Journal of Education for Pure Science- University of Thi-Qar
Vol.16, No. 1(2026)
DOI: https://doi.org/10.32792/jeps.v16i1.754
Website: jceps.utq.edu.iq Email: jceps@eps.utq.edu.iq

Cohomological Analysis of the Orlik-Solomon Algebra Associated with Graphs Free of 4-Cycles $H^*(A_*(\mathcal{A}_G), a_1 - a_t)$

Numan Y. Neamah[1], **Hana' M. Ali**[2]

[1] Department of Mathematics, College of Science, University of Basrah, Basrah, Iraq

[2] Department of Mathematics, College of Science, University of Basrah, Basrah, Iraq

[*] Corresponding email: nomanzied@gmail.com

Received 17/ 7 /2025, Accepted 30 / 9 /2025, Published 1/ 3 /2026

Abstract:

In this paper, the first non-vanishing cohomology of the Orlik-Solomon algebra, $\boldsymbol{A}_*(G)$, for a graph G with no triangles was investigated where $a = a_1 - a_t, 1 < t \leq \ell$, and ℓ is the number of edges in G. Particularly, $H^3(\boldsymbol{A}_*(\mathcal{A}_G), a)$, did not vanish if G has chordless 5 -cycles that contain the edges e_1 and e_t.

Keywords: Hyperplane arrangement, hypersolvable arrangement, Orlik-Solomon algebra, cohomology of the Orlik-Solomon algebra, graph theory, hypersolvable graph.

1. Introduction:

Combinatorics, a branch of Mathematics that organizes and calculates elements under specific constraints, has implications in a variety of fields, including Computer Science, Physics, and Statistics [1], [2]. The matroid serves as one of its essential concepts, since it abstracts the concept of independence in contexts such as Graphs, Networks, and Algebraic structures [3]. Matroids offer a connection between combinatorics and algebraic topology under their association with pure simplicial complexes, which are crucial for the analysis of topological spaces [4].
Hyperplane arrangement $\mathcal{A}$ (shortened to "arrangements"), which are collections of affine subspaces of codimension one in a vector space V over a field $\mathcal{K}$, have a topology specified by invariants obtained combinatorially from their intersection lattice $L(\mathcal{A}) = \{\bigcap_{H\in B} H \mid B \subseteq \mathcal{A}\}$ [5], [6]. Orlik and Solomon's construction of the Orlik-Solomon algebra $\boldsymbol{A}_*(\mathcal{A})$ of an arrangement $\mathcal{A}$ [7], calculates the cohomology $H^*(M(\mathcal{A}), \mathcal{K})$ of the complement of the arrangement $M(\mathcal{A}) = V \setminus \cup_{H\in\mathcal{A}} H$ and presents techniques for studying cohomology. $H^*(M(\mathcal{A}), \mathcal{L}(a))$ with the rank one local system's coefficients $\mathcal{L}(a)$, $a \in \boldsymbol{A}_1(\mathcal{A})$ [9, 10]. It is possible to generalize $H^*(M(\mathcal{A}), \mathcal{K})$ to cohomology with local system $\mathcal{L}(a)$, if the non-vanishing cohomology $H^*(\boldsymbol{A}_*(\mathcal{A}), a))$ of the Orlik-Solomon algebra $\boldsymbol{A}_*(\mathcal{A})$ exists [8], [9].

Our study integrates two important fields in mathematics with a wide range of applications, “Arrangements” and “Graphs,” by using matroids for transferring the structure used to construct the Orlik-Solomon algebra $\boldsymbol{A}_*(\mathcal{A}_G)$ of the graphic arrangement $\mathcal{A}_G$ to reconstruct the Orlik-Solomon algebra $\boldsymbol{A}_*(G)$ of a graph G using the duality between these two fields [6]. Accordingly, the structure of the third cohomological group of the Orlik-Solomon algebra for a free triangles graph $H^3(\boldsymbol{A}_*(\mathcal{A}_G); a)$ was investigated for $a = a_1 - a_t$, $1 < t \leq \ell$. We proved that $H^3(\boldsymbol{A}_*(\mathcal{A}_G); a)$ not vanished if G has chordless 5-cycles contains the edges e_1 and e_t simultaneously where m the number of vertices in G. Finally, we provided a few illustrations of our results.

2. Basic Concepts:

This section is motivated to recall the concepts and structures that we need in our work.

2.1 Matroids:

Matroid, a crucial concept in combinatorics, encapsulates the idea of independence in various contexts and was first introduced by Hassler Whitney in 1935 [10]. Its foundational structure is a pure simplicial complex, which emerged in the late 19th century via the work of Henri Poincaré [11]. The formal notion of simplicial complexes was later introduced by J. H. C. Whitehead [4].

Definition 2.1.1: [3]

By a finite matroid, we mean a pair $M = (A, \Delta)$, such that Δ is a collection of subsets of a finite set A, satisfying:

1. Δ is an abstract, non-empty simplicial complex, this means, $\Delta \neq \emptyset$ and if $\Delta' \in \Delta$ and $\Delta'' \subset \Delta'$, then $\Delta'' \in \Delta$.
2. All of Δ's induced subcomplexes are pure, i.e. if $B \subseteq A$, the maximal elements of $\Delta \cap 2^B$ have the same cardinality, where $2^B = \{C \subseteq A \mid C \subseteq B\}$.

Members of Δ are referred to as independent matroid sets, it is thought that the facets represent the matroid's bases, can be write $\upsilon \in M$ to mean $\upsilon \in A$.

It claims that two matroids, $M_1 = (A_1, \Delta_1)$ and $M_2 = (A_2, \Delta_2)$, are isomorphic if and only if $\{\psi(\upsilon_1), \dots, \psi(\upsilon_k)\} \in \Delta_2$ and a bijection $\psi : A_1 \longrightarrow A_2$ such that $\{\upsilon_1, \dots, \upsilon_k\} \in \Delta_1$.

A minimally dependent set is a circuit $C \subseteq A$, i.e. when we remove any 0-simplex from C, it becomes independent. Can be define the rank of B by;

,$rk(B) = \max\{|B'| \,|\, B' \subseteq B \text{ and } B' \in \Delta\}$

where $B \subseteq A$. Specifically, $rk(\phi) = 0$ and define the following:

1. The matroid M itself's rank by $rk(M) = rk(A) = \dim(\Delta) + 1 = |F|$, where M's facet is F. Matroid's level is $l(M) = |A| - rk(M)$-1.
2. A maximum subset of rank k is called a $k - flat$ of M. Here, it is worth pointing, If a matroid M has flats B and B', then $B \cap B'$ is also a matroid's flat. The closure $\bar{B}$ of a subset $B \subseteq A$, is the smallest flat containing B, i.e. $\bar{B} = \bigcap_{flats\, B' \supseteq B} B'$.
3. The poset of matroid M denoted by $L(M)$, ordered by inclusions. $L(M)$ is a lattice, since it has a top element A, which calls the lattice of flats of M. A unique minimal element $\hat{0} = \emptyset$ has been seen in $L(M)$.
4. Can be define the characteristic polynomial $\chi_M(t)$ of M, as:

;$\chi_M(t) = \sum_{X \in L(M)} \mu(\hat{0}, X)\, t^{r - rk(X)}$

denotes to the Möbius function of $L(M)$ and $r = rk(M).\mu$

5. The set $\bar{C} = C \backslash v$ is a broken circuit of an ordered matroid $M_{\trianglelefteq}$, where v is the minimal element of a circuit C via a total order $\trianglelefteq$ on A. The simplicial complex is the definition of the broken circuit complex, also known as the BC-complex.

   $NBC_{\trianglelefteq}(M) = \{B \subseteq A \mid B \text{ contains no broken circuit}\}$.

   For $0 \le k \le rk(M)$, set

;$NBC^k_{\trianglelefteq}(M) = \{B \subseteq A \mid B \text{ contains no broken circuit and } |B| = k+1\}$

to be the k^{th}- skeleton of $NBC_{\trianglelefteq}(M)$. It has been noticed that, if $f^{\Delta} = (f_0^{\Delta}, f_1^{\Delta}, \dots, f_{r-1}^{\Delta})$ be the f-vector of $NBC_{\trianglelefteq}(M)$, then $\left|NBC^k_{\trianglelefteq}(M)\right| = f_k^{\Delta}$ and by applying a result of Rota [12];

;$\chi_M(t) = f_{-1}^{\Delta} t^r - f_0^{\Delta} t^{r-1} + \dots + (-1)^r f_{r-1}^{\Delta}$

where$f_{-1}^{\Delta} = 1$.

Remark 2.1.2:

It is worth noting the following:

1. For a field $K = \mathbb{C}$, let $\mathcal{A}$ be a hyperplane arrangement of a finite dimensional vector space V, we will use $M_{\mathcal{A}} = (\mathcal{A}, \Delta_{\mathcal{A}})$ to denote its related matroid, such that $\Delta_{\mathcal{A}} = \{B \subseteq \mathcal{A}: rk(B) = |B|\}$, i.e. all of the linearly independent subarrangements of $\mathcal{A}$ are contained in $\Delta_{\mathcal{A}}$, and the lattice of flats $L(M_{\mathcal{A}})$ will be the lattice intersection $L(\mathcal{A})$, for more information, see [13].
2. If $G = (V, \mathcal{E})$ is a simple connected graph such that V and $\mathcal{E}$ be its vertices set and edges set respectively, then $M_G = (\mathcal{E}, \Delta_G)$, will denote its graphic matroid, where Δ_G is the set of all subgraphs that contain no cycle and the lattice of flats $L(M_G)$ will be the lattice of bonds $L(G)$ and its characteristic polynomial $\chi_{M_G}(t)$ represent the chromatic function of G, i.e. $\chi_{M_G}(t)$ is the number of colorings of G with t colors, see [14].
3. The hyperplane arrangements class has a dual subclass within the class of graphs called the graphic arrangements class. Precisely, for every simple connected graph $G = (V, \mathcal{E})$, there is an associated hyperplane arrangement $\mathcal{A}_G$ via the bijection $[v_i, v_j] \in \mathcal{E} \leftrightarrow H_{i,j} \in \mathcal{A}_G$, where $H_{i,j} = \{(x_1, \dots x_r) \in \mathbb{C}^r: x_i = x_j\}$. Consequently, their matroids M_G and $M_{\mathcal{A}_G}$ are isomorphic. So, their lattices of flats $L(G)$ and $L(\mathcal{A}_G)$ are isomorphic and $\chi_{M_G}(t) = \chi_{M_{\mathcal{A}_G}}(t)$.
4. It is known, the Poincare polynomial of $\mathcal{A}_G$, $P(\mathcal{A}_G, t) = \chi_{M_G}(-t)$. Consequently, if b_j is a j^{th} Betti number of the Poincare polynomial $P(\mathcal{A}_G, t)$, then for $1 \le j \le \ell$, $b_j =$ The coloring's number of j vertices of G with t colors.

Definition 2.1.3:

By a partition $\Pi = (\Pi_1, \dots, \Pi_\ell)$ of a matroid $M = (A, \Delta)$ we mean, a partition of A and by a section S of Π, that mean for each $1 \le k \le \ell$, a subset S of A satisfied, either $S \cap \Pi_k$ is empty or a singleton. Let denoted to the set all sections of Π, by $S(\Pi)$, and denoted to the set of all sections of Π with $|S| = k$, by $S_k(\Pi)$. So, can be call the sections of Π, as k-sections. The empty section of Π is a 0-sections. Moreover, can be call Π nice if every choice of ℓ-section of Π is independent. If Π is nice, then $\ell = rk(M)$.

2.2 Hypersolvable Partition of a Graph:

In addition to hypersolvable graphs, our research emphasizes triangle-free graphs. Papadima and Suciu initially introduced hypersolvable graphs in 2002 [15]. The hypersolvable partition was subsequently devised by Fadhil and Ali as a necessary and sufficient condition for any graph to be considered hypersolvable [16]. We shall utilize the following terminology for defining this concept.

Definition 2.2.1: [16]

The undirected connected simple graph $G = (V, \mathcal{E})$, refer to two finite sets of m-vertices and n-edges respectively, V and $\mathcal{E}$. The hypersolvable partition of G, is a pair $\Pi^G = (\Pi^V, \Pi^{\mathcal{E}})$, and it is denoted by HP, the following facts are satisfy, if $\Pi^V = (\Pi_1^V, \dots, \Pi_{m-1}^V)$ and $\Pi^{\mathcal{E}} = (\Pi_1^{\mathcal{E}}, \dots, \Pi_\ell^{\mathcal{E}})$ are partitions of V and $\mathcal{E}$ respectively:

HP1: $\Pi_1^V = \{v_1, v_2\}$ and $\Pi_1^{\mathcal{E}} = \{e_1\}$, where $e_1 = [v_1, v_2]$, i.e. $\Pi_1^{\mathcal{E}}$ is a singleton.

HP2: The block Π_j^V is a singleton, for every $2 \le j \le m-1$.

HP3: For every $2 \le k \le \ell$, the block $\Pi_k^{\mathcal{E}}$ satisfying the following:

HP$_3 i$: For every $e_{i_1}, e_{i_2} \in \Pi_1^{\mathcal{E}} \cup \dots \cup \Pi_k^{\mathcal{E}}$, there is no edge $e \in \Pi_{k+1}^{\mathcal{E}} \cup \dots \cup \Pi_\ell^{\mathcal{E}}$ such that $\{e_{i_1}, e_{i_2}, e\}$ forms a triangle.

HP$_3 ii$: There is a positive integer $1 < m_k \le m-1$, such that $V_k = \Pi_1^V \cup \dots \cup \Pi_{m_k}^V$ is a subset of V that is contains every endpoints of the edges in $\Pi_1^{\mathcal{E}} \cup \dots \cup \Pi_k^{\mathcal{E}}$, i.e. $G_k = (V_k, \Pi_1^{\mathcal{E}} \cup \dots \cup \Pi_k^{\mathcal{E}})$ makes up a subgraph of G. Then, either.

1. $\Pi_k^{\mathcal{E}} = \{e\}$ such that $V_k = V_{k-1}$, or;
2. $\Pi_k^{\mathcal{E}} = \{e_{i_1}, \dots, e_{i_{d_k}}\}$, such that $V_k \backslash V_{k-1} = \Pi_{m_{k-1}+1}^V = \Pi_{m_k}^V = \{v\}$ and for $1 \le j \le d_k$, $e_{i_j} = [v_{i_j}, v]$, for some $v_{i_j} \in \Pi_1^V \cup \dots \cup \Pi_{m_{k-1}}^V$, where $\{v_{i_1}, \dots, v_{i_{d_k}}\} \subseteq V_{k-1} = \Pi_1 \cup \dots \cup \Pi_{m_{k-1}}$ creates a complete subgraph of G.

The number of the blocks of $\Pi^{\mathcal{E}}$ is called the length of Π^G and denoted by $\ell(G) = \ell$. The vector $d = (d_1, \dots, d_\ell)$ is said to be the exponent vector (or d-vector) of Π if $d_k = |\Pi_k^{\mathcal{E}}|$ for $1 \le k \le \ell$. The rank of $\Pi_k^{\mathcal{E}}$ defined as $rk\Pi_k^{\mathcal{E}} = |V_k| - 1$ and $rk(G) = rk\, \Pi_\ell^{\mathcal{E}} = m - 1$. The block $\Pi_k^{\mathcal{E}}$ can be considered singular if $|V_{k-1}| = |V_k|$, thus it is non-singular if $|V_k \backslash V_{k-1}| = 1$. Can be consider a hypersolvable partition Π^G is a supersolvable if and only if, $\Pi^{\mathcal{E}}$ has no singular blocks. Also, a hypersolvable partition Π^G, is called m-generic if $\ell \ge m$, the exponent vector $d = (1, \dots, 1)$ and every k-eadges of $\mathcal{E}$ cannot be an k-cycle, $3 < k \le m-1$.

Theorem 2.2.2: [5]

A graph $G = (V, \mathcal{E})$ is said to be supersolvable if and only if there exists a sequence $v_1, v_2, \dots, v_m$ that arranges its vertices so that, for any indices $1 \le i < j < k \le m$, if $[v_i, v_k] \in \mathcal{E}$ and $[v_j, v_k] \in \mathcal{E}$, then $[v_i, v_j] \in \mathcal{E}$. The neighborhood of v_i forms a clique when considering only the vertices $v_1, \dots, v_i$ in G.

Proposition 2.2.3: [16]

A graphic arrangement $\mathcal{A}_G$ is hypersolvable (supersolvable or generic), if and only if, it is graph G is hypersolvable (supersolvable or generic).

Note 2.2.4:

Let G be a hypersolvable graph. There are important points to take into consideration:

1. The related graphic arrangement $\mathcal{A}_G$ has a hypersolvable partition $\Pi^{\mathcal{A}_G} = (\Pi_1 \dots \Pi_\ell)$ derived from the hypersolvable partition Π^G=$(\Pi^V, \Pi^{\mathcal{E}})$ on G as; for $1 \le k \le \ell$, $H_{ij} \in \Pi_k$ if, and only if, $[i,j] \in \Pi_k^{\mathcal{E}}$. We will call $\Pi^{\mathcal{A}_G}$ the induced partition of Π^G [16].
2. According to remark (2.1.2) and definition (2.1.3), there is a bijection between $S(\Pi^{\mathcal{A}_G})$ and the set $S(\Pi^G)$. There is also a bijection between $S_k(\Pi^G)$ and $S_k(\Pi^{\mathcal{A}_G})$, for $1 \le k \le \ell$.

2.3 Cohomology of Orlik-Solomon Algebra:

Recalling isomorphic structures of the Orlik-Solomon algebra $\boldsymbol{A}_*(\mathcal{A})$ is the main goal of this section.

Definition 2.3.1: [13]

Suppose that K is a commutative ring and $\trianglelefteq$ is an arbitrary total order defined on the hyperplanes of a ℓ-arrangement $\mathcal{A}$. Define the Orlik-Solomon algebra (or simplicity OS algebra) $\boldsymbol{A}_*(\mathcal{A})$ to be the quotient of the exterior K-algebra $E_* = \wedge_{K\geq 0} (\bigoplus_{H\in\mathcal{A}} K_{e_H})$, by the homogenous ideal $I_*(\mathcal{A})$ is generated by the relations;

$$\sum_{j=1}^{K}(-1)^{k-1} e_{H_{i_1}} \cdots \hat{e}_{H_{i_j}} \cdots e_{H_{i_k}} \text{ for all } 1 \leq i_1 < \cdots < i_k \leq n;$$

where $\{H_{i_1}, \dots, H_{i_k}\}$ is a dependent subarrangement of $\mathcal{A}$ and the circumflex "^"indicates the deletion of $e_{H_{i_j}}$.

The broken circuit module $\boldsymbol{NBC}_*(\mathcal{A})$ is a submodule of the exterior algebra E_* is defined as; $\boldsymbol{NBC}_0(\mathcal{A}) = K$ and $\boldsymbol{NBC}_k(\mathcal{A})$ for $1 \leq k \leq \ell$, be the free K-module of E_k with an NBC (no broken circuit) monomial basis $\{e_C | C \in NBC_k(\mathcal{A})\} \subseteq E_k$, i.e.;

$$\boldsymbol{NBC}_k(\mathcal{A}) = \bigoplus_{C\in NBC_k(\mathcal{A})} Ke_C \text{ and } \boldsymbol{NBC}_*(\mathcal{A}) = \bigoplus_{k=0}^{\ell} \boldsymbol{NBC}_k(\mathcal{A}).$$

Theorem 2.3.2: [13]

On the broken circuit submodule $\boldsymbol{NBC}_*(\mathcal{A})$, the restriction of canonical chain map $\varphi_*: E_* \to \boldsymbol{A}_*(\mathcal{A})$, is an isomorphism defined as; for $1 \leq k \leq \ell$, $\varphi_k(e_C) = e_C + I_k(\mathcal{A}) = a_C$, $C \in NBC_k(\mathcal{A})$. Consequently, the OS algebra $\boldsymbol{A}_*(\mathcal{A})$ embedded by the following structure as a free K-submodule of the exterior algebra: $\boldsymbol{A}_*(\mathcal{A}) = \bigoplus_{k=0}^{\ell} (\bigoplus_{C\in NBC_{K(\mathcal{A})}} Ka_C)$.

Definition 2.3.3: [9]

Assume that $a \in \boldsymbol{A}_1(\mathcal{A})$ with $a = \sum_{s=1}^{n} \lambda_s a_s$ for $\lambda_s \in K$.

Multiplication by a give a differentiation $d_k: \boldsymbol{A}_k(\mathcal{A}) \overset{a.}{\to} \boldsymbol{A}_{k+1}(\mathcal{A})$ such that $(\boldsymbol{A}_*(\mathcal{A}), a)$ forms a complex. The cohomology of this complex denoted by $H^*(\boldsymbol{A}(\mathcal{A}), a)$ and is said to be the cohomology of the Orlik-Solomon algebra.

3. Main Results:

Our work involves two parts. First, we reconstruct the Orlik-Solomon algebra for a graph by utilizing its hypersolvable partition structure. Second, we examine the first non-vanishing cohomology for a triangle-free graph.

3.1 The Orlik-Solomon Algebra of a Graph:

Here, we use the hypersolvable partition structure of a triangle-free graph to constrain the cohomology structure of the Orlik-Solomon algebra. We review the Orlik-Solomon algebra's structure as offered in [17] in order to make our methodology clearer.

3.1.1 Construction:

Let Q_m be a complete graph with m vertisies and assume $\trianglelefteq_{Q_m}^{V}$ is a total order defined on the set of its vertices say as; $V_{Q_m} = \{v_1, \dots, v_m\}$. Via $\trianglelefteq_{Q_m}^{V}$ we will define a total order on its edges that is induced from its supersolvability. We conduct this by recalling the strong geometrical features of its supersolvable partition on $\mathcal{E}_{Q_m}$:

$\Pi^{\mathcal{E}_{Q_m}} = \left(\Pi_1^{\mathcal{E}_{Q_m}}, \Pi_2^{\mathcal{E}_{Q_m}}, \Pi_3^{\mathcal{E}_{Q_m}}, \dots, \Pi_{m-1}^{\mathcal{E}_{Q_m}}\right)$

$= (\{[v_1, v_2]\}, \{[v_1, v_3], [v_2, v_3]\}, \{[v_1, v_4], [v_2, v_4], [v_3, v_4]\}, \dots, \{[v_1, v_m], [v_2, v_m], \dots, [v_{m-1}, v_m]\})$

to define an order $\trianglelefteq_{Q_m}^{\mathcal{E}}$ on $\mathcal{E}_{Q_m}$ that respect this partition as follows: $[v_1, v_2] \trianglelefteq_{Q_m}^{\mathcal{E}} [v_1, v_3] \trianglelefteq_{Q_m}^{\mathcal{E}} [v_2, v_3] \trianglelefteq_{Q_m}^{\mathcal{E}} [v_1, v_4] \trianglelefteq_{Q_m}^{\mathcal{E}} [v_2, v_4] \trianglelefteq_{Q_m}^{\mathcal{E}} [v_3, v_4] \trianglelefteq_{Q_m}^{\mathcal{E}} \dots$

$\trianglelefteq_{Q_m}^{\mathcal{E}} [v_1, v_m] \trianglelefteq_{Q_m}^{\mathcal{E}} [v_2, v_m] \trianglelefteq_{Q_m}^{\mathcal{E}} \dots \trianglelefteq_{Q_m}^{\mathcal{E}} [v_{m-1}, v_m]$.

Straight to the definition of $\trianglelefteq_{Q_m} = (\trianglelefteq_{Q_m}^{V}, \trianglelefteq_{Q_m}^{\mathcal{E}})$, we have the following:

1. If $i < j$, $[v_{k_i}, v_{i+1}] \in \Pi_i^{\mathcal{E}_{Q_m}}$ and $[v_{k_j}, v_{j+1}] \in \Pi_j^{\mathcal{E}_{Q_m}}$, then $[v_{k_i}, v_{i+1}] \trianglelefteq_{Q_m}^{\mathcal{E}} [v_{k_j}, v_{j+1}]$, for each $1 \le k_i \le i$ and $1 \le k_j \le j$.
2. If $1 \le k_1 < k_2 < k_3 \le k$, then in block $\Pi_k^{\mathcal{E}_{Q_m}}$ we have:
   - ✓ $[v_{k_1}, v_{k+1}] \trianglelefteq_{Q_m}^{\mathcal{E}} [v_{k_2}, v_{k+1}] \trianglelefteq_{Q_m}^{\mathcal{E}} [v_{k_3}, v_{k+1}]$ and;
   - ✓ $[v_{k_1}, v_{k_2}] \trianglelefteq_{Q_m}^{\mathcal{E}} [v_{k_1}, v_{k_3}] \trianglelefteq_{Q_m}^{\mathcal{E}} [v_{k_2}, v_{k_3}]$.

We will employ the above structure of $\trianglelefteq_{Q_m}$ on the complete graph Q_m to establish an order $\trianglelefteq_G$ defined on any hypersolvable graph $G = (V, \mathcal{E})$ with m vertices through its HP $\Pi^G = (\Pi^V, \Pi^{\mathcal{E}})$ via the inclusion map $i_G: G \hookrightarrow Q_m$, which we designate as hypersolvable order. That is, the order $\trianglelefteq_G$ will satisfy:

1. If $v_i \in \Pi_i^V$ and $v_j \in \Pi_j^V$ such that; $i < j$, then $v_i \trianglelefteq_G^V v_j$.
2. If $e \in \Pi_i^{\mathcal{E}}$ and $e' \in \Pi_j^{\mathcal{E}}$ such that; $i < j$, then $e \trianglelefteq_G^{\mathcal{E}} e'$.
3. If $e_{i_1}, e_{i_2}, e_{i_3} \in \Pi_k^{\mathcal{E}}$ and $e_{i_1} \trianglelefteq_G^{\mathcal{E}} e_{i_2} \trianglelefteq_G^{\mathcal{E}} e_{i_3}$, then $e_{i_1,i_2} \trianglelefteq_G^{\mathcal{E}} e_{i_1,i_3} \trianglelefteq_G^{\mathcal{E}} e_{i_2,i_3}$,where $\{e_{i_1,i_2}, e_{i_1}, e_{i_2}\}$,$\{e_{i_1,i_3}, e_{i_1}, e_{i_3}\}$, $\{e_{i_2,i_3}, e_{i_2}, e_{i_3}\}$ are the triangles in G via the solvable property of Π^G.

3.1.2 Construction:

From this point on, we assume that $G = (V, \mathcal{E})$ be a simple connected triangle-free graph such that the number of its vertices is $m \ge 4$, the number of its edges is ℓ and $M_G = (\mathcal{E}, \Delta_G)$ represent its Matroid. Therefore, G is hypersolvable with an HP say $\Pi^G = (\Pi^V, \Pi^{\mathcal{E}})$. Consequently, G has an exponent vector $w = (1, \dots, 1)$ such that the length of Π^G is $\ell(G) = \ell = |\mathcal{E}| = |\mathcal{A}_G|$.

In construction (3.1.1), it is crucial to recollect the hypersolvable order $\trianglelefteq_G$ on G to establish an order $\trianglelefteq_{\mathcal{A}_G}$ on the hyperplanes of $\mathcal{A}_G$, based on the structure of the induced partition $\Pi^{\mathcal{A}_G}$ as outlined in note (1.2.4). Consequently, due to remark (2.1.2) and the isomorphism between the lattice of bonds $L(G)$ and the lattice of intersections $L(\mathcal{A}_G)$, we shall derive the following bijections via the hypersolvable order $\trianglelefteq_G$:

1. $f: NBC_{\trianglelefteq_G}(M_G) \to NBC_{\trianglelefteq_{\mathcal{A}_G}}(M_{\mathcal{A}_G})$.
2. For $0 \le k \le rk(M_G)$, $f_k: NBC_{\trianglelefteq_G}^k(M_G) \to NBC_{\trianglelefteq_{\mathcal{A}_G}}^k(M_{\mathcal{A}_G})$ will be the restriction of f on the k^{th}- skeleton of $NBC_{\trianglelefteq_G}^k(M_G)$.
3. $g: S_{\trianglelefteq_G}(\Pi^G) \to S_{\trianglelefteq_{\mathcal{A}_G}}(\Pi^{\mathcal{A}_G})$.
4. For $0 \le k \le \ell(G)$, $g_k: S_{\trianglelefteq_G}^k(\Pi^G) \to S_{\trianglelefteq_{\mathcal{A}_G}}^k(\Pi^{\mathcal{A}_G})$ will be the restriction of g on $S_{\trianglelefteq_G}^k(\Pi^G)$.

According [17], we have,

$p(M_G) = Max\{k \,|\, |NBC_{\trianglelefteq_G}^k(M_G)| = |S_{\trianglelefteq_G}^k(\Pi^G)|\} = c - 2;$

where $c = c(G) = Min\{|C|: C \text{ is a } j - \text{cycle with no chord}, j \geq 4\}$. As a result, we can categorize the class of triangle-free graphs into three subclasses, and we can reconstruct the Orlik-Solomon algebra as free K-module and will be denoted by $\boldsymbol{A}_*(G)$ using the following formula:

1. If $\ell(G) = m - 1$, then G is a tree which is supersolvable, and the Orlik-Solomon algebra reconstructed as: $\boldsymbol{A}_*(G) \cong \oplus_{k=0}^{m-1} \left(\oplus_{C \in S_{\trianglelefteq_G}^k(\Pi^G)} Ka_C \right)$ and for $1 \leq k \leq m-1$, $\boldsymbol{A_k}(G) \cong \oplus_{C \in S_{\trianglelefteq_G}^k(\Pi^G)} Ka_C$ and, $b_k(\boldsymbol{A}_*(G)) = \binom{m-1}{k}$.
2. If $c(G) = \ell(G) = m$, then G is generic that form's m-cycle with no chord, and the Orlik-Solomon algebra reconstructed as: for $1 \leq k \leq m-2$, $\boldsymbol{A_k}(G) \cong \oplus_{C \in S_{\trianglelefteq_G}^k(\Pi^G)} Ka_C$ with $b_k(\boldsymbol{A}_*(G)) = \binom{m}{k}$ and, $\boldsymbol{A_{m-1}}(G) \cong \oplus_{C \in NBC_{\trianglelefteq_G}^{m-1}(M_G)} Ka_C$, where $NBC_{\trianglelefteq_G}^{m-1}(M_G) = S_{\trianglelefteq_G}^{m-1}(\Pi^G) - \{\mathcal{E} - \{[v_1, v_2]\}\}$ and, $b_{m-1}(\boldsymbol{A}_*(G)) = m - 1$. Thus,
$\boldsymbol{A}_*(G) \cong \oplus_{k=0}^{m-2} \left(\oplus_{C \in S_{\trianglelefteq_G}^k(\Pi^G)} Ka_C \right) \oplus \left(\oplus_{C \in S_{\trianglelefteq_G}^{m-1}(\Pi^G) - \{\mathcal{E} - \{[v_1, v_2]\}\}} Ka_C \right)$.
3. If $c(G) = c \leq m - 1 < \ell$, then G is neither supersolvable nor generic and for $1 \leq k \leq c-2$, $\boldsymbol{A_k}(G) \cong \oplus_{C \in S_{\trianglelefteq_G}^k(\Pi^G)} Ka_C$ with $b_j(\boldsymbol{A}_*(G)) = |NBC_k(\mathcal{A}_G)| = \binom{m}{k}$ and $\boldsymbol{A}_{c-1}(G) \cong \oplus_{C \in NBC_{\trianglelefteq_G}^{c-1}(M_G)} Ka_C$, where $NBC_{\trianglelefteq_G}^{c-1}(M_G) = S_{\trianglelefteq_G}^{c-1}(\Pi^G) - BC_{\trianglelefteq_G}^{c-1}(M_G)$, $BC_{\trianglelefteq_G}^{c-1}(M_G)$ is the set of all broken c-cycles via $\trianglelefteq_G^{\mathcal{E}}$, and $b_{c-1}(\boldsymbol{A}_*(G)) = \binom{m}{c-1} - \left|BC_{\trianglelefteq_G}^{c-1}(M_G)\right|$. Thus:

$$\boldsymbol{A}_*(G) \cong \oplus_{k=0}^{m-1} \left(\oplus_{C \in NBC_{\trianglelefteq_G}^k(M_G)} Ka_C \right)$$
$$\cong \oplus_{k=0}^{c-2} \left(\oplus_{C \in S_{\trianglelefteq_G}^k(\Pi^G)} Ka_C \right) \oplus \left(\oplus_{C \in S_{\trianglelefteq_G}^{c-1}(\Pi^G) - BC_{\trianglelefteq_G}^{c-1}(M_G)} Ka_C \right) \oplus_{k=c}^{m-1} \left(\oplus_{C \in NBC_{\trianglelefteq_G}^k(M_G)} Ka_C \right).$$

3.2 The Structure of $\boldsymbol{H}^3(\boldsymbol{A}_*(G), \boldsymbol{a})$:

From now on, assume G is a triangle-free graph and we reordered its set of edges $\mathcal{E} = \{e_1, e_2, \cdots, e_\ell\}$ of G by the hypersolvable order $\trianglelefteq_G$, i.e. $e_1 \trianglelefteq_G^{\mathcal{E}} e_2 \trianglelefteq_G^{\mathcal{E}} \cdots \trianglelefteq_G^{\mathcal{E}} e_\ell$. Furthermore, for a fixed $a \in \boldsymbol{A}_1(G) \cong \oplus_{e_t \in \mathcal{E}} Ka_{e_t}$, recall the complex $(\boldsymbol{A}_*(G), a)$ due definition (2.3.6) for constructing the cohomology of the Orlik-Solomon algebra $H^*(\boldsymbol{A}_*(G), a)$.

Lemma 3.2.1:

If $a = a_{e_1} - a_{e_t}$, for $2 \leq t \leq \ell$, then $\dim(\mathrm{Im}\, d_1) = \ell - 1$;

Proof:

Refer to construction (3.1.2), specifically, $\boldsymbol{A_2}(G) \cong \oplus_{C \in S_{\trianglelefteq_G}^2(\Pi^G)} Ka_C$. To achieve our aim, we will analyze the homomorphism, $d_1: \boldsymbol{A}_1(G) \xrightarrow{a} \boldsymbol{A_2}(G)$. It is significant to highlight that the basis for $\boldsymbol{A_2}(G)$ is $\{a_C: C \in S_{\trianglelefteq_G}^2(\Pi^G)\}$ which includes all the monomials of the 2-sections of $\Pi^{\mathcal{E}}$. Thus, for $1 \leq s \leq \ell$, we have:

$$d_1(a_{e_s}) = a_{e_s}(a_{e_1} - a_{e_t}) = \begin{cases} -a_{e_1}a_{e_t} & : s = 1 \text{ or } t \\ -a_{e_1}a_{e_s} - a_{e_s}a_{e_t} & : 1 < s < t \leq \ell \\ -a_{e_1}a_{e_s} + a_{e_t}a_{e_s} & : 1 < t < s \leq \ell \end{cases}$$

Indeed G is hypersolvable, hence every 2-section of $\Pi^{\mathcal{E}}$ is an NBC base of Δ_G. Therefore, $d_1(a_{e_s}) \neq 0_{\boldsymbol{A_2}(G)}$, since it is expressed as a combination of NBC monomials. Moreover, considering that our graph G is devoid of triangles, it implies that each 3-section of $\Pi^{\mathcal{E}}$ is either an NBC base or broken circuit of Δ_G. So, for $1 < s < t \leq \ell$,

$$\partial_3^{\boldsymbol{A}_*(G)}(a_{e_1}a_{e_s}a_{e_t}) = a_{e_s}a_{e_t} - a_{e_1}a_{e_t} + a_{e_1}a_{e_s} = -d_1(a_{e_s}) - a_{e_1}a_{e_t} \neq 0_{\boldsymbol{A_2}(G)}$$
$$\Rightarrow d_1(a_{e_s}) \neq -a_{e_1}a_{e_t}$$

As well as, for $1 < t < s \leq \ell$,

$$\partial_3^{\boldsymbol{A}_*(G)}(a_{e_1}a_{e_t}a_{e_s}) = a_{e_t}a_{e_s} - a_{e_1}a_{e_s} + a_{e_1}a_{e_t} = d_1(a_{e_s}) + a_{e_1}a_{e_t} \neq 0_{\boldsymbol{A_2}(G)}$$
$$\Rightarrow d_1(a_{e_s}) \neq -a_{e_1}a_{e_t}$$

Therefore, the basis of Im d_1 included $\ell - 1$ monomials and $\dim(\text{Im } d_1) = \ell - 1$.

Proposition 3.2.2:

If $a = a_{e_1} - a_{e_t}$, for $2 \le t \le \ell$, then $\text{Im } d_1 = \ker d_2$ and,

$$\dim(\text{Im } d_2) = \binom{\ell - 2}{2} + (\ell - 2);$$

Proof:

We will analyze the homomorphism, $d_2: \boldsymbol{A}_2(G) \xrightarrow{a} \boldsymbol{A}_3(G)$. It is significant to highlight that the basis for $\boldsymbol{A}_3(G)$ is $\{a_C : C \in S^3_{\trianglelefteq_G}(\Pi^G)\}$ which includes all the monomials of the 3-sections of $\Pi^{\mathcal{E}}$. Thus, for $1 \le k_1 < k_2 \le \ell$, we have;

$$d_2\left(a_{e_{k_1}} a_{e_{k_2}}\right) = a_{e_{k_1}} a_{e_{k_2}} (a_{e_1} - a_{e_t})$$

$$= \begin{cases} 0_{\boldsymbol{A}_3(G)} & : k_1 = 1 \text{ and } k_2 = t \\ -a_{e_1} a_{e_{k_2}} a_{e_t} & : 1 = k_1 < k_2 < t \le \ell \\ a_{e_1} a_{e_t} a_{e_{k_2}} & : 1 = k_1 < t < k_2 \le \ell \\ a_{e_1} a_{e_{k_1}} a_{e_t} & : 1 < k_1 < k_2 = t \le \ell \\ a_{e_1} a_{e_t} a_{e_{k_2}} & : 1 < k_1 = t < k_2 \le \ell \\ a_{e_1} a_{e_{k_1}} a_{e_{k_2}} - a_{e_{k_1}} a_{e_{k_2}} a_{e_t} & : 1 < k_1 < k_2 < t \le \ell \\ a_{e_1} a_{e_{k_1}} a_{e_{k_2}} + a_{e_{k_1}} a_{e_t} a_{e_{k_2}} & : 1 < k_1 < t < k_2 \le \ell \\ a_{e_1} a_{e_{k_1}} a_{e_{k_2}} - a_{e_t} a_{e_{k_1}} a_{e_{k_2}} & : 1 < t < k_1 < k_2 \le \ell \end{cases} \quad \ldots (3.2.2.1)$$

It is clear, for $1 < k < t$ or $t < k \le \ell$, $d_2(a_{e_1} a_{e_k}) = -d_2(a_{e_k} a_{e_t}) \ne 0_{\boldsymbol{A}_3(G)}$, Formula (3.2.2.1) states that, $d_2\left(a_{e_{k_1}} a_{e_{k_2}}\right) = 0_{\boldsymbol{A}_3(G)}$ for $k_1 = 1$ and $k_2 = t$, and for the other cases it constitutes a combination of NBC-monomials. Thus, $a_{e_1} a_{e_t} \in \ker d_2$ but $a_{e_1} a_{e_t} \in \text{Im } d_1$ And since, $(\boldsymbol{A}_*(\mathcal{A}), a)$ is a complex, therefore, $\text{Im } d_1 = \ker d_2$ and $\dim(\text{Im } d_2) = \binom{\ell - 2}{2} + (\ell - 2)$.

Propostion 3.2.3:

If $a = a_{e_1} - a_{e_t}$, for $2 \le t \le \ell$, then $\ker d_3 = \text{Im } d_2$ and;

1. If G is tree or $c(G) > 5$, then $\dim(\text{Im } d_3) = \binom{\ell - 2}{3} + \binom{\ell - 2}{2}$.
2. If $c(G) = 5$, then $\dim(\text{Im } d_3) = \binom{\ell - 2}{3} + \binom{\ell - 2}{2} - u_5(G)$, such that $u_5(G)$ be the number of chordless 5-cycles that includes e_1 and e_t.

Proof:

Our objective will drive us to study the homomorphism, $d_3: \boldsymbol{A}_3(G) \xrightarrow{a} \boldsymbol{A}_4(G)$. Due construction (3.1.2), the structure of $\boldsymbol{A}_4(G)$ depend on the value of $c(G) = Min\{|C| : C \text{ is a } j - \text{cycle with no chord}, j \ge 5\}$. As previously mentioned, $m \ge 4$, hence either (G is tree or $c(G) > 5$) or ($c(G) = 5$). So, either:

$NBC^4_{\trianglelefteq_G}(M_G) = S^4_{\trianglelefteq_G}(\Pi^G)$ and $\boldsymbol{A}_4(G) \cong \bigoplus_{C \in S^4_{\trianglelefteq_G}(\Pi^G)} K a_C$, or;

$NBC^4_{\trianglelefteq_G}(M_G) = S^4_{\trianglelefteq_G}(\Pi^G) - BC^4_{\trianglelefteq_G}(M_G)$ and $\boldsymbol{A}_4(G) \cong \bigoplus_{C \in S^4_{\trianglelefteq_G}(\Pi^G) - BC^4_{\trianglelefteq_G}(M_G)} K a_C$;

where $BC^4_{\trianglelefteq_G}(M_G)$ obtained by removing the minimal edge via $\trianglelefteq_G$ from the 5-cycles with no chord. So, the basis for $\boldsymbol{A_4}(G)$ depend on $c(G)$. In general, for $1 \le k_1 < k_2 < k_3 \le \ell$,

$$d_3\left(a_{e_{k_1}}a_{e_{k_2}}a_{e_{k_3}}\right) = a_{e_{k_1}}a_{e_{k_2}}a_{e_{k_3}}\left(a_{e_1} - a_{e_t}\right)$$

$$= \begin{cases} 0_{\boldsymbol{A_4}(G)} & : k_i = 1 \text{ and } k_j = t \text{ for some } 1 \le i < j \le 3 \\ -a_{e_1}a_{e_{k_2}}a_{e_{k_3}}a_{e_t} & : 1 = k_1 < k_2 < k_3 < t \le \ell \\ a_{e_1}a_{e_{k_2}}a_{e_t}a_{e_{k_3}} & : 1 = k_1 < k_2 < t < k_3 \le \ell \\ -a_{e_1}a_{e_t}a_{e_{k_2}}a_{e_{k_3}} & : 1 = k_1 < t < k_2 < k_3 \le \ell \\ -a_{e_1}a_{e_{k_1}}a_{e_{k_2}}a_{e_t} & : 1 < k_1 < k_2 < k_3 = t \le \ell \\ -a_{e_1}a_{e_{k_1}}a_{e_t}a_{e_{k_3}} & : 1 < k_1 < k_2 = t < k_3 \le \ell \\ -a_{e_1}a_{e_t}a_{e_{k_2}}a_{e_{k_3}} & : 1 < k_1 = t < k_2 < k_3 \le \ell \\ a_{e_1}a_{e_{k_1}}a_{e_{k_2}}a_{e_{k_3}} - a_{e_{k_1}}a_{e_{k_2}}a_{e_{k_3}}a_{e_t} & : 1 < k_1 < k_2 < k_3 < t \le \ell \\ a_{e_1}a_{e_{k_1}}a_{e_{k_2}}a_{e_{k_3}} + a_{e_{k_1}}a_{e_{k_2}}a_{e_t}a_{e_{k_3}} & : 1 < k_1 < k_2 < t < k_3 \le \ell \\ a_{e_1}a_{e_{k_1}}a_{e_{k_2}}a_{e_{k_3}} - a_{e_{k_1}}a_{e_t}a_{e_{k_2}}a_{e_{k_3}} & : 1 < k_1 < t < k_2 < k_3 \le \ell \\ a_{e_1}a_{e_{k_1}}a_{e_{k_2}}a_{e_{k_3}} + a_{e_t}a_{e_{k_1}}a_{e_{k_2}}a_{e_{k_3}} & : 1 < t < k_1 < k_2 < k_3 \le \ell \end{cases} \quad \ldots(3.2.3.1)$$

It is clear,

1. The relation $NBC^4_{\trianglelefteq_G}(M_G) = S^4_{\trianglelefteq_G}(\Pi^G)$ holds if G is a tree or $c(G) > 5$. If $k_i = 1$ and $k_j = t$ for some $1 \le i < j \le 3$, then the formula (2.2.3.1) indicates that $d_3\left(a_{e_{k_1}}a_{e_{k_2}}a_{e_{k_3}}\right) = 0_{\boldsymbol{A_4}(G)}$, and in all other cases, $d_3\left(a_{e_{k_1}}a_{e_{k_2}}a_{e_{k_3}}\right)$ forms a combination of NBC-monomials. Hence, for any $1 < k \le \ell$ and $k \ne t$, $\mp a_{e_1}a_{e_t}a_{e_k} \in \ker d_3$. But $\mp a_{e_1}a_{e_t}a_{e_k} \in \operatorname{Im} d_2$, as shown in proposition (3.2.2). Consequently, $\ker d_3 = \operatorname{Im} d_2$ and $\dim(\operatorname{Im} d_3) = \binom{\ell-2}{3} + \binom{\ell-2}{2}$.

2. In case $c(G) = 5$, there are two possibilities:

a. If $m = 5$, then G is 5-generic. Assume $\mathcal{E} = \{e_1 = [v_1, v_2], e_2 = [v_2, v_3], e_3 = [v_1, v_4], e_4 = [v_3, v_4], e_5 = [v_4, v_5]\}$. So, the broken 5-cycle with no chord written as:

$$a_{e_2}a_{e_3}a_{e_4}a_{e_5} = a_{e_1}a_{e_3}a_{e_4}a_{e_5} - a_{e_1}a_{e_2}a_{e_4}a_{e_5} + a_{e_1}a_{e_2}a_{e_3}a_{e_5} - a_{e_1}a_{e_2}a_{e_3}a_{e_4} \quad \ldots(3.2.3.2)$$

Below are four cases for values of t:

a.1. If $t = 2$, then:

$d_3(a_{e_1}a_{e_2}a_{e_3}) = d_3(a_{e_1}a_{e_2}a_{e_4}) = d_3(a_{e_1}a_{e_2}a_{e_5}) = 0_{\boldsymbol{A_4}(G)}$,
$d_3(a_{e_1}a_{e_3}a_{e_4}) = d_3(a_{e_2}a_{e_3}a_{e_4}) = -a_{e_1}a_{e_2}a_{e_3}a_{e_4}$,
$d_3(a_{e_1}a_{e_3}a_{e_5}) = d_3(a_{e_2}a_{e_3}a_{e_5}) = -a_{e_1}a_{e_2}a_{e_3}a_{e_5}$,
$d_3(a_{e_1}a_{e_4}a_{e_5}) = d_3(a_{e_2}a_{e_4}a_{e_5}) = -a_{e_1}a_{e_2}a_{e_4}a_{e_5}$,
and $d_3(a_{e_3}a_{e_4}a_{e_5}) = d_3(a_{e_1}a_{e_3}a_{e_4}) - d_3(a_{e_1}a_{e_3}a_{e_5}) + d_3(a_{e_1}a_{e_4}a_{e_5})$.

a.2. If $t = 3$, then:

$d_3(a_{e_1}a_{e_2}a_{e_3}) = d_3(a_{e_1}a_{e_3}a_{e_4}) = d_3(a_{e_1}a_{e_3}a_{e_5}) = 0_{\boldsymbol{A_4}(G)}$,
$d_3(a_{e_1}a_{e_2}a_{e_4}) = -d_3(a_{e_2}a_{e_3}a_{e_4}) = a_{e_1}a_{e_2}a_{e_3}a_{e_4}$,
$d_3(a_{e_1}a_{e_2}a_{e_5}) = -d_3(a_{e_2}a_{e_3}a_{e_5}) = a_{e_1}a_{e_2}a_{e_3}a_{e_5}$,
$d_3(a_{e_1}a_{e_4}a_{e_5}) = d_3(a_{e_3}a_{e_4}a_{e_5}) = -a_{e_1}a_{e_3}a_{e_4}a_{e_5}$,
and $d_3(a_{e_2}a_{e_4}a_{e_5}) = d_3(a_{e_1}a_{e_2}a_{e_4}) - d_3(a_{e_1}a_{e_2}a_{e_5}) + d_3(a_{e_1}a_{e_4}a_{e_5})$.

a.3. If $t = 4$, then:

$d_3(a_{e_1}a_{e_2}a_{e_4}) = d_3(a_{e_1}a_{e_3}a_{e_4}) = d_3(a_{e_1}a_{e_4}a_{e_5}) = 0_{\boldsymbol{A_4}(G)}$,
$d_3(a_{e_1}a_{e_2}a_{e_3}) = d_3(a_{e_2}a_{e_3}a_{e_4}) = -a_{e_1}a_{e_2}a_{e_3}a_{e_4}$,
$d_3(a_{e_1}a_{e_2}a_{e_5}) = -d_3(a_{e_2}a_{e_4}a_{e_5}) = a_{e_1}a_{e_2}a_{e_4}a_{e_5}$,

$d_3(a_{e_1}a_{e_3}a_{e_5}) = -d_3(a_{e_3}a_{e_4}a_{e_5}) = a_{e_1}a_{e_3}a_{e_4}a_{e_5}$,
and $d_3(a_{e_2}a_{e_3}a_{e_5}) = d_3(a_{e_1}a_{e_2}a_{e_3}) - d_3(a_{e_1}a_{e_2}a_{e_5}) - d_3(a_{e_1}a_{e_3}a_{e_5})$.

a.4. If $t = 5$, then:
$d_3(a_{e_1}a_{e_2}a_{e_5}) = d_3(a_{e_1}a_{e_3}a_{e_5}) = d_3(a_{e_1}a_{e_4}a_{e_5}) = 0_{\boldsymbol{A_4}(G)}$,
$d_3(a_{e_1}a_{e_2}a_{e_3}) = d_3(a_{e_2}a_{e_3}a_{e_5}) = -a_{e_1}a_{e_2}a_{e_3}a_{e_5}$,
$d_3(a_{e_1}a_{e_2}a_{e_4}) = d_3(a_{e_2}a_{e_4}a_{e_5}) = -a_{e_1}a_{e_2}a_{e_4}a_{e_5}$,
$d_3(a_{e_1}a_{e_3}a_{e_4}) = d_3(a_{e_3}a_{e_4}a_{e_5}) = -a_{e_1}a_{e_3}a_{e_4}a_{e_5}$,
and $d_3(a_{e_2}a_{e_3}a_{e_4}) = d_3(a_{e_1}a_{e_2}a_{e_3}) - d_3(a_{e_1}a_{e_2}a_{e_4}) + d_3(a_{e_1}a_{e_3}a_{e_4})$.

Hence, for any $1 < k \le \ell$ and $k \neq t$, $\mp a_{e_1}a_{e_t}a_{e_k} \in \ker d_3$. But $\mp a_{e_1}a_{e_t}a_{e_k} \in \operatorname{Im} d_2$, as shown in (3.4.1). Consequently, $\ker d_3 = \operatorname{Im} d_2$. It is clear, $\dim(\operatorname{Im} d_3) = \binom{5-2}{3} + \binom{5-2}{2} - 1 = 3$ for this case.

b. If $m > 5$ and $c_5(G)$ be the number of chordless 5-cycles then for this case,
$|NBC^3_{\trianglelefteq_G}(M_G)| = |S^3_{\trianglelefteq_G}(\Pi^G)| = \binom{\ell}{3}$, $|BC^4_{\trianglelefteq_G}(M_G)| = c_5(G)$ and;
$|NBC^4_{\trianglelefteq_G}(M_G)| = |S^4_{\trianglelefteq_G}(\Pi^G)| - |BC^4_{\trianglelefteq_G}(M_G)| = \binom{\ell}{4} - c_5(G)$.
Suppose $S = \{e_{q_1}, e_{q_2}, e_{q_3}, e_{q_4}, e_{q_5}\}$ be a chordless 5-cycle such that $1 \le q_1 < q_2 < q_3 < q_4 < q_5 \le \ell$. Hence, it's broken cycle is $C = \{e_{q_2}, e_{q_3}, e_{q_4}, e_{q_5}\} \in BC^4_{\trianglelefteq_G}(M_G)$. It is known that every monomial of a broken 5-circuit written as a combination of NBC-monomials as:

$$a_{e_{q_2}}a_{e_{q_3}}a_{e_{q_4}}a_{e_{q_5}} = a_{e_{q_1}}a_{e_{q_3}}a_{e_{q_4}}a_{e_{q_5}} - a_{e_{q_1}}a_{e_{q_2}}a_{e_{q_4}}a_{e_{q_5}} + a_{e_{q_1}}a_{e_{q_2}}a_{e_{q_3}}a_{e_{q_5}} - a_{e_{q_1}}a_{e_{q_2}}a_{e_{q_3}}a_{e_{q_4}} \quad \dots (3.2.3.3)$$

Therefore, we have the following possible cases:

b.1. If S is one of $(c_5 - u_5)$ of chordless 5-cycles that do not simultaneously include e_1 and e_t, then:

b.1.1. If $e_1, e_t \notin S$, then formula (2.2.3.1) shows that $d_3\left(a_{e_1}a_{e_{q_i}}a_{e_{q_j}}\right)$ and $d_3(a_{e_{q_i}}a_{e_{q_j}}a_{e_{q_k}})$ are written as a combination of NBC-monomials, for $1 \le i < j < k \le 5$.

b.1.2. For the second case assume either ($e_1 \in S$ and $e_t \notin S$) or ($e_1 \notin S$ and $e_t \in S$). If ($e_1 \in S$ and $e_t \notin S$), then, $q_1 = 1$ and $q_2, q_3, q_4, q_5 \neq t$. Accordingly to formula (2.2.3.1), we get either $d_3\left(a_{e_1}a_{e_{q_i}}a_{e_{q_j}}\right) = \mp d_3\left(a_{e_{q_i}}a_{e_{q_j}}a_{e_t}\right) = \mp a_{e_1}a_{e_{q_i}}a_{e_{q_j}}a_{e_t}$ is an NBC monomial or $d_2\left(a_{e_{q_i}}a_{e_{q_j}}a_{e_{q_k}}\right) = a_{e_1}a_{e_{q_i}}a_{e_{q_j}}a_{e_{q_k}} \mp a_{e_{q_i}}a_{e_{q_j}}a_{e_{q_k}}a_{e_t}$, is written as a combination of an NBC monomials, for $1 < i < j < k \le 5$. Therefore, the number of generators that combine the basis of $\operatorname{Im} d_3$ is 4, determined as: $\binom{5-2}{3} + \binom{5-2}{2} = 4$. Similarly, it's possible to conclude that four generators will be obtained to be included in $\operatorname{Im} d_3$'s basis, if ($e_1 \notin S$ and $e_t \in S$).

b.2. Assume S is one of the u_5 chordless 5-cycles, that including e_1 and e_t, for the purpose of seeing how one of the 2-NBC monomial images cannot be including in $\operatorname{Im} d_2$'s basis, as follows:

b.2.1. If $t = q_2$:
$d_3\left(a_{e_{q_3}}a_{e_{q_4}}a_{e_{q_5}}\right) = d_3\left(a_{e_1}a_{e_{q_2}}a_{e_{q_4}}\right) - d_3\left(a_{e_1}a_{e_{q_3}}a_{e_{q_5}}\right) + d_3\left(a_{e_1}a_{e_{q_4}}a_{e_{q_5}}\right)$.

b.2.2. If $t = q_3$:
$d_3\left(a_{e_{q_2}}a_{e_{q_4}}a_{e_{q_5}}\right) = d_3\left(a_{e_1}a_{e_{q_2}}a_{e_{q_4}}\right) - d_3\left(a_{e_1}a_{e_{q_2}}a_{e_{q_5}}\right) + d_3\left(a_{e_1}a_{e_{q_4}}a_{e_{q_5}}\right)$.

b.2.3. If $t = q_4$:
$d_3\left(a_{e_{q_2}}a_{e_{q_3}}a_{e_{q_5}}\right) = d_3\left(a_{e_1}a_{e_{q_2}}a_{e_{q_3}}\right) - d_3\left(a_{e_1}a_{e_{q_2}}a_{e_{q_5}}\right) - d_3\left(a_{e_1}a_{e_{q_3}}a_{e_{q_5}}\right)$.

b.2.4. If $t = q_5$:
$d_3\left(a_{e_{q_2}}a_{e_{q_3}}a_{e_{q_4}}\right) = d_3\left(a_{e_1}a_{e_{q_2}}a_{e_{q_3}}\right) - d_3\left(a_{e_1}a_{e_{q_2}}a_{e_{q_4}}\right) + d_3\left(a_{e_1}a_{e_{q_3}}a_{e_{q_4}}\right)$.

In this case, the generators joining the basis of Im d_3, are exactly three elements calculated as: $\binom{5-2}{3} + \binom{5-2}{2} - 1 = 3$. For general case, the number of such generators is, $\binom{\ell-2}{3} + \binom{\ell-2}{2} - u_5(G)$.

Theorem 3.2.4:

If $a = a_{e_1} - a_{e_t}$, for $2 \le t \le \ell$, then:

1. If G is tree or $c(G) > 6$, then kerd_4 = Im d_3 and dim(Im d_4) = $\binom{\ell-2}{4} + \binom{\ell-2}{3}$.
2. If $c(G) = 6$, then kerd_4 = Im d_3 and dim(Im d_4) = $\binom{\ell-2}{4} + \binom{\ell-2}{3} + -u_6(G)$.
3. If G is 5-generic graph, then ker$d_4 = \boldsymbol{A_4}(G)$.
4. If $c(G) = 5$, then dim(ker d_4) = dim(Im d_3) + u_5 and:

$$\dim(Imd_4) = \left(\binom{\ell-2}{4} - (u_5(\ell-5) - b_5) - (c_5 - u_5)\right) + \left(\binom{\ell-2}{3} - (u_5 + v_5)\right) - u_6.$$

where c_5 be the number of chordless 5-cycles, u_5 and u_6 be the number of chordless 5-cycles and chordless 6-cycles that includes e_1 and e_t respectively, v_5 is the number of chordless 5-cycles that includes e_t as not minimal edge via $\trianglelefteq_G$ and b_5 is the number of broken circuits that contains $B - \{e_1, e_t\}$, and B is a chordless 5-cycle that simultaneously include e_1 and e_t.

Proof:

According to construction (3.1.2), the structure of $\boldsymbol{A_5}(G)$ depends on the value of $c(G)$. As we mentioned, the graph G has no triangles, hence either (G is tree or $c(G) > 5$) or ($c(G) = 5$). So:

1. If G is tree or $c(G) > 6$, then $NBC^5_{\trianglelefteq_G}(M_G) = S^5_{\trianglelefteq_G}(\Pi^G)$ and $\boldsymbol{A_5}(G) \cong \bigoplus_{C \in S^5_{\trianglelefteq_G}(\Pi^G)} K a_C$.
2. If $c(G) =$ 6, then $NBC^5_{\trianglelefteq_G}(M_G) = S^5_{\trianglelefteq_G}(\Pi^G) - BC^5_{\trianglelefteq_G}(M_G)$ and;
   $\boldsymbol{A_5}(G) \cong \bigoplus_{C \in S^5_{\trianglelefteq_G}(\Pi^G) - BC^5_{\trianglelefteq_G}(M_G)} K a_C$,
3. If $c(G) =$ 5, then $NBC^5_{\trianglelefteq_G}(M_G) = S^5_{\trianglelefteq_G}(\Pi^G) - (BC^5_{\trianglelefteq_G}(M_G) \cup C^5_{\trianglelefteq_G}(M_G))$ and;
   $\boldsymbol{A_5}(G) \cong \bigoplus_{C \in S^5_{\trianglelefteq_G}(\Pi^G) - (BC^5_{\trianglelefteq_G}(M_G) \cup C^5_{\trianglelefteq_G}(M_G))} K a_C$,

where, where $BC^5_{\trianglelefteq_G}(M_G)$ contains the sections that obtained by deleting the minimal edge via $\trianglelefteq_G$ from the chordless 6-cycles and $C^5_{\trianglelefteq_G}(M_G)$ include the sections that related to chordless 5-cycles ordered via $\trianglelefteq_G$.

Now, we need to examine the homomorphism, $d_4: \boldsymbol{A}_4(G) \overset{a}{\rightarrow} \boldsymbol{A_5}(G)$, according to the value of $c(G)$. In general, for $1 \le k_1 < k_2 < k_3 < k_4 \le \ell$,

$$d_4\left(a_{e_{k_1}} a_{e_{k_2}} a_{e_{k_3}} a_{e_{k_4}}\right) = a_{e_{k_1}} a_{e_{k_2}} a_{e_{k_3}} a_{e_{k_4}} \left(a_{e_1} - a_{e_t}\right)$$

$$= \begin{cases} 0_{A_4(G)} & : k_i = 1 \text{ and } k_j = t \text{ for some } 1 \le i < j \le 4 \\ -a_{e_1}a_{e_{k_2}}a_{e_{k_3}}a_{e_{k_4}}a_{e_t} & : 1 = k_1 < k_2 < k_3 < k_4 < t \le \ell \\ a_{e_1}a_{e_{k_2}}a_{e_{k_3}}a_{e_t}a_{e_{k_4}} & : 1 = k_1 < k_2 < k_3 < t < k_4 \le \ell \\ -a_{e_1}a_{e_{k_2}}a_{e_t}a_{e_{k_3}}a_{e_{k_4}} & : 1 = k_1 < k_2 < t < k_3 < k_4 \le \ell \\ a_{e_1}a_{e_t}a_{e_{k_2}}a_{e_{k_3}}a_{e_{k_4}} & : 1 = k_1 < t < k_2 < k_3 < k_4 \le \ell \\ a_{e_1}a_{e_{k_1}}a_{e_{k_2}}a_{e_{k_3}}a_{e_t} & : 1 < k_1 < k_2 < k_3 < k_4 = t \le \ell \\ a_{e_1}a_{e_{k_1}}a_{e_{k_2}}a_{e_t}a_{e_{k_4}} & : 1 < k_1 < k_2 < k_3 = t < k_4 \le \ell \\ a_{e_1}a_{e_{k_1}}a_{e_t}a_{e_{k_3}}a_{e_{k_4}} & : 1 < k_1 < k_2 = t < k_3 < k_4 \le \ell \\ a_{e_1}a_{e_t}a_{e_{k_2}}a_{e_{k_3}}a_{e_{k_4}} & : 1 < k_1 = t < k_2 < k_3 < k_4 \le \ell \\ a_{e_1}a_{e_{k_1}}a_{e_{k_2}}a_{e_{k_3}}a_{e_{k_4}} - a_{e_{k_1}}a_{e_{k_2}}a_{e_{k_3}}a_{e_{k_4}}a_{e_t} & : 1 < k_1 < k_2 < k_3 < k_4 < t \le \ell \\ a_{e_1}a_{e_{k_1}}a_{e_{k_2}}a_{e_{k_3}}a_{e_{k_4}} + a_{e_{k_1}}a_{e_{k_2}}a_{e_{k_3}}a_{e_t}a_{e_{k_4}} & : 1 < k_1 < k_2 < k_3 < t < k_4 \le \ell \\ a_{e_1}a_{e_{k_1}}a_{e_{k_2}}a_{e_{k_3}}a_{e_{k_4}} - a_{e_{k_1}}a_{e_{k_2}}a_{e_t}a_{e_{k_3}}a_{e_{k_4}} & : 1 < k_1 < k_2 < t < k_3 < k_4 \le \ell \\ a_{e_1}a_{e_{k_1}}a_{e_{k_2}}a_{e_{k_3}}a_{e_{k_4}} + a_{e_{k_1}}a_{e_t}a_{e_{k_2}}a_{e_{k_3}}a_{e_{k_4}} & : 1 < k_1 < t < k_2 < k_3 < k_4 \le \ell \\ a_{e_1}a_{e_{k_1}}a_{e_{k_2}}a_{e_{k_3}}a_{e_{k_4}} - a_{e_t}a_{e_{k_1}}a_{e_{k_2}}a_{e_{k_3}}a_{e_{k_4}} & : 1 < t < k_1 < k_2 < k_3 < k_4 \le \ell \end{cases}$$

…(3.2.4.1)

Within the type G, we will look at all possible cases that follow:

1. The relation $NBC^5_{\trianglelefteq_G}(M_G) = S^5_{\trianglelefteq_G}(\Pi^G)$ holds if G is a tree or $c(G) > 6$. If $k_i = 1$ and $k_j = t$ for some $1 \le i < j \le 4$, then the formula (3.2.4.1) indicates that $d_4\left(a_{e_{k_1}}a_{e_{k_2}}a_{e_{k_3}}a_{e_{k_4}}\right) = 0_{A_5(G)}$, and in all other cases, $d_4\left(a_{e_{k_1}}a_{e_{k_2}}a_{e_{k_3}}a_{e_{k_4}}\right)$ forms a combination of NBC-monomials. Hence, for any $1 < k_i < k_j \le \ell$ and $k_i, k_j \ne t$, $\mp a_{e_1}a_{e_t}a_{e_{k_i}}a_{e_{k_j}} \in \ker d_4$, But $\mp a_{e_1}a_{e_t}a_{e_{k_i}}a_{e_{k_j}} \in \operatorname{Im} d_3$, as shown in proposition (2.2.3). Consequently, $\ker d_4 = \operatorname{Im} d_3$ and $\dim(\operatorname{Im} d_4) = \binom{\ell-2}{4} + \binom{\ell-2}{3}$.

2. In case $c(G) = 6$, there are two possibilities:

a. If $m = 6$, then G is 6-generic. Assume $\mathcal{E} = \{e_1 = [v_1, v_2], e_2 = [v_2, v_3], e_3 = [v_3, v_4], e_4 = [v_4, v_5], e_5 = [v_1, v_6], e_6 = [v_5, v_6]\}$. So, the broken 6-cycle with no chord written as:

$a_{e_2}a_{e_3}a_{e_4}a_{e_5}a_{e_6} = a_{e_1}a_{e_3}a_{e_4}a_{e_5}a_{e_6} - a_{e_1}a_{e_2}a_{e_4}a_{e_5}a_{e_6} + a_{e_1}a_{e_2}a_{e_3}a_{e_5}a_{e_6} - a_{e_1}a_{e_2}a_{e_3}a_{e_4}a_{e_6} + a_{e_1}a_{e_2}a_{e_3}a_{e_4}a_{e_5}$ …(2.2.4.2)

Below are five cases for values of t:

a.1. If $t = 2$, then:

$d_4(a_{e_1}a_{e_2}a_{e_3}a_{e_4}) = d_4(a_{e_1}a_{e_2}a_{e_3}a_{e_5}) = d_4(a_{e_1}a_{e_2}a_{e_3}a_{e_6}) = d_4(a_{e_1}a_{e_2}a_{e_4}a_{e_5}) =$
$d_4(a_{e_1}a_{e_2}a_{e_4}a_{e_6}) = d_4(a_{e_1}a_{e_2}a_{e_5}a_{e_6}) = 0_{A_5(G)}$,
$d_4(a_{e_1}a_{e_3}a_{e_4}a_{e_5}) = d_4(a_{e_2}a_{e_3}a_{e_4}a_{e_5}) = a_{e_1}a_{e_2}a_{e_3}a_{e_4}a_{e_5}$,
$d_4(a_{e_1}a_{e_3}a_{e_4}a_{e_6}) = d_4(a_{e_2}a_{e_3}a_{e_4}a_{e_6}) = a_{e_1}a_{e_2}a_{e_3}a_{e_4}a_{e_6}$,
$d_4(a_{e_1}a_{e_3}a_{e_5}a_{e_6}) = d_4(a_{e_2}a_{e_3}a_{e_5}a_{e_6}) = a_{e_1}a_{e_2}a_{e_3}a_{e_5}a_{e_6}$,
$d_4(a_{e_1}a_{e_4}a_{e_5}a_{e_6}) = d_4(a_{e_2}a_{e_4}a_{e_5}a_{e_6}) = a_{e_1}a_{e_2}a_{e_4}a_{e_5}a_{e_6}$, and
$d_4(a_{e_3}a_{e_4}a_{e_5}a_{e_6}) = d_4(a_{e_1}a_{e_3}a_{e_4}a_{e_5}) - d_4(a_{e_1}a_{e_3}a_{e_4}a_{e_6}) + d_4(a_{e_1}a_{e_3}a_{e_5}a_{e_6}) - d_4(a_{e_1}a_{e_4}a_{e_5}a_{e_6})$.

a.2. If $t = 3$, then:

$d_4(a_{e_1}a_{e_2}a_{e_3}a_{e_4}) = d_4(a_{e_1}a_{e_2}a_{e_3}a_{e_5}) = d_4(a_{e_1}a_{e_2}a_{e_3}a_{e_6}) = d_4(a_{e_1}a_{e_3}a_{e_4}a_{e_5}) =$
$d_4(a_{e_1}a_{e_3}a_{e_4}a_{e_6}) = d_4(a_{e_1}a_{e_3}a_{e_5}a_{e_6}) = 0_{A_5(G)}$,
$d_4(a_{e_1}a_{e_2}a_{e_4}a_{e_5}) = -d_4(a_{e_2}a_{e_3}a_{e_4}a_{e_5}) = -a_{e_1}a_{e_2}a_{e_3}a_{e_4}a_{e_5}$,
$d_4(a_{e_1}a_{e_2}a_{e_4}a_{e_6}) = -d_4(a_{e_2}a_{e_3}a_{e_4}a_{e_6}) = -a_{e_1}a_{e_2}a_{e_3}a_{e_4}a_{e_6}$,
$d_4(a_{e_1}a_{e_2}a_{e_5}a_{e_6}) = -d_4(a_{e_2}a_{e_3}a_{e_5}a_{e_6}) = -a_{e_1}a_{e_2}a_{e_3}a_{e_5}a_{e_6}$,

$d_4(a_{e_1}a_{e_4}a_{e_5}a_{e_6}) = -d_4(a_{e_3}a_{e_4}a_{e_5}a_{e_6}) = -a_{e_1}a_{e_3}a_{e_4}a_{e_5}a_{e_6}$, and
$d_4(a_{e_2}a_{e_4}a_{e_5}a_{e_6}) = d_4(a_{e_1}a_{e_2}a_{e_4}a_{e_5}) - d_4(a_{e_1}a_{e_2}a_{e_4}a_{e_6}) + d_4(a_{e_1}a_{e_2}a_{e_5}a_{e_6}) - d_4(a_{e_1}a_{e_4}a_{e_5}a_{e_6})$.

a.3. If $t = 4$, then:
$d_4(a_{e_1}a_{e_2}a_{e_3}a_{e_4}) = d_4(a_{e_1}a_{e_2}a_{e_4}a_{e_5}) = d_4(a_{e_1}a_{e_2}a_{e_4}a_{e_6}) = d_4(a_{e_1}a_{e_3}a_{e_4}a_{e_5}) =$
$d_4(a_{e_1}a_{e_3}a_{e_4}a_{e_6}) = d_4(a_{e_1}a_{e_4}a_{e_5}a_{e_6}) = 0_{\boldsymbol{A_5}(G)}$,
$d_4(a_{e_1}a_{e_2}a_{e_3}a_{e_5}) = d_4(a_{e_2}a_{e_3}a_{e_4}a_{e_5}) = a_{e_1}a_{e_2}a_{e_3}a_{e_4}a_{e_5}$,
$d_4(a_{e_1}a_{e_2}a_{e_3}a_{e_6}) = d_4(a_{e_2}a_{e_3}a_{e_4}a_{e_6}) = a_{e_1}a_{e_2}a_{e_3}a_{e_4}a_{e_6}$,
$d_4(a_{e_1}a_{e_3}a_{e_5}a_{e_6}) = d_4(a_{e_3}a_{e_4}a_{e_5}a_{e_6}) = a_{e_1}a_{e_3}a_{e_4}a_{e_5}a_{e_6}$,
$d_4(a_{e_1}a_{e_2}a_{e_5}a_{e_6}) = d_4(a_{e_2}a_{e_4}a_{e_5}a_{e_6}) = a_{e_1}a_{e_2}a_{e_4}a_{e_5}a_{e_6}$
and $d_4(a_{e_2}a_{e_3}a_{e_5}a_{e_6}) = d_4(a_{e_1}a_{e_2}a_{e_3}a_{e_5}) - d_4(a_{e_1}a_{e_2}a_{e_3}a_{e_6}) + d_4(a_{e_1}a_{e_2}a_{e_5}a_{e_6}) -$
$d_4(a_{e_1}a_{e_3}a_{e_5}a_{e_6})$.

a.4. If $t = 5$, then:
$d_4(a_{e_1}a_{e_2}a_{e_3}a_{e_5}) = d_4(a_{e_1}a_{e_2}a_{e_4}a_{e_5}) = d_4(a_{e_1}a_{e_2}a_{e_5}a_{e_6}) = d_4(a_{e_1}a_{e_3}a_{e_4}a_{e_5}) =$
$d_4(a_{e_1}a_{e_3}a_{e_5}a_{e_6}) = d_4(a_{e_1}a_{e_4}a_{e_5}a_{e_6}) = 0_{\boldsymbol{A_5}(G)}$,
$d_4(a_{e_1}a_{e_2}a_{e_3}a_{e_4}) = -d_4(a_{e_2}a_{e_3}a_{e_4}a_{e_5}) = -a_{e_1}a_{e_2}a_{e_3}a_{e_4}a_{e_5}$,
$d_4(a_{e_1}a_{e_2}a_{e_3}a_{e_6}) = d_4(a_{e_2}a_{e_3}a_{e_5}a_{e_6}) = a_{e_1}a_{e_2}a_{e_3}a_{e_5}a_{e_6}$,
$d_4(a_{e_1}a_{e_2}a_{e_4}a_{e_6}) = d_4(a_{e_2}a_{e_4}a_{e_5}a_{e_6}) = a_{e_1}a_{e_2}a_{e_4}a_{e_5}a_{e_6}$,
$d_4(a_{e_1}a_{e_3}a_{e_4}a_{e_6}) = d_4(a_{e_3}a_{e_4}a_{e_5}a_{e_6}) = a_{e_1}a_{e_3}a_{e_4}a_{e_5}a_{e_6}$, and
$d_4(a_{e_2}a_{e_3}a_{e_4}a_{e_6}) = d_4(a_{e_1}a_{e_2}a_{e_3}a_{e_4}) - d_4(a_{e_1}a_{e_2}a_{e_4}a_{e_6}) + d_4(a_{e_1}a_{e_2}a_{e_3}a_{e_6}) - d_4(a_{e_1}a_{e_3}a_{e_4}a_{e_6})$.

a.5. $t = 6$, then:
$d_4(a_{e_1}a_{e_2}a_{e_3}a_{e_6}) = d_4(a_{e_1}a_{e_2}a_{e_4}a_{e_6}) = d_4(a_{e_1}a_{e_2}a_{e_5}a_{e_6}) = d_4(a_{e_1}a_{e_3}a_{e_4}a_{e_6}) =$
$d_4(a_{e_1}a_{e_3}a_{e_5}a_{e_6}) = d_4(a_{e_1}a_{e_4}a_{e_5}a_{e_6}) = 0_{\boldsymbol{A_5}(G)}$,
$d_4(a_{e_1}a_{e_2}a_{e_3}a_{e_4}) = -d_4(a_{e_2}a_{e_3}a_{e_4}a_{e_6}) = -a_{e_1}a_{e_2}a_{e_3}a_{e_4}a_{e_6}$,
$d_4(a_{e_1}a_{e_2}a_{e_3}a_{e_5}) = -d_4(a_{e_2}a_{e_3}a_{e_5}a_{e_6}) = -a_{e_1}a_{e_2}a_{e_3}a_{e_5}a_{e_6}$,
$d_4(a_{e_1}a_{e_2}a_{e_4}a_{e_5}) = -d_4(a_{e_2}a_{e_4}a_{e_5}a_{e_6}) = -a_{e_1}a_{e_2}a_{e_4}a_{e_5}a_{e_6}$,
$d_4(a_{e_1}a_{e_3}a_{e_4}a_{e_5}) = -d_4(a_{e_3}a_{e_4}a_{e_5}a_{e_6}) = -a_{e_1}a_{e_3}a_{e_4}a_{e_5}a_{e_6}$, and

$$d_4(a_{e_2}a_{e_3}a_{e_4}a_{e_5}) = d_4(a_{e_1}a_{e_2}a_{e_3}a_{e_4}) - d_4(a_{e_1}a_{e_2}a_{e_3}a_{e_5}) + d_4(a_{e_1}a_{e_2}a_{e_4}a_{e_5}) - d_4(a_{e_1}a_{e_3}a_{e_4}a_{e_5}).$$

Hence, for any $1 < k_i < k_j \leq \ell$ and $k_i, k_j \neq t$, $\mp a_{e_1}a_{e_t}a_{e_{k_i}}a_{e_{k_j}} \in \ker d_4$. But $\mp a_{e_1}a_{e_t}a_{e_k}a_{e_k} \in \operatorname{Im} d_3$, as shown in proposition (3.4.2). Consequently, $\ker d_4 = \operatorname{Im} d_3$. It is clear, $\dim(\operatorname{Im} d_4) = \binom{6-2}{4} + \binom{6-2}{3} - 1 = 4$ for this case.

b. If $m > 6$ and $c_6(G)$ be the number of chordless 6-cycles then for this case,
$\left|NBC^4_{\trianglelefteq_G}(M_G)\right| = \left|S^4_{\trianglelefteq_G}(\Pi^G)\right| = \binom{\ell}{4}$, $\left|BC^5_{\trianglelefteq_G}(M_G)\right| = c_6(G)$ and;
$\left|NBC^5_{\trianglelefteq_G}(M_G)\right| = \left|S^5_{\trianglelefteq_G}(\Pi^G)\right| - \left|BC^5_{\trianglelefteq_G}(M_G)\right| = \binom{\ell}{5} - c_6(G)$.
Suppose $S = \{e_{q_1}, e_{q_2}, e_{q_3}, e_{q_4}, e_{q_5}, e_{q_6}\}$ be a chordless 6-cycle such that $1 \leq q_1 < q_2 < q_3 < q_4 < q_5 < q_6 \leq \ell$. Hence, it's broken cycle is $C = \{e_{q_2}, e_{q_3}, e_{q_4}, e_{q_5}, e_{q_6}\} \in BC^4_{\trianglelefteq_G}(M_G)$. It is known that every monomial of a broken 6-circuit written as a combination of NBC-monomials as:

$$a_{e_{q_2}}a_{e_{q_3}}a_{e_{q_4}}a_{e_{q_5}}a_{q_6} = a_{e_{q_1}}a_{e_{q_3}}a_{e_{q_4}}a_{e_{q_5}}a_{q_6} - a_{e_{q_1}}a_{e_{q_2}}a_{e_{q_4}}a_{e_{q_5}}a_{q_6} + a_{e_{q_1}}a_{e_{q_2}}a_{e_{q_3}}a_{e_{q_5}}a_{q_6} - a_{e_{q_1}}a_{e_{q_2}}a_{e_{q_3}}a_{e_{q_4}}a_{q_6} + a_{e_{q_1}}a_{e_{q_2}}a_{e_{q_3}}a_{e_{q_4}}a_{q_5} \quad \dots (3.2.4.2)$$

Therefore, we have the following possible cases:
b.1. If S is one of $(c_6 - u_6)$ of chordless 6-cycles that do not simultaneously include e_1 and e_t, then:

b.1.1. If $e_1, e_t \notin S$, then formula (3.2.4.2) shows that $d_4\left(a_{e_1}a_{e_{q_i}}a_{e_{q_j}}a_{e_{q_k}}\right)$ and $d_3(a_{e_{q_i}}a_{e_{q_j}}a_{e_{q_k}}a_{e_{q_n}})$ are written as a combination of NBC-monomials, for $1 \leq i < j < k < n \leq 6$.

b.1.2. For the second case assume either ($e_1 \in S$ and $e_t \notin S$) or ($e_1 \notin S$ and $e_t \in S$). If ($e_1 \in S$ and $e_t \notin S$), then, $q_1 = 1$ and $q_2, q_3, q_4, q_5, q_6 \neq t$. Accordingly to formula (3.2.4.2), we get either $d_4\left(a_{e_1}a_{e_{q_i}}a_{e_{q_j}}a_{e_{q_k}}\right) = \mp d_4\left(a_{e_{q_i}}a_{e_{q_j}}a_{e_{q_k}}a_{e_t}\right) = \mp a_{e_1}a_{e_{q_i}}a_{e_{q_j}}a_{e_{q_k}}a_{e_t}$ is an NBC monomial or $d_4\left(a_{e_{q_i}}a_{e_{q_j}}a_{e_{q_k}}a_{e_{q_n}}\right) = a_{e_1}a_{e_{q_i}}a_{e_{q_j}}a_{e_{q_k}}a_{e_{q_n}} \mp a_{e_{q_i}}a_{e_{q_j}}a_{e_{q_k}}a_{e_{q_n}}a_{e_t}$, is written as a combination of an NBC monomials, for $1 < i < j < k < n \leq 6$. Therefore, the number of generators that combine the basis of $\operatorname{Im} d_4$ is 4, determined as: $\binom{6-2}{4} + \binom{6-2}{3} = 4$. Similarly, it's possible to conclude that four generators will be obtained to be included in $\operatorname{Im} d_4$'s basis, if ($e_1 \notin S$ and $e_t \in S$).

b.2. Assume S is one of the u_6 chordless 6-cycles, that including e_1 and e_t, for the purpose of seeing how one of the 2-NBC monomial images cannot be including in $\operatorname{Im} d_3$'s basis, as follows:

b.2.1. If $t = q_2$:

$$d_4\left(a_{e_{q_3}}a_{e_{q_4}}a_{e_{q_5}}a_{e_{q_6}}\right) = d_4\left(a_{e_1}a_{e_{q_3}}a_{e_{q_4}}a_{e_{q_5}}\right) - d_4\left(a_{e_1}a_{e_{q_3}}a_{e_{q_4}}a_{e_{q_6}}\right) + d_4\left(a_{e_1}a_{e_{q_3}}a_{e_{q_5}}a_{e_{q_6}}\right) - d_4\left(a_{e_1}a_{e_{q_4}}a_{e_{q_5}}a_{e_{q_6}}\right).$$

b.2.2. If $t = q_3$:

$$d_4\left(a_{e_{q_2}}a_{e_{q_4}}a_{e_{q_5}}a_{e_{q_6}}\right) = d_4\left(a_{e_1}a_{e_{q_2}}a_{e_{q_4}}a_{e_{q_5}}\right) - d_4\left(a_{e_1}a_{e_{q_2}}a_{e_{q_4}}a_{e_{q_6}}\right) + d_4\left(a_{e_1}a_{e_{q_2}}a_{e_{q_5}}a_{e_{q_6}}\right) - d_4\left(a_{e_1}a_{e_{q_4}}a_{e_{q_5}}a_{e_{q_6}}\right).$$

b.2.3. If $t = q_4$:

$$d_4\left(a_{e_{q_2}}a_{e_{q_3}}a_{e_{q_5}}a_{e_{q_6}}\right) = d_4\left(a_{e_1}a_{e_{q_2}}a_{e_{q_3}}a_{e_{q_5}}\right) - d_4\left(a_{e_1}a_{e_{q_2}}a_{e_{q_3}}a_{e_{q_6}}\right) + d_4\left(a_{e_1}a_{e_{q_2}}a_{e_{q_5}}a_{e_{q_6}}\right) - d_4\left(a_{e_1}a_{e_{q_3}}a_{e_{q_5}}a_{e_{q_6}}\right).$$

b.2.4. If $t = q_5$:

$$d_3\left(a_{e_{q_2}}a_{e_{q_3}}a_{e_{q_4}}a_{e_{q_6}}\right) = d_3\left(a_{e_1}a_{e_{q_2}}a_{e_{q_3}}a_{e_{q_4}}\right) - d_3\left(a_{e_1}a_{e_{q_2}}a_{e_{q_3}}a_{e_{q_6}}\right) + d_3\left(a_{e_1}a_{e_{q_2}}a_{e_{q_4}}a_{e_{q_6}}\right) - d_3\left(a_{e_1}a_{e_{q_3}}a_{e_{q_4}}a_{e_{q_6}}\right).$$

b.2.5. If $t = q_6$:

$$d_3\left(a_{e_{q_2}}a_{e_{q_3}}a_{e_{q_4}}a_{e_{q_5}}\right) = d_3\left(a_{e_1}a_{e_{q_2}}a_{e_{q_3}}a_{e_{q_4}}\right) - d_3\left(a_{e_1}a_{e_{q_2}}a_{e_{q_3}}a_{e_{q_5}}\right) + d_3\left(a_{e_1}a_{e_{q_2}}a_{e_{q_4}}a_{e_{q_5}}\right) - d_3\left(a_{e_1}a_{e_{q_3}}a_{e_{q_4}}a_{e_{q_5}}\right).$$

In this case, the generators joining the basis of $\operatorname{Im} d_3$, are exactly four elements calculated as: $\binom{6-2}{4} + \binom{6-2}{3} - 1 = 4$. For general case, the number of such generators is, $\binom{\ell-2}{4} + \binom{\ell-2}{3} - u_6(G)$.

3. If $c(G) = 5$, then,

$$\left|NBC^4_{\trianglelefteq_G}(M_G)\right| = \left|S^4_{\trianglelefteq_G}(\Pi^G)\right| - \left|BC^4_{\trianglelefteq_G}(M_G)\right| = \binom{\ell}{4} - c_5(G) \text{ and;}$$

$$\left|NBC^5_{\trianglelefteq_G}(M_G)\right| = \left|S^5_{\trianglelefteq_G}(\Pi^G)\right| - (\left|BC^5_{\trianglelefteq_G}(M_G)\right| + c_5(G) = \binom{\ell}{5} - (c_6(G) + c_5(G)).$$

where $c_5(G)$ and $c_6(G)$ be the numbers of chordless 5-cycles and chordless 6-cycles respectively. Here, we need to know wither the G is generic or not:

a. If G is generic, then $\boldsymbol{A_5}(G) = 0$. Thus, $\ker d_4 = \boldsymbol{A_4}(G)$, since $d_4: \boldsymbol{A_4}(G) \to 0$ is the zero homomorphism.

b. If G is not 5-generic (i.e. $c_5(G) > 1$), we will be partitioned $NBC^4_{\trianglelefteq_G}(M_G)$ by using the type of the chordless 5 cycles. Put $NBC^{4,0}_{\trianglelefteq_G}(M_G)$ to be the set of all NBC bases that not related to any chordless 5- cycles or related

to chordless 5 cycle that contain e_1 or (e_t as minimal edge). Put $NBC^{4,t}_{\trianglelefteq_G}(M_G)$ and $BC^{4,t}_{\trianglelefteq_G}(M_G)$ to be the set of all NBC bases and broken circuits that related to chordless 5 cycle that contains e_t as not minimal edge via $\trianglelefteq_G$, respectively, and let $v_5(G) = \left|BC^{4,t}_{\trianglelefteq_G}(M_G)\right|$. Put $NBC^{4,1,t}_{\trianglelefteq_G}(M_G)$ be the set of all NBC bases that related to chordless 5-cycles that contains each of e_1 and e_t. It is clear:
$NBC^{4}_{\trianglelefteq_G}(M_G) = NBC^{4,0}_{\trianglelefteq_G}(M_G) \cup NBC^{4,t}_{\trianglelefteq_G}(M_G) \cup NBC^{4,1,t}_{\trianglelefteq_G}(M_G)$.
Now, we will study the behavior of d_4 as follows:

b.1. If $c_6(G) = 0$, then $BC^{5}_{\trianglelefteq_G}(M_G) = \emptyset$ since every 5 edges either they are in a chordless 5-cycle or they cannot be broken circuit for a chordless 6-cycle. Thus, for this case $NBC^{5}_{\trianglelefteq_G}(M_G) = S^{5}_{\trianglelefteq_G}(\Pi^G) - C^{5}_{\trianglelefteq_G}(M_G)$, where $C^{5}_{\trianglelefteq_G}(M_G)$ is the set of all chordless 5-cycles. According to the formula (3.2.4.1):

b.1.1. For $1 < k_i < k_j \leq \ell$, $1 \leq i < j \leq 4$ and $k_i, k_j \neq t$, $\{e_1, e_{k_i}, e_{k_j}, e_t\} \in NBC^{4}_{\trianglelefteq_G}(M_G)$ and $d_4\left(\pm a_{e_1} a_{e_{k_1}} a_{e_{k_2}} a_{e_t}\right) = 0_{A_5(G)}$. These monomials contained in the basis of Imd_3, and their number is $(\ell - 2)$.

b.1.2 For $1 < k_i < k_j < k_n \leq \ell$, $1 \leq i < j < n \leq 4$ and $k_i, k_j, k_n \neq t$, if $\{e_1, e_{k_i}, e_{k_j}, e_{k_n}\} \in NBC^{4,1,t}_{\trianglelefteq_G}(M_G)$ then $d_4\left(a_{e_1} a_{e_{k_i}} a_{e_{k_j}} a_{e_{k_n}}\right) = 0_{A_5(G)}$. There are $u_5(G)$ such monomials, and Imd_3 does not include them.

b.1.3 If $\{e_{k_1}, e_{k_2}, e_{k_3}, e_{k_4}\} \in NBC^{4,0}_{\trianglelefteq_G}(M_G)$, such that $1 < k_1 < k_2 < k_3 < k_4 \leq \ell$ and $k_1, k_2, k_3, k_4 \neq t$, as a direct result to formula (2.2.4.1), each of $d_4\left(a_{e_1} a_{e_{k_1}} a_{e_{k_2}} a_{e_{k_3}}\right)$, $d_4\left(a_{e_1} a_{e_{k_1}} a_{e_{k_2}} a_{e_{k_4}}\right)$, $d_4\left(a_{e_1} a_{e_{k_2}} a_{e_{k_3}} a_{e_{k_4}}\right)$, $d_4\left(a_{e_1} a_{e_{k_2}} a_{e_{k_3}} a_{e_{k_4}}\right)$ $d_4\left(a_{e_{k_1}} a_{e_{k_2}} a_{e_{k_3}} a_{e_{k_4}}\right)$ are written as a combination of NBC monomials.

b.1.4 If $\{e_{k_1}, e_{k_2}, e_{k_3}, e_{k_4}\} \in NBC^{4,t}_{\trianglelefteq_G}(M_G)$, such that $1 < k_1 < k_2 < k_3 < k_4 \leq \ell$ and $k_1, k_2, k_3, k_4 \neq t$, hence $d_3\left(a_{e_{k_1}} a_{e_{k_2}} a_{e_{k_3}} a_{e_{k_4}}\right) = a_{e_1} a_{e_{k_1}} a_{e_{k_2}} a_{e_{k_3}} a_{e_{k_4}}$ is an NBC monomials and,

$$d_4\left(a_{e_1} a_{e_{k_2}} a_{e_{k_3}} a_{e_{k_4}}\right) = d_4\left(a_{e_1} a_{e_{k_1}} a_{e_{k_3}} a_{e_{k_4}}\right) - d_4\left(a_{e_1} a_{e_{k_1}} a_{e_{k_2}} a_{e_{k_4}}\right) + d_4\left(a_{e_1} a_{e_{k_1}} a_{e_{k_2}} a_{e_{k_3}}\right) - d_4\left(a_{e_{k_1}} a_{e_{k_1}} a_{e_{k_2}} a_{e_{k_3}}\right).$$

The number of such monomials is $\left|NBC^{4,t}_{\trianglelefteq_G}(M_G)\right| = v_5$.

b.1.5 If $B = \{e_1, e_{k_i}, e_{k_j}, e_{k_n}\} \in NBC^{4,1,t}_{\trianglelefteq_G}(M_G)$, $1 < k_i < k_j < k_n \leq \ell$ and $k_i, k_j, k_n \neq t$, then for all $1 < k \leq \ell$ with $k \neq k_i, k_j, k_n, t$ and $\{e_{k_1}, e_{k_2}, e_{k_3}, e_k\}$ not a broken 4-crcuits;

$$d_4\left(a_{e_{k_1}} a_{e_{k_2}} a_{e_{k_3}} a_{e_k}\right) = \pm d_4\left(a_{e_1} a_{e_{k_1}} a_{e_{k_2}} a_{e_k}\right) \mp d_4\left(a_{e_1} a_{e_{k_1}} a_{e_{k_3}} a_{e_k}\right) \mp d_4\left(a_{e_1} a_{e_{k_2}} a_{e_{k_3}} a_{e_k}\right).$$

The number of such monomials is $(u_5(\ell - 5) - b_5)$, where b_5 is the number of broken circuits that contains $B - \{e_1\}$, and $B \in NBC^{4,1,t}_{\trianglelefteq_G}(M_G)$.

For this case, $\dim(kerd_4) = \dim(Imd_3) + u_5(G)$ and:

$$\dim(Imd_4) = \left(\binom{\ell - 2}{4} - (u_5(\ell - 5) - b_5) - (c_5 - u_5)\right) + \left(\binom{\ell - 2}{3} - (u_5 + v_5)\right).$$

b.2. The discussion of the case that $c_6(G) \neq 0$ is a mixture among all the cases that we analyzed in item 2 ($c(G) = 6$) and item b.1 ($c(G) = 5$ and $c_6(G) = 0$) above. Thus, $\dim(kerd_4) = \dim(Imd_3) + u_5$and:

$$\dim(Imd_4) = \left(\binom{\ell-2}{4} - (u_5(\ell-5) - b_5) - (c_5 - u_5)\right) + \left(\binom{\ell-2}{3} - (u_5 + v_5)\right) - u_6.$$

■

Theorem 2.2.5:

If $a = a_{e_1} - a_{e_t}$, for $2 \leq t \leq \ell$, then:

1. The Orlik-Solomon algebra $\boldsymbol{A}_*(G)$ has vanished second cohomological group, i.e. $H^2(\boldsymbol{A}_*(G); a) = 0$.
2. The structure of $H^3(\boldsymbol{A}_*(G); a)$ depending on the value of $c(G)$, as follows:
   i. If G is tree or $c(G) \geq 6$, then $\boldsymbol{A}_*(G)$ has vanished $H^3(\boldsymbol{A}_*(G); a)$.
   ii. If $c(G) = 5$, then the Orlik-Solomon algebra $\boldsymbol{A}_*(G)$ has non vanished second cohomological group with dim $(H^3(\boldsymbol{A}_*(G); a)) = u_5(G)$, where $u_5(G)$ be the number of chordless 5-cycles that includes e_1 and e_t.

Proof:

This is a direct result of proposition (3.2.3) and theorem (3.2.4).

4. Illustrations:

We will illustrate our results as follows:

4.1 Trees:

All the trees in figure (4.1) are hypersolvable with 15 vertices, 14 edges and each one of them has hypersolvable partition with exponent vector $(1,1,\ldots,1)$. According to theorem (3.2.4), they have vanished second and third cohomology of the Orlik-Solomon algebra:

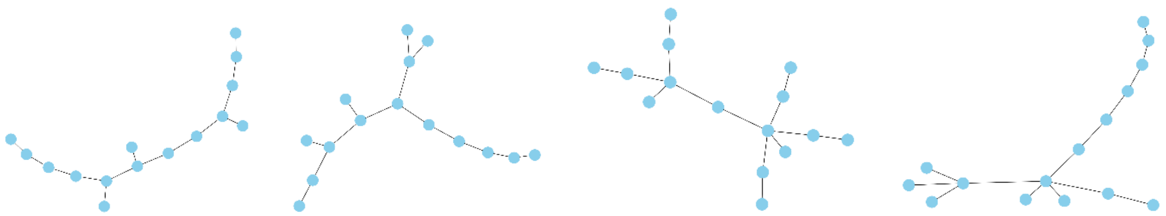

Figure (4.1): Trees with 15 vertices and 14 edges.

4.2 generic Graph:

In figure (4.2) G is a hypersolvable 5-generic Graph with 5 vertices, 5 edges and has a hypersolvable partitionwith exponent vector $(1,1,\ldots,1)$, where $u_5(G) = 1$ According to theorem (3.2.4), it has vanished second cohomology of the Orlik-Solomon algebra and dim $(H^3(\boldsymbol{A}_*(G); a_{e_1} - a_{e_t})) = 1$.

v_1

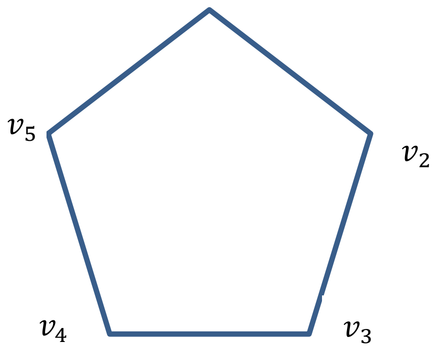

Figure (4.2): 5-generic Graph with 5 vertices and 5 edge

5. Conclusion

This study is intended to investigate the first non-vanishing cohomology $H^*(\boldsymbol{A}_*(G); a)$, of the Orlik-Solomon algebra $\boldsymbol{A}_*(G)$ for a free-tringles graph G such that $a = a_{e_1} - a_{e_t}$, $2 \le t \le \ell$, specifically that has chordless 5-cycles and conclusion was driven based on the results:

1. The graph G has the second vanishing cohomology $H^2(\boldsymbol{A}_*(G); a)$.
2. If a graph G with no chordless 5-cycles, then $H^3(\boldsymbol{A}_*(G); a)$ vanished.
3. If G has no chordless 5-cycles that includes e_1 and e_t, then $H^3(\boldsymbol{A}_*(G); a)$ vanished.
4. If the number of chordless 5-cycles that includes e_1 and e_t, $u_5(G) > 0$, then $\dim\left(H^3(\boldsymbol{A}_*(G); a)\right) = u_5(G)$.

Our strategy generally reconstructs the Orlik-Solomon algebra by combining techniques from graph theory and arrangement theory. These findings not only improve our understanding of the underlying algebraic and combinatorial structures but also open opportunities for investigating the potential uses of these concepts in a variety of networks and associated mathematical fields.

References

[1] **K. P. Bogart,** " Combinatorics through guided discovery, " pp. 1-21, Kenneth P, Bogart, 2004, doi:10.1073/ panas.1013213108.

[2] **M. J. Erickson,** " Introduction to combinatorics, " John Wiley & Sons, 2013, doi:10.1002/9781118032640.

[3] **W. T. Tutte**, " Matroids and Graphs,", Trans Am Math Soc, vol. 90, no. 3, pp.527,1959, doi:10.2307 /1993185.

[4] **J. H. C. Whitehead,** "Simplicial spaces," nuclei and m group, Proceedings of the London mathematical society, vol. 2, no.1, pp. 243-327, 1939, doi:org/10.1090/S0002-9939-1979-0524319-8.

[5] **R. P. Stanly**, "An Introduction to Hyperplane Arrangements," IAS/Park City Mathematics Series, vol. 14,2004, doi:10.1090/pcms/013/08.

[6] **P. Orlik, H. Terao**, "Arrangements of hyperplanes,", vol. 300. Springer Science \& Business Media, 2013, doi.org/10.1007/978-3-662-02772-1.

[7] **P. Orlik and L. Solomon,** " Combinatories and Topology of Complements of Hyperplanes,", lnventiones math, vol. 56, pp. 167–189, 1980, doi.org/10.1007/BF01392549.

[8] **S. Yuzvinsky**, "Cohomology of the Brieskorn-Orlik-Solomon algebras,", 2006, doi:10.1006/aama.2001.0779.

[9] **K. J. Pearson,** "Cohomology of Orlik-Solomon algebras for quadratic arrangements,", Lect. Mat., vol. 22, no. 2, pp. 103–134, 2001.

[10] **H. Whitney**, " On the abstract properties of linear dependence,", Hassler Whitney Collected Papers, 147-171,1992, doi:10.1007/978-0-8176-4842-8-5.

[11] **H. Poincaré**, " Analysis situs and its five supplements", Papers on Topology, 22-28, 2009, doi.org/10.1007/BF01001956.

[12] **G. C. Rota**, " On the foundations of combinatorial theory",: I. Theory of Möbius functions. In Classic Papers in Combinatorics (pp. 332-360), Boston, MA: Birkhäuser Boston. 1964, doi.org/10.1007/978-3-031-41420-6_74.

[13] **A. Dimca**, "Hyperplane arrangements,", Springer International Publishing, 2017, doi.org/10.1007/978-3-319-56221-6.

[14] **C. Chun, I. Moffatt, S. D. Noble and R. Rueckriemen,** " Matroids, delta-matroids and embedded graphs,", Journal of Combinatorial Theory, Series A, vol. 167, pp. 7-59, 2019, doi.org/10.1016/j.jcta. 2019.02.023

[15] **S. Papadima and A. I. Suciu**, " Higher Homotopy Groups of Complements of Complex Hyperplane Arrangements,", Adv Math (N Y), vol. 165, no. 1, pp. 71–100, Jan. 2002, doi: 10.1006/AIMA.2001.2023.

[16] **A. G. Fadhil. and H. M. Ali,** "On the hypersolvable graphic arrangements,", A M. Sc. thesis submitted to College of Science/ University of Basrah, 2012.

[17] **H. M. Ali,** " On the CW Complex of the Complement of a Hypersolvable Graphic Arrangement,", Mathematical Theory and Modeling, vol.6, no.5, pp. 112-131, 2016.