left(\left(\mathcal{T}_i^{q_i}\mathcal{T}_j^{q_j}-\mathcal{T}_j^{q_j}\mathcal{T}_i^{q_i}\right)u,t\right)$

$If\ q_i>1\ \forall\ i\in\{1,\ldots,p\}$ In light of (Lemma (4.3)), It is obtained that

$$p_{\mu,v}\left(\left(\mathcal{T}_i^{q_i}\mathcal{T}_j^{q_j}-\mathcal{T}_j^{q_j}\mathcal{T}_i^{q_i}\right)u,t\right)=\sum_{\substack{\alpha+\alpha'=\rho i-1\\ \beta+\beta'=\rho j-1}}\mathcal{T}_i^{\alpha}\mathcal{T}_j^{\beta}\,p_{\mu,v}\left(\left(\mathcal{T}_i\mathcal{T}_j-\mathcal{T}_j\mathcal{T}_i\right)u,t\right)\mathcal{T}_j^{\alpha'}\mathcal{T}_i^{\beta'}$$

Provided that we assume $\mathcal{T}\in I\mathcal{F}(n,m)\text{-}N\mathcal{B}_d[\mathcal{X}]^p$, This implies that

$$p_{\mu,v}\left(\left(\mathcal{T}_i^{q_i}\mathcal{T}_j^{q_j}-\mathcal{T}_j^{q_j}\mathcal{T}_i^{q_i}\right)u,t\right)=\sum_{\substack{\alpha+\alpha'=\rho i-1\\ \beta+\beta'=\rho j-1}}\mathcal{T}_i^{\alpha}\mathcal{T}_j^{\beta}\,p_{\mu,v}\left(\left(\mathcal{T}_i\mathcal{T}_j-\mathcal{T}_j\mathcal{T}_i\right)u,t\right)\mathcal{T}_j^{\alpha'}\mathcal{T}_i^{\beta'}$$

$\forall(i,j)\in\{1,\ldots,p\}^2$

Considering that $T_i\in I\mathcal{F}(n,m)\text{-}N\mathcal{B}_d[\mathcal{X}]$, then from (Proposition 4.4), we obtain that $\mathcal{T}_i^{q_i}$ is an (n,m)-$\mathcal{D}$-intuitionistic Fuzzy normal $\forall i\in\{1,\ldots,p\}$. This means that

$$\left(\mathcal{T}_1^{q_1},\ldots\mathcal{T}_p^{q_p}\right)\in I\mathcal{F}(n,m)\text{-}N\mathcal{B}_d[\mathcal{X}]^p$$

(iii) By Definition 4.1 we have under the condition that $\boldsymbol{\mathcal{T}}\in I\mathcal{F}(n,m)\text{-}N\mathcal{B}_d[\mathcal{X}]^p$ that is

$$\begin{cases} p_{\mu,v}\left(\left(\mathcal{T}_i\mathcal{T}_j-\mathcal{T}_j\mathcal{T}_i\right)u,t\right)=0 & i,j\in\{1,\ldots,p\}^2\\ p_{\mu,v}\left(\left(\left(\mathcal{T}_i^{D}\right)^n\mathcal{T}_i^{*m}-\mathcal{T}_i^{*m}\left(\mathcal{T}_i^{D}\right)^n\right)u,t\right)=0 & i=1,\ldots,p\end{cases}$$

and therefore

$$\begin{cases} p_{\mu,v}\left(\left(\mathcal{T}_i^*\mathcal{T}_j^*-\mathcal{T}_j^*\mathcal{T}_i^*\right)u,t\right)=0\\ p_{\mu,v}\left(\left(\left(\mathcal{T}_i^{\mathcal{D}}\right)^{*n}\mathcal{T}_i^{m}-\mathcal{T}_i^{m}\left(\mathcal{T}_i^{\mathcal{D}}\right)^{*n}\right)u,t\right)=0\end{cases}$$

Therefore $\boldsymbol{\mathcal{T}}^*\in I\mathcal{F}(n,m)\text{-}N\mathcal{B}_d[\mathcal{X}]^p$

(iv) It is obvious that

$$p_{\mu,v}\left(\left((\mathcal{U}^*\mathcal{T}_i\mathcal{U})(\mathcal{U}^*\mathcal{T}_j\mathcal{U})-(\mathcal{U}^*\mathcal{T}_j\mathcal{U})(\mathcal{U}^*\mathcal{T}_i\mathcal{U})\right)u,t\right)$$

$$=p_{\mu,v}\left(\left(\mathcal{U}^*\mathcal{T}_i\mathcal{T}_j\mathcal{U}-(\mathcal{U}^*\mathcal{T}_j\mathcal{U})(\mathcal{U}^*\mathcal{T}_i\mathcal{U})\right)u,t\right)$$

$$= p_{\mu,\nu}\left(\left(\mathcal{U}^*\mathcal{T}_j\mathcal{T}_i\mathcal{U} - (\mathcal{U}^*\mathcal{T}_j\mathcal{U})(\mathcal{U}^*\mathcal{T}_i\mathcal{U})\right)u, t\right)$$

$$= p_{\mu,\nu}\left(\left((\mathcal{U}^*\mathcal{T}_j\mathcal{U})(\mathcal{U}^*\mathcal{T}_i\mathcal{U}) - (\mathcal{U}^*\mathcal{T}_j\mathcal{U})(\mathcal{U}^*\mathcal{T}_i\mathcal{U})\right)u, t\right)$$

$$= 0$$

Moreover

$$p_{\mu,\nu}\left(((\mathcal{U}^*\mathcal{T}_i\,\mathcal{U})^{Dn}(\mathcal{U}^*\mathcal{T}_j\mathcal{U})^{*m} - (\mathcal{U}^*\mathcal{T}_j\mathcal{U})^{*m}(\mathcal{U}^*\mathcal{T}_i\mathcal{U})^{Dn})u, t\right)$$

$$p_{\mu,\nu}\left((\mathcal{U}^*(\mathcal{T}_i^D)^n\mathcal{U}\mathcal{U}^*\mathcal{T}_i^{*m}\mathcal{U} - (\mathcal{U}^*\mathcal{T}_j\mathcal{U})^{*m}(\mathcal{U}^*\mathcal{T}_i\mathcal{U})^{Dn})u, t\right)$$

$$p_{\mu,\nu}\left((\mathcal{U}^*(\mathcal{T}_i^D)^n\mathcal{T}_i^{*m}\mathcal{U} - (\mathcal{U}^*\mathcal{T}_j\mathcal{U})^{*m}(\mathcal{U}^*\mathcal{T}_i\mathcal{U})^{Dn})u, t\right)$$

$$p_{\mu,\nu}\left((\mathcal{U}^*\mathcal{T}_i^{*m}(\mathcal{T}_i^D)^n\mathcal{U} - (\mathcal{U}^*\mathcal{T}_j\mathcal{U})^{*m}(\mathcal{U}^*\mathcal{T}_i\mathcal{U})^{Dn})u, t\right)$$

$$p_{\mu,\nu}\left(((\mathcal{U}^*\mathcal{T}_j\mathcal{U})^{*m}(\mathcal{U}^*\mathcal{T}_i\mathcal{U})^{\mathcal{D}n} - (\mathcal{U}^*\mathcal{T}_j\mathcal{U})^{*m}(\mathcal{U}^*\mathcal{T}_i\mathcal{U})^{\mathcal{D}n})u, t\right) = 0$$

Therefore

$$\mathcal{U}^*\mathcal{T}\mathcal{U} \in \ I\mathcal{F}(n, m)\text{-}\ N\mathcal{B}_d[\mathcal{X}]^{\wp}$$

5.CONCIUSION

"The concept of a $I\mathcal{F}(n, m)$- $N\mathcal{B}_d[\mathcal{X}]^{\wp}$ in (IFH-space) represents a relatively recent contribution to this field. This study explored certain properties related to this class of operators. The findings offer valuable insights that can contribute to the development and refinement of fuzzy functional analysis, thereby enhancing the theoretical understanding of this branch of mathematics."

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