Journal of Education for Pure Science- University of Thi-Qar
Vol.15, No.4 (2025)
DOI: https://doi.org/10.32792/jeps.v15i4.741
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Solving first and second-order linear differential equations by using the Lie symmetry method

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Received 13 / 6 / 2025, Accepted 5 / 8 / 2025, Published 1 / 12 / 2025

Abstract:

In this study, we apply Lie group theory to find solutions to first- and second–ordersecond–order ordinary differential equations. The method is based on identifying a one-parameter Lie group of symmetries admitted by the equations. Additionally, we perform symmetry-based separation of variables using coordinate systems that correspond to the identified symmetries.
Keywords: Fixed Point, Common Fixed Point, Coupled Fixed Point, Cone S-Metric Spaces.

1-Introduction

The theory of Lie groups, initially developed by the Norwegian mathematician Sophus Lie, was inspired by his exploration of the structural methods used in Galois theory for solving algebraic equations. Lie envisioned a similar framework for analyzing and solving ordinary differential equations (ODEs) through symmetry-based methods. Although initially underappreciated, his pioneering ideas were later advanced by mathematicians such as Élie Vessiot, Élie Cartan, and George Birkhoff, who successfully applied Lie group theory to the study of differential equations.

Lie's approach centers on identifying continuous transformation groups—known as Lie groups—that leave a differential equation invariant. These symmetries, represented by their infinitesimal generators, can be used to reduce the order of the equation or to simplify its structure, often enabling the derivation of exact solutions. The method transforms complex nonlinear differential equations into more tractable forms by leveraging invariance properties under symmetry transformations.

This paper demonstrates the application of Lie symmetry methods to the solution of first- and second-order ordinary differential equations. We explore how infinitesimal transformations, canonical coordinates, and symmetry conditions can systematically reduce differential equations and lead to exact solutions. Several illustrative examples, including Riccati-type equations, are provided to highlight the method's effectiveness.

Definition 1.1 Lie Pioint Tiransformation [5]

A one-parameter group constructed through a Lie point transformation G, comprises a set of transformations.$A_\lambda : (x, y) \to (\hat{x}, \hat{y}) = X$ that depends on a parameter λ in R and that brings a point (x, y) to X and satisfies the following conditions.

- $A_\lambda\ is\ bijective.$
- $A_{\lambda_1} o\, A_{\lambda_2} = A_{\lambda_1+\lambda_2}$
- $A_0 = I$
- For each λ_1 there exists a unique $\lambda_1 = -\lambda_2$ such that $A_{\lambda_1} o\, A_{\lambda_2} = A_0 = I$.

Definition 1.2 (Orbits [1]).

It is an important tool for solving differential equations using symmetry methods. Suppose there is a point A on the solution curve of a differential equation. Given the symmetry, the orbit of A is the set of all points that A can be mapped to for all possible values of λ.

Definition 1.3 (The Taingent Vectors [1,7]).

The tangent line to the curve at any point $(\hat{x}, \hat{y})$ can be characterized by two tangent vectors: one in the x direction, represented by $\zeta(\hat{x}, \hat{y})$, and the other in the y-direction, represented by thus, we have: $\varphi(\hat{x}, \hat{y}.)$. So

$$\frac{\partial \hat{x}}{\partial \lambda} = \zeta(x, y)$$

And

$$\frac{\partial \hat{y}}{\partial \lambda} = \varphi(x, y)$$

At the initial points (x, y), Since λ equals zero, we have :

$$\left(\frac{\partial \hat{x}}{\partial \lambda}|\lambda = 0, \frac{\partial \hat{y}}{\partial \lambda}|\lambda = 0\right) = \left(\zeta(x, yi), .\, \varphi(x, y)\right)$$

Definition 1.4 (Infinitesimal, Operator [2]):

We define the infinitesimal operator as follows:

$$X = \zeta(x,y)\frac{\partial}{\partial x.} + \varphi(x,y)\frac{\partial}{\partial y}$$

Definition 1.5 (Symmetry Condition [3,7])

The differential operator is important in understanding the symmetry conditions. The total differential operator is:

$$D_x = \frac{\partial}{\partial x} + \frac{dy}{dx}\frac{\partial}{\partial y} + \frac{d^2y}{dx^2}\frac{\partial}{\partial \acute{y}} + \cdots$$

Generally speaking, we study the form of:

$$\frac{dy}{dx} = \rho(x,y)$$

To solve the equation to order to meet the symmetry condition, the point is required to satisfy $(\hat{x},\hat{y})$. The function must satisfy the differential equation $\frac{dy}{dx} = \rho(x,y)$. Writing it using the derivative operator, we get:

$$\frac{d\hat{y}}{d\hat{x}} = \frac{D_x\hat{y}}{D_x\hat{x}}$$

$$= \frac{\hat{y}_{x.} + y'\hat{y}_y}{\hat{x}_x + y'\hat{x}_y}$$

$$= \rho(\hat{x},\hat{y})$$

Since

$$y' = \rho(x,y)$$

Therefore :

$$\frac{\hat{y}_x + \rho(x,y)\hat{y}_y}{\hat{x}_x + \rho(x,y)\hat{x}_y} = \rho(x,y)$$

The symmetry condition must be satisfied to solve the equation.

Definition 1.6 (Linearized Symmetry Condition [1]).
To find symmetry, the symmetry condition must be solved.

$$\frac{\hat{y}_{x..} + \rho(x,y)\hat{y}_y}{\hat{x}_{x..} + \rho(x,y)\hat{x}_y} = \rho(x,y).$$

This equation shows the symmetry $(x,y) \to (\hat{x},\hat{y})$. If we can solve this equation to get $\hat{x}$and $\hat{y}$. We can. determine the tangent vectors $\zeta$and φ . The symmetry condition can be simplified by applying a Taylor series expansion.

We intend to expand $\hat{x}$, $\hat{y}$i and $\mu(\hat{x},\hat{y})$about $\lambda = 0$.

i$\hat{x} = x + \lambda\zeta(x,yi) + O(\lambda^2)$, i$\hat{y}$. $= x + \lambda\varphi(x,y) + O(\lambda^2)$,

$$\rho(\hat{x},\hat{y}) = \rho(x,y) + \lambda\left(\rho_x\zeta(x,y) + \rho_y\varphi(x,y)\right) + O(\lambda^2)$$

We want to find the constants of:

$$\frac{dy}{dx} = \rho(x,y)$$

Under infinitesimal transformations, $\hat{x}$ and $\hat{y}$, we obtain the condition of invariance of

$$\varphi_x + (\varphi_y - \zeta_x)\rho - \zeta_y\rho^2 = \zeta\rho_x + \varphi\rho_y,$$

Definition 1.7 (The Extended Operator [4]).

The first extended operator code is

$$\Gamma^{(1)} = \rho(x,y)\frac{\partial}{\partial x} + \varphi(x,y)\frac{\partial}{\partial y} + \delta^{(1)}\frac{\partial}{\partial y'},$$

In which

$$\delta^{(1)} = D_x(\varphi) - y'D_x(\zeta)$$
$$= \varphi_x + (\varphi_y - \zeta_x)\rho - \zeta_y\rho^2$$

This condition is known as the linear, The condition of symmetry for the first-order ordinary differential equation.

If we label our ordinary differential equation with Δ such that,

$$\Delta = \frac{dy}{dx} - \rho(x,y) = 0$$

So, the condition of stability is as follows:

$$\Gamma^{(1)}\Delta|_{\Delta=0} = 0$$

The second extended operator code is

$$\Gamma^{(2)} = \zeta(x,y)\frac{\partial}{\partial x} + \varphi(x,y)\frac{\partial}{\partial y} + \delta^{(1)}\frac{\partial}{\partial y'} + \delta^{(2)}\frac{\partial}{\partial y''}$$

In which

$$\delta^{(2)} = D_x(\delta^{(1)}) - y''D_x(\zeta)$$
$$= \varphi_{xx} + (2\varphi_{xy} - \zeta_{xx})y' + (\varphi_{yy} - 2\zeta_{xy})y'^2 - \zeta_{yy}y'^3$$

This condition is known as the linear condition of symmetry applied to second-order differential equations.

If we label our ordinary differential equation with Δ,

$$\Delta = y'' - \rho(x, y, y'),$$

Then the steady state is:

$$\Gamma^{(2)}\Delta|_{\Delta=0} = 0.,$$

Definition 1.8 (The Canonical Coordinates [2,6]).

In the simplest case, we look for coordinates that allow for symmetry:

$$A_\lambda : (t, s) \to (\hat{t}, \hat{s}) = (t, s + \lambda)$$

when

$$(t, s) = \big(t(x, y)i, s(x, y)\big),$$

such that

$$(\hat{t}, \hat{s})i = \big(t(\hat{x}, \hat{y}), s(\hat{x}, \hat{y})\big),$$

Consequently, the vector that is tangent to the given point(t, s) when λi = 0i isi

$$\left(\frac{\partial \hat{t}}{\partial \lambda} i\bigg|_{\lambda.=i0}, \frac{\partial \hat{s}}{\partial \lambda}\bigg|_{\lambda=0}\right) = (0,1)$$

Taking derivatives with respect to λ when $\lambda = 0$, we get

$$\frac{\partial \hat{t}}{\partial \lambda}\bigg|_{\lambda=0} = \frac{\partial t}{\partial \hat{x}}\frac{\partial \hat{x}}{\partial \lambda}\bigg|_{\lambda=0} + \frac{\partial t}{\partial \hat{y}}\frac{\partial \hat{y}}{\partial \lambda}\bigg|_{\lambda=0} = \frac{\partial t}{\partial x}\zeta(x, y) + \frac{\partial t}{\partial y}\varphi(x, y) = 0$$

$$\frac{\partial \hat{s}}{\partial \lambda}\bigg|_{\lambda=0} = \frac{\partial s}{\partial \hat{x}}\frac{\partial \hat{x}}{\partial \lambda}\bigg|_{\lambda=0} + \frac{\partial s}{\partial \hat{y}}\frac{\partial \hat{y}}{\partial \lambda}\bigg|_{\lambda=0} = \frac{\partial s}{\partial x}\zeta(x, y) + \frac{\partial s}{\partial y}\varphi(x, y) = 1.$$

Or

$$t_x\zeta(x, y.) + t_y\varphi(x, y) = 0 \qquad (1)$$

$$s_x\zeta(x, y) + s_y\varphi(x, y) = 1 \qquad (2)$$

Equations (1) and (2) allow for the determination of the canonical coordinates using the characteristic method. The characteristic equations are:

$$\frac{dx}{\zeta(x, y)} = \frac{dy}{\varphi(x, y)} = ds \qquad (3)$$

First integral of a Differential equation

$$\frac{dy}{dx} = f(x, y) \qquad (4)$$

Although the function is non-constant, its value remains constant along the solution trajectories of the equation. Therefore, the general solution is as follows:

$$\psi(x, y) = c$$

Where c is a constant.

By applying the derivative operator to the function $\psi(x, y)$, we get the following equation:

$$\psi_x + f(x, y)\psi_y = 0 , \psi_y \neq 0 \quad (5)$$

Assume that $\zeta(x, y) \neq 0$, is not equal to zero. If we divide equation (1) by $\zeta(.x, y)i$ we get:

$$t_x + \frac{\varphi(x, y)}{\zeta(x, y)} t_y = 0$$

When comparing this result with equation (5)

$$\frac{dy}{dx} = \frac{\varphi(x, y)}{\zeta(x, y)} \quad (6)$$

So, the value of t is determined by solving equation (6) where $t(x, y) = c$, while equation (3) can be used to determine the coordinates $s(x, y)$.

Hence,

$$s = \int \frac{dy}{\varphi(x, y)} = \int \frac{dx}{\zeta(x, y)},$$

There. Is. a special case when $\zeta(x, y) = 0$ and $\varphi(x, y) \neq 0$.

Then $t = x \text{ and } s = \int \frac{dy}{\varphi(x,y)}$

The goal the change is w.r.ite. The differential equation in terms of $t(x, y)$ and $s(x, y)$ to make it easier to solve. Next, we can calculate $\frac{ds}{dt}$ using Cartesian coordinates. Then, we apply the total derivative operator to get:

$$\frac{ds}{dt} = \frac{s_x + \mu(x, y)s_y}{t_s + \mu(x, y)t_y} \quad (7)$$

1. Solving first-order differential equations using the least squares method.

Example 2.1 Consider the. differential equation:

$$\frac{dy}{dx} = e^{-x}y^2 + y + e^x \quad (8)$$

We intend to apply the linearized symmetry condition, Equation (8) to find the tangent vectors $\zeta(x, y) \text{ and } \varphi(x, y)$. First, keep in mind that

$$\rho_x = -e^{-x}y^2 + e^x$$

And

$$\rho_y = 2e^{-x}y + 1$$

We will make an ansatz about the form of $\zeta(x, y)$ *and* $\varphi(x, y)$. Suppose that $\zeta = 1$ *and* φ is ifunctiion. of y only. Therefore, Equation (8) looks like

$$\varphi_y\rho - \zeta\rho_y - \varphi\rho_y = 0$$

$$\varphi_y(e^{-x}y^2 + y + e^x) - (-e^{-x}y^2 + e^x) - \varphi(2e^{-x}y + 1) = 0 \quad (9)$$

Simplifying Equation (9) yields

$$e^{-x}y(\varphi_y y + y - 2\varphi) + e^x(\varphi_y - 1) + \varphi_y y - \varphi = 0$$

Therefore, we know that

$$\varphi_y y + y - 2\varphi = 0 \quad (10)$$

$$\varphi_y y - \varphi = 0 \quad (11)$$

And

$$\varphi_y - 1 = 0 \quad (12)$$

Solving Equation (12),

$$\frac{dy}{d\varphi} = 1$$

we get $\varphi = y$. This is consistent with the Equations (10) and (11). Now we can find the canonical coordinates. To find r, solve

$$\frac{dy}{dx} = \frac{\varphi}{\zeta} = y$$

The equation is separable; by integrating, we get $\ln y = x + c_0$, which simplifies to

$$y = ce^x$$

and therefore,

$$t = \frac{y}{e^x}$$

To find s:

$$s = \int dx = x$$

As a result, the standard coordinates are

$$(t,s) = \left(\frac{y}{e^x}, x\right)$$

By inserting the canonical coordinates into Equation (7), we obtain

$$\frac{ds}{dt} = \frac{1}{-ye^{-x} + e^{-x}(e^{-x}y^2 + y + e^x)} = \frac{1}{y^2e^{-2x} + 1}$$

Therefore:

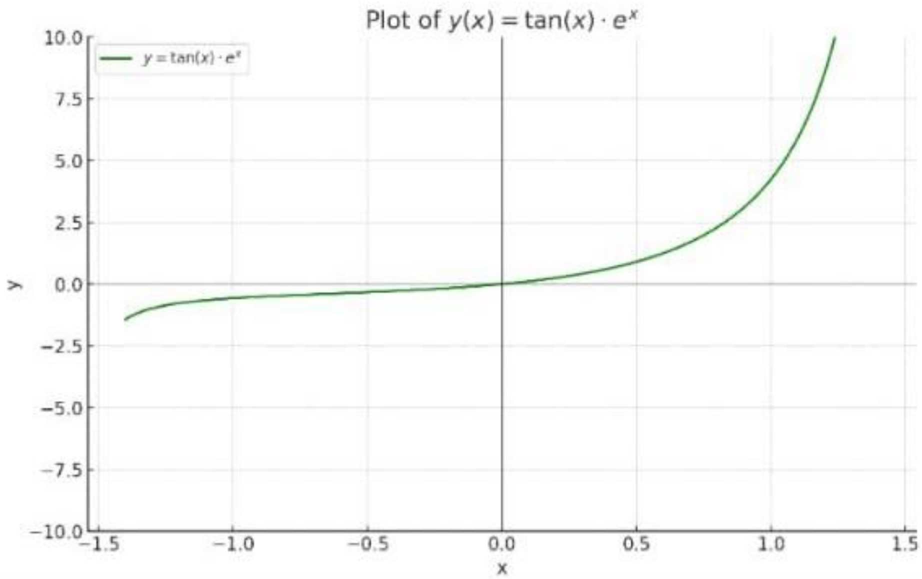
$$\frac{ds}{dt} = \frac{1}{t^2+1} \tag{13}$$

$$s = \int \frac{1}{t^2+1}dt = \arctan(r)$$

$$x = \arctan\left(\frac{y}{e^x}\right)$$

and

$$y = \tan.(x)e^x.$$



The graph above represents the function:

$$y(x) = \tan(x) \cdot e^x$$

As you can see, the graph shows oscillatory behavior due to the $\tan(x)$ term, which has vertical asymptotes (discontinuities) at

$$x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$$

To avoid these discontinuities, the plot is limited to the range from -1.4 to 1.4.

2. Solving second-order differential equations using the least squares method.

E.xample 3.1. Let us consider the following second-order differential equation:

$$y'' = 0 \tag{14}$$

The linearized. Symmetry condition for this ODE is

$\delta^{(2)} = 0$ when $y'' = 0$

That. is,

$$\varphi_{xx} + (2\varphi_{xy} - \zeta_{xx})y' + (\varphi_{yy} - 2\zeta_{xy})y'^2 - \zeta_{yy}y'^3 = 0 \tag{15}$$

As ζ *and* φ a.re Iind ependent, of y' , to Leneariized symmetry. conditiion spliits. into the following. system of determining equations:

$$\varphi_{xx} = 0, \quad 2\varphi_{xy} - \zeta_{xx} = 0 \ , \quad \varphi_{yy} - 2\zeta_{xy} = 0 \ , \quad \zeta_{yy} = 0 \tag{16}$$

The general. solution. of the last of (16) is

$$\zeta(x, yi) = A(x)y + B(x),$$

For general functions A, B, the third equation in (16) gives:

$i\varphi(x, yi) = A'(x)y^2 + C(x)y + D(x),$

Here, C and D represent arbitrary functions as well, and the other equations in (16) lead to:

$iA'''(x)y^2 + iC''(x)y + D''(x) = 0\ ,\ 3A''(x)y + 2C'(x) - B''(x) = 0 \qquad (17)$

Equating the powers. Of y in (17), we obtain a system of ODEs for the unknown functions A, B, C, and D:

$iA'''(x) = 0,\ C''(x)0,\ D''(x) = 0\ ,\ B''(x) = 2C'(x)$i

These ordinary differential equations can be solved with ease to leading. to a following result for each one - parameter Lie group of symmetries of (14), the functions ζ *and* φ are of tihe form

$$\zeta(x, y) = c_1 + c_3x + c_5y + c_7x^2x^2 + c_8xy\ ,$$

$$\varphi(x, y) = c_2 + c_4y + c_6x + c_7xy + c_8y^2\ ,$$

Where $c_1, \dots, , c_8$, they are a constant, hence, the most general expression for the infinitesimal generator is

$$X = \sum_{i=1}^{8} c_i x_i$$

Where,

$X_{1.} = \partial x, X_2 = \partial y_3 = x\partial x, X_4 = y\partial y\ , X_5 = y\partial x, X_6 = x\partial y, X_7 = x^2\partial x + xy\partial y, X_8 = xy\partial x + y^2\partial y$

We have chosen

$$X_2 = \partial y$$

Then, we have the tangent vectors

$$\big(\zeta(x, y), \varphi(x, y)\big) = (0,1)$$

The simplest canonical coordinates are

$$t(x, yi) = x$$

$$s(x, y) = y$$

Which prolong to

$$\frac{ds}{dt} = y', \frac{d^2s}{dt^2} = \frac{D_x(y')}{D_x(x)} = y''$$

Let

$$v = \frac{ds}{dt} = y'$$

Thus

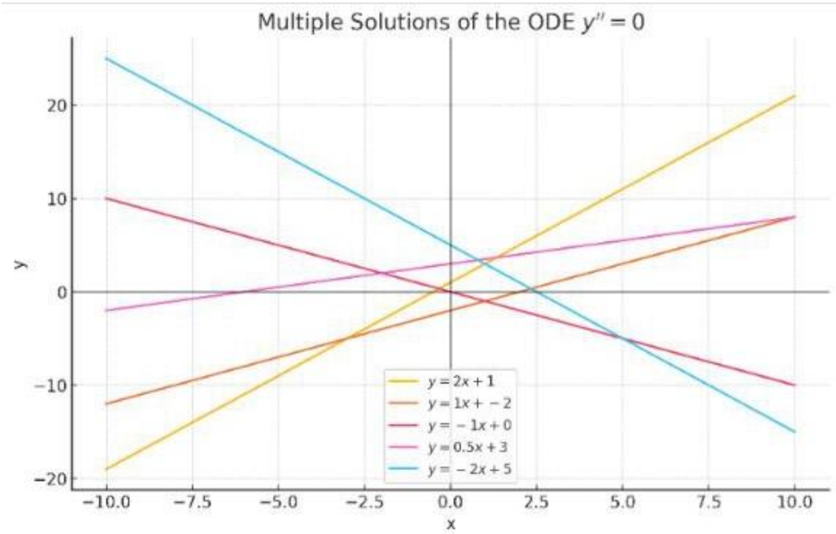
$$\frac{dv}{dt} = y', y'' = 0.$$

Hence Equation (14) simplifies to the following equation:

$$\frac{dv}{dt} = 0$$

The general solution of (14) is

$$y = c_1x + c_2$$



In this plot, several solutions of the differential equation:

$$y'' = 0$$

are displayed. Each solution represents a straight line of the form:

$$y(x) = C_1 x + C_2$$

Each pair (C_1, C_2) gives a different curve.

This shows that the equation admits a complete family of linear solutions.

3. Riccaiti Equations [4,6]

This general from. of a Riccati. ODE is:

$$\frac{dy}{dx} = T(x)y^2 + Z(x)y + R(x) \tag{18}$$

Our goal is to find $\zeta(x, y)$ and $\varphi(x, y)$. It satisfies the invariance condition proposed by Lie. We will assume $\zeta(x, y) = 0$, giving. Lie's invariance. condition as

$$\varphi_x + \left(T(x)y^2 + Z(x)y + R(x)\right)\varphi_y = \left(2T(x)y + Z(x)\right)\varphi \tag{19}$$

One solution of equations (19) is

$$\varphi = (y - y_1)^2 F(x)$$

Where y_1 is one solution. to equations (18) and F satisfies

$$F' + (2Ty_1 + Z)Fi = 0 \tag{20}$$

In order to solve for the canonical variables t and s. It is necessary to solve

$$(y - y_1)^2 F(x) t_y = 0$$

$$(y - y_1)^2 F(x) s_y = 0$$

As a result, we obtain

$$t = R(x)\,, \qquad s = S(x) - \frac{1}{(y - y_1)iF}$$

Where R , S are. arbitrary, functions setting

$$R(x) = x, and\ S(x) = 0$$

Yields

$$x = t \tag{21}$$

$$y = y_1 - \frac{1}{sF(t)} \tag{22}$$

Thus, the original Riccati equations (18) are transformed into

$$\frac{ds}{dt} = \frac{a(t)}{F(t)}$$

Using Lie's method, one can recover the conventional linearizing transformation.

Example 4.1: consider the riccte equation

$$\frac{dy}{dx} = \frac{y+1}{x} + \frac{y^2}{x^3} \tag{23}$$

It. has the. following symmetry.

$\mathrm{i}(\hat{x}, \hat{y}) = \left(\frac{x}{1-\lambda x}, \frac{y}{1-\lambda x}\right)$

The. Tangent vectors. are

$$\zeta(x,y) = \left.\frac{\partial \hat{x}}{\partial \lambda}\right|_{\lambda=0} = x^2$$

And.

$$\varphi(x,y) = \left.\frac{\partial \hat{y}}{\partial \lambda}\right|_{\lambda=0} = xy$$

$$\big(\zeta(x,y), \varphi(x,y)\big) = (x^2, xy)$$

Now. We find t using the following.

$$\frac{dy}{dx} = \frac{\varphi(x,y)}{\zeta(x,y)} = \frac{xy}{x^2} = \frac{y}{x} \tag{24}$$

$$t = \frac{y}{x}$$

This equation (24) the differential equation is separable, and therefore we can integrate it :

$$S = \int \frac{dx}{\delta(x,y)} \qquad or \qquad S = \int \frac{dy}{\varphi(x,y)}$$

$$= \int \frac{dx}{x^2} = \frac{-1}{x}$$

Then.

$$(t,s) = \left(\frac{y}{x}, \frac{-1}{x}\right)$$

Then the equation is reduced to

$$\frac{ds}{dt} = \frac{1}{1+t^2} \qquad where \qquad t = \frac{y}{x}$$

Where, $\Omega(t) = \frac{1}{1+t^2}$

$$C = S - \int^t \Omega(t)\, dt$$

$$= \frac{-1}{x} - \int^t \frac{1}{1+t^2}\, dt$$

$$= \frac{-1}{x} - \tan^{-1}(t)$$

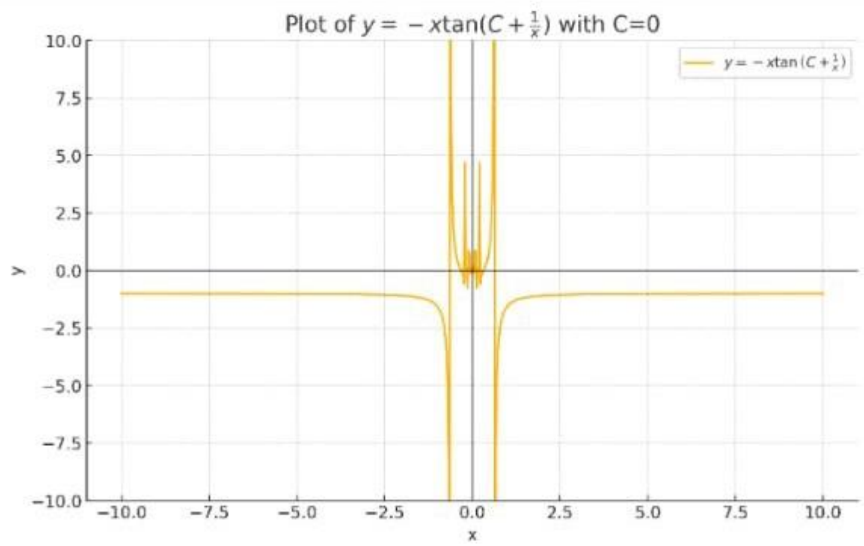
$$C = \frac{-1}{x} - \tan^{-1}\left(\frac{y}{x}\right)$$

$$\tan^{-1}\frac{y}{x} = -C - \frac{1}{x}$$

$$\frac{y}{x} = \tan\left(-C - \frac{1}{x}\right)$$

The general solution is:

$$y = -x\tan\left(C + \frac{1}{x}\right) \qquad (25)$$



Here is the plot of the function:

$$y = -x \tan\left(C + \frac{1}{x}\right)$$

with $C = 0$.

Key Features:

- The function has vertical asymptotes where $\tan\left(\frac{1}{x}\right)$ is undefined

$(i.e., \frac{1}{x} = \frac{\pi}{2} + n\pi)$.

- Around $x = 0$, the function becomes highly oscillatory due to the nature of the tangent function as its argument tends to infinity.
- For large $|x|$, the function tends to a linear behavior since $\frac{1}{x} \to 0$ and $\tan\left(\frac{1}{x}\right) \to \frac{1}{1}$.

4. Conclusion

The Lie symmetry method offers a robust, systematic approach to solving ordinary differential equations by exploiting the equations' underlying symmetry properties. Rooted in the robust framework of group theory, this method facilitates symmetry reduction and often yields exact analytical solutions. By identifying infinitesimal generators and constructing canonical coordinates, complex differential equations can be transformed into simpler forms.

In this work, we demonstrated the application of the Lie symmetry method to both first- and second-order differential equations, including Riccati equations. The examples presented confirm the method's ability to simplify and solve a broad class of equations. Furthermore, with the aid of symbolic computation tools like Maple, the process of finding symmetries and solving the resulting determining equations becomes significantly more accessible.

The continued development of symmetry-based techniques holds great promise for addressing more complex and nonlinear systems in mathematics, physics, and engineering. Future work may extend these methods to partial differential equations and explore their integration with numerical techniques for hybrid analytical-computational solutions.

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