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Study of the Geometric Structure of General Rotation Matrices Using the Fubini–Study Metric

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Abstract:

The research aims to study how to measure the shape of the abstract mathematical tools used in the analysis of important quantum rotation gates and to understand them better. The components of the matrix representing rotation gates $U(\theta, \emptyset, \lambda)$ on the initial qubit state $|0\rangle$, $|1\rangle$ were analyzed using the Fubini–Study Metric (dS^2). The results showed that they are related to the pinching constant 1/4, which is the condition with the ratio gives a sphere in topological considerations for a complete Riemannian manifold. The study also proved that the metric includes only matrices with non-zero elements (superposition gates).

Keywords: Qubit state, Rotation gates, Fubini–Study metric, Matrix analysis

1-Introduction

Through the known algebra, the elements of a vector space can be expressed $X_i = (x_i^1, \dots, x_i^n)$ in $\mathfrak{R}^n$ and X_j in space $\mathfrak{R}^m$, and transformations between them $X_i \in \mathfrak{R}^n \mapsto X_j \in \mathfrak{R}^m : T$ [1]. one of the most important applications of these transformations is Riemannian geometry, which depends on quantizing and not just operations on coordinates [2]. The analysis to be studied is in the field of booming because of its wonderful results in linking the concepts and methods of Riemannian differential spaces with quantum states, such as analyzing the effect of the geometric structure of rotation matrices on qubit states, which can represent an electron or a photon, etc [3]. especially tensors $(X_i \otimes X_j \in \mathfrak{R}^{\otimes(m+n)})$, which play a major role in descry. This research aims to analyze the components of a particular operator $\left(\hat{U}|\psi\rangle = g_j^i \psi^i\right)$ to study how it senses the initial states of a binary quantum system {0,1} through certain parameters using dS^2 a differential form, considering that the states are transformable [5],[6]. The analysis of quantum states using random matrix theory has wide applications in quantum computing, such as the analysis of quantum entanglement produced in quantum circuits in which Rotation gates have been used (the geometric estimator is the Fubini-Study metric) [7],[8]. The analysis of quantum gate matrices has been addressed in many ways, such as bit-flip and phase - flip gates, using permutation $(\prod_i)$ tools [9]. we will analyze Rotation gates $\left(R_x(\theta), R_y(\theta), R_z(\theta)\right)$ which have a wide application in quantum algorithms, using the Fubini–Study metric, which gives a clear understanding and methodology of how gates affect the initial qubit states $\{\,|0\rangle, |1\rangle\,\}$ [10].

2- Theoretical concepts

In Hilbert space $(\mathcal{H})$, the complex vector space: $\psi_t \Rightarrow \alpha\, {\psi_0}^n + \beta\, {\psi_1}^n \in \mathcal{H}$, $\forall\, n \in [0,1]$, where ψ_t is a single qubit state and α, β are complex numbers whose norm is equal to 1 [11] .

$${\psi_0}^n = \binom{1}{0} \quad , \quad {\psi_1}^n = \binom{0}{1} \tag{1}$$

The qubit in pure state : $\psi_t = e^{i\lambda}cos(\theta/2)\,{\psi_0}^n + e^{i\lambda+i\emptyset}sin(\theta/2)\,{\psi_1}^n$, this is the general state of the qubit in quantum superposition, where λ is deleted due to the lack of influence on the Bloch sphere, and the remaining values range $0 \leq \theta \leq \pi$, $0 \leq \emptyset \leq 2\pi$, $\theta, \emptyset$ and λ are real numbers [12]. Since we will be dealing with a single qubit $|q\rangle_i$, we do not need a tensor product ($\otimes$), which saves more space in the Hilbert space $\{\mathcal{H}_t = \mathcal{H}_1 \otimes \mathcal{H}_2\}$ when dealing with n-qubit ($|q_0\rangle \otimes |q_1\rangle \otimes |q_2\rangle \otimes |q_3\rangle$) [13].

The Conjugate transpose (ψ_t);

$$(\psi_t)^\dagger = e^{-i\lambda}cos^*(\theta/2)\,({\psi_0}^n)^T + e^{-i\lambda-i\emptyset}sin^*(\theta/2)({\psi_1}^n)^T$$

By using chain rule;

$$\frac{d\psi_t}{d\tau} = \partial_x\psi_t\frac{dx}{d\tau} + \partial_y\psi_t\frac{dy}{d\tau} \Rightarrow d\psi_t = \partial_x\psi_t dx + \partial_y\psi_t dy$$

Inner product [14];

$$(d\psi_t)^\dagger d\psi_t = \sum_{j=1}^{n}(d\psi_t)^\dagger{}_j\,(d\psi_t)_j \in \mathcal{H} = \Re^1 \tag{2}$$

Absolute square of the inner product between (ψ_t) and $(d\psi_t)$;

$$\left|(\psi_t)^\dagger\, d\psi_t\right|^2 = \left|\sum_{j=1}^{n}(\psi_t)^\dagger{}_j\,(d\psi_t)_j \cdot \sum_{j=1}^{n}(\psi_t)^\dagger{}_j\,(d\psi_t)_j\right| \tag{3}$$

The scalar line element from Fubini–Study metric [15]:

$$d\mathcal{S}^2 = (d\psi_t)^\dagger d\psi_t - \left|\sum_{j=1}^{n}(\psi_t)^\dagger{}_j\,(d\psi_t)_j \cdot \sum_{j=1}^{n}(\psi_t)^\dagger{}_j\,(d\psi_t)_j\right| \tag{4}$$

We also use the general formula for a global gate with 3-parameters [16]:

$$U(\theta,\emptyset,\lambda) = \begin{pmatrix} \cos(\theta/2) & -e^{i\lambda}sin(\theta/2) \\ e^{i\emptyset}sin(\theta/2) & e^{i\lambda+i\emptyset}cos(\theta/2) \end{pmatrix} \tag{5}$$

Similar to the Similarity Metrics methodology between two operators, we can use the results of mathematical structures on qubit states to better understand their digital nature and open the door to quantum algebra and its applications [17]. The matrices in question are Hamiltonian unitary quantum gates, which are used to give a phase shift on the Bloch sphere axes representing the qubit states on them. As in Eq.(5), we will also perform an analysis on the rotation gates $\{R_x(\theta), R_y(\theta), R_z(\theta)\}$ to study them using the Fubini–Study metric [18].

$$R_x(\theta) = \begin{pmatrix} \cos(\theta/2) & -i\,sin(\theta/2) \\ -i\,sin(\theta/2) & \cos(\theta/2) \end{pmatrix}, R_y(\theta) = \begin{pmatrix} \cos(\theta/2) & -\,sin(\theta/2) \\ sin(\theta/2) & \cos(\theta/2) \end{pmatrix}, R_z(\theta) = \begin{pmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{pmatrix} \tag{6}$$

The quantum superposition of qubit states depends on the probability associated with the composite states of the final state, and what makes them collapse, remain, or change the value of the probability % of the superposition state is determined by the quantum gates, which are matrices that we need to develop [19]. Since the phenomenon of quantum entanglement cannot be created without matrices representing quantum superposition gates, such as the Hadamard gate and rotation gates [20].

3- Results and Discussion

From Eq.(5), and for all matrices of gate rotations, geometrically, the unitary linear transformations in the quantum state space are rotations of the vector space [21]. We will calculate the effect of the above gates using the Fubini–Study metric for qubit states ${\psi_0}^n$ and ${\psi_1}^n$. ($\Phi_{0,1}^{i_1}$) it is the output state resulting from the matrix (operator U) -vector (qubit state) product [22]. For mathematical calculations, we must know the property : $U^{\dagger}U = {\psi_0}^n({\psi_0}^n)^{\dagger} + {\psi_1}^n({\psi_1}^n)^{\dagger} =$ Identity matrix, and as follows:

Unitary state transformation: $\Phi_{0,1}^{i_1} = U(\theta,\emptyset,\lambda)\, {\psi_1}^n = -e^{i\lambda} sin\left(\frac{\theta}{2}\right) {\psi_0}^n + e^{i\lambda+i\emptyset} cos\left(\frac{\theta}{2}\right) {\psi_1}^n$

The chain rule:
$$d\Phi_{0,1}^{i_1} = -\frac{1}{2} e^{i\lambda} cos\left(\frac{\theta}{2}\right) d\theta\, {\psi_0}^n - ie^{i\lambda} sin\left(\frac{\theta}{2}\right) d\lambda\, {\psi_0}^n - \frac{1}{2} e^{i\lambda+i\emptyset} sin\left(\frac{\theta}{2}\right) d\theta\, {\psi_1}^n$$
$$+ ie^{i\lambda+i\emptyset} cos\left(\frac{\theta}{2}\right) d\lambda\, {\psi_1}^n + ie^{i\lambda+i\emptyset} cos\left(\frac{\theta}{2}\right) d\emptyset\, {\psi_1}^n$$

Components of the Fubini–Study metric :

$$\left(d\Phi_{0,1}^{i_1}\right)^{\dagger} d\Phi_{0,1}^{i_1} = \frac{1}{4}\, sin^2\left(\frac{\theta}{2}\right) d\theta^2 + cos^2\left(\frac{\theta}{2}\right)(d\emptyset + d\lambda)^2, \quad \left|\left(\Phi_{0,1}^{i_1}\right)^{\dagger} d\Phi_{0,1}^{i_1}\right|^2 = d\lambda^2 + cos^4\left(\frac{\theta}{2}\right) d\emptyset^2$$

Using Eq.(4) , this is the matrix result $U(\theta,\emptyset,\lambda)$ for qubit state $|1\rangle = {\psi_1}^n$.

$$d{\mathcal{S}_{i_1}}^2 = \frac{1}{4}\, sin^2\left(\frac{\theta}{2}\right) d\theta^2 + \left\{cos^2\left(\frac{\theta}{2}\right) - cos^4\left(\frac{\theta}{2}\right)\right\} d\emptyset^2 + \left\{cos^2\left(\frac{\theta}{2}\right) - 1\right\} d\lambda^2 + 2cos^2\left(\frac{\theta}{2}\right) d\lambda d\emptyset$$

$$d{\mathcal{S}_{i_1}}^2 = \frac{1}{4}\, sin^2\left(\frac{\theta}{2}\right) d\theta^2 + cos^2\left(\frac{\theta}{2}\right) sin^2\left(\frac{\theta}{2}\right) d\emptyset^2 - sin^2\left(\frac{\theta}{2}\right) d\lambda^2 + 2cos^2\left(\frac{\theta}{2}\right) d\lambda d\emptyset$$

$$d{\mathcal{S}_{i_1}}^2 = \frac{1}{4}\, sin^2\left(\frac{\theta}{2}\right) d\theta^2 + \frac{1}{4}\, sin^2(\theta) d\emptyset^2 - sin^2\left(\frac{\theta}{2}\right) d\lambda^2 + 2cos^2\left(\frac{\theta}{2}\right) d\lambda d\emptyset$$

$$d{\mathcal{S}_{i_1}}^2 = \frac{1}{4}\left\{sin^2\left(\frac{\theta}{2}\right)(d\theta^2 - 4d\lambda^2) + sin^2(\theta) d\emptyset^2\right\} + 2cos^2\left(\frac{\theta}{2}\right) d\lambda d\emptyset \tag{7}$$

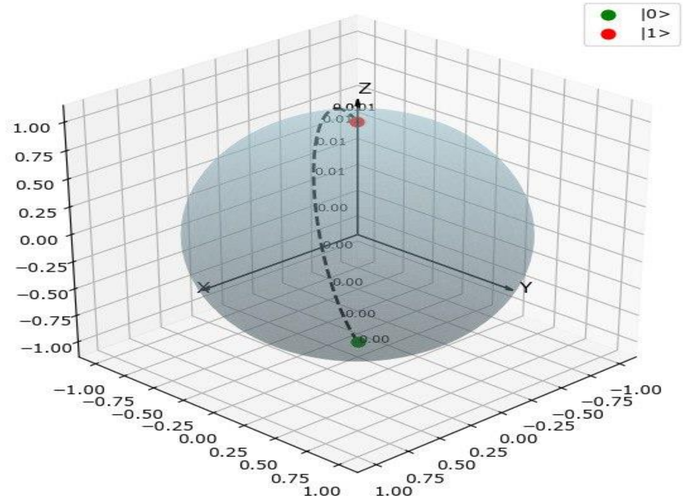

Fig1: Trajectory of the qubit state ${\psi_1}^n$ for the matrix $U(\theta,\emptyset,\lambda)$ with metric $d{\mathcal{S}_{i_1}}^2$ on the Bloch sphere.

In the same way we get the result for qubit state $|0\rangle = {\psi_0}^n$

$$\Phi_{0,1}^{i_2} = U(\theta,\emptyset,\lambda)\,\psi_0{}^{n} = cos\left(\frac{\theta}{2}\right)\psi_0{}^{n} + e^{i\emptyset}sin^2\left(\frac{\theta}{2}\right)\psi_1{}^{n}$$

$$\left(d\Phi_{0,1}^{i_2}\right)^{\dagger} d\Phi_{0,1}^{i_2} = \frac{1}{4}\,d\theta^2 + sin^2\left(\frac{\theta}{2}\right)d\emptyset^2 \;\;,\;\; \left|\left(\Phi_{0,1}^{i_2}\right)^{\dagger}\, d\Phi_{0,1}^{i_2}\right|^2 = sin^4\left(\frac{\theta}{2}\right)d\emptyset^2$$

$$d\mathcal{S}_{i_2}{}^2 = \frac{1}{4}\,d\theta^2 + sin^2\left(\frac{\theta}{2}\right)d\emptyset^2 - sin^4\left(\frac{\theta}{2}\right)d\emptyset^2 = \frac{1}{4}\,d\theta^2 + \left\{sin^2\left(\frac{\theta}{2}\right)(1 - sin^2\left(\frac{\theta}{2}\right))\right\}d\emptyset^2$$

$$d\mathcal{S}_{i_2}{}^2 = \frac{1}{4}\,d\theta^2 + sin^2\left(\frac{\theta}{2}\right)cos^2\left(\frac{\theta}{2}\right)d\emptyset^2$$

$$d\mathcal{S}_{i_2}{}^2 = \frac{1}{4}d\theta^2 + \frac{1}{4}sin^2(\theta)d\emptyset^2 \tag{8}$$

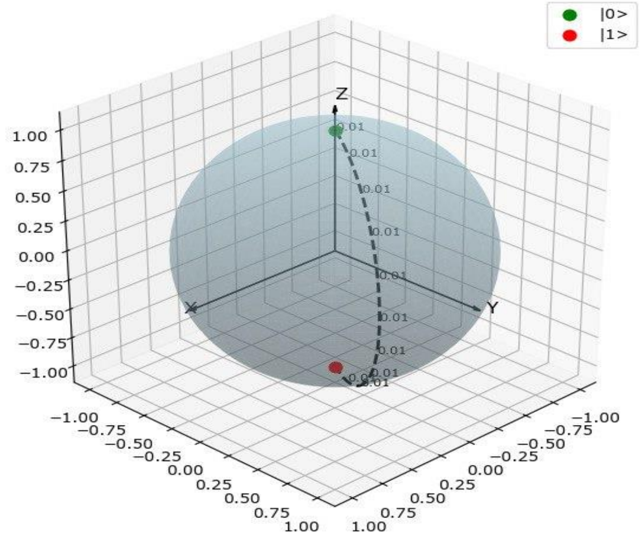

Fig2: Trajectory of the qubit state $\psi_1{}^n$ for the matrix $U(\theta,\emptyset,\lambda)$ with metric $d\mathcal{S}_{i_2}{}^2$ on the Bloch sphere.

From Eq.(7) and Eq. (8), we notice the disappearance of the terms$\{d\lambda^2, d\lambda d\emptyset\}$ and this shows the results of changing the inner product on space and in geometry and it is similar to the contraction of the tensor [23], where the difference of the transformation between $(\psi_0{}^n)$ and $(\psi_1{}^n)$ is like a projection [24].

To summarize, we will give the results of the Rotations gates $[R_{x,y,z}(\theta)]$, which will be similar to the qubit states $|0\rangle$ and $|1\rangle$ for the Line element Fubini–Study metric :

- $R_x(\theta)$ rotation around the X-axis on Bloch sphere : $\psi_0{}^n \neq \psi_1{}^n \rightarrow \psi_{0,1}{}^n$

$$\left(\Phi_{0,1}^{i_3}\right)_0 = R_x(\theta)\,\psi_0{}^n \Rightarrow \left(d\Phi_{0,1}^{i_3}\right)_0 = \left\{\frac{-sin\left(\frac{\theta}{2}\right)\,\psi_0{}^n}{2} - \frac{i\,cos\left(\frac{\theta}{2}\right)\psi_1{}^n}{2}\right\}d\theta$$

$$\left(\Phi_{0,1}^{i_3}\right)_1 = R_x(\theta)\,\psi_1{}^n \Rightarrow \left(d\Phi_{0,1}^{i_3}\right)_1 = \left\{\frac{-icos\left(\frac{\theta}{2}\right)\,\psi_0{}^n}{2} - \frac{sin\left(\frac{\theta}{2}\right)\psi_1{}^n}{2}\right\}d\theta$$

- $R_y(\theta)$ rotation around the Y-axis on Bloch sphere:

$$\left(\Phi_{0,1}^{i_4}\right)_0 = R_y(\theta)\,\psi_0{}^n \Rightarrow \left(d\Phi_{0,1}^{i_4}\right)_0 = \left\{\frac{-sin\left(\frac{\theta}{2}\right)\psi_0{}^n}{2} + \frac{cos\left(\frac{\theta}{2}\right)\psi_1{}^n}{2}\right\} d\theta$$

$$\left(\Phi_{0,1}^{i_4}\right)_1 = R_y(\theta)\,\psi_1{}^n \Rightarrow \left(d\Phi_{0,1}^{i_4}\right)_1 = \left\{\frac{-cos\left(\frac{\theta}{2}\right)\psi_0{}^n}{2} - \frac{sin\left(\frac{\theta}{2}\right)\psi_1{}^n}{2}\right\} d\theta$$

$\Phi_{0,1}^{i_3}$ and $\Phi_{0,1}^{i_4}$ will have the same values because there is no parameter other than(θ).

$$\left(d\Phi_{0,1}^{i_3}\right)_{0,1}^{\dagger}\left(d\Phi_{0,1}^{i_3}\right)_{0,1} = \left(d\Phi_{0,1}^{i_4}\right)_{0,1}^{\dagger}\left(d\Phi_{0,1}^{i_4}\right)_{0,1} = \frac{1}{4}d\theta^2$$

$$\left|\left(\Phi_{0,1}^{i_3}\right)^{\dagger}{}_{0,1}\left(d\Phi_{0,1}^{i_3}\right)_{0,1}\right|^2 = \left|\left(\Phi_{0,1}^{i_4}\right)^{\dagger}{}_{0,1}\left(d\Phi_{0,1}^{i_4}\right)_{0,1}\right|^2 = \frac{1}{4}sin^2(\theta)d\theta^2$$

$$d\mathcal{S}_{i_3}{}^2 = d\mathcal{S}_{i_4}{}^2 = \frac{1}{4}d\theta^2 - \frac{1}{4}sin^2(\theta)d\theta^2 = \frac{1}{4}cos^2(\theta)d\theta^2 \tag{9}$$

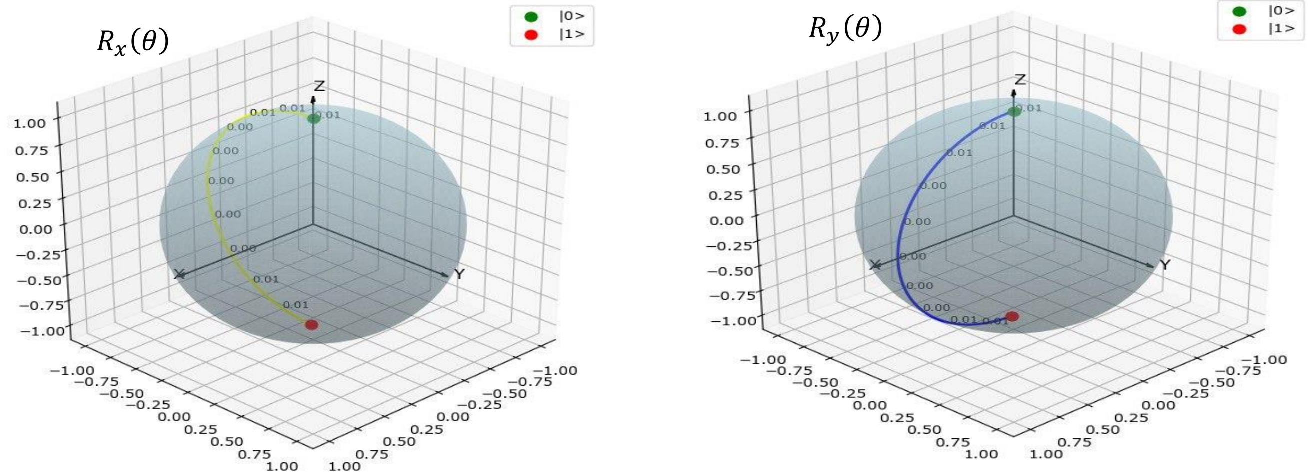

Fig3: Trajectory of the qubit state $\psi_0{}^n$ and $\psi_1{}^n$ for the matrix $R_x(\theta)$ with metric $d\mathcal{S}_{i_3}{}^2$ and the matrix $R_y(\theta)$ with metric $d\mathcal{S}_{i_4}{}^2$on the Bloch sphere.

Fig1 and Fig2 show that the Trajectory gives a clear rotation on the sphere and that the scale indicates that the state of the qubit is stable in the Trajectory at (0.00) and moves slightly at (0.01), and the same applies to Fig3.

- $R_z(\theta)$ rotation around the Z-axis on Bloch sphere:

Eq. (2) and Eq.(3) are equal, which includes the qubit states $|0\rangle$ and $|1\rangle$.

$$d\mathcal{S}_{i_5}{}^2 = 0 \begin{cases} \Phi_0^{i_5} = e^{-i\frac{\theta}{2}}\psi_0{}^n \\ \Phi_1^{i_5} = e^{-i\frac{\theta}{2}}\psi_1{}^n \end{cases} \tag{10}$$

This indicates that $d\mathcal{S}_{i_5}{}^2$ has an invariant effect on all components of the metric and has several interpretations in space geometry. this is because the $R_z(\theta)$ gate is not a gate that does not give a superposition state [25]. The results we obtained are related to the results of The Sphere Theorem in Riemannian geometry, as all equations except Eq.(10) contain $1/4$ in the term $d\theta^2$, which represents the pinching constant, where when the ratio $>\ 1/4$ gives a sphere in topological considerations for a complete Riemannian manifold [26]. We find many Riemannian applications of the 1/4-pinched, which corresponds to 1/2 of the actual radius of the sphere versus the coordinates on CP^n,While introducing the Fubini–Study metric to quantum projective spaces [27],[28],[29].

4- Conclusion

The study is only on gates that give a superposition of the qubit state because the Fubini–Study metric depends on matrices with non-zero elements $U = u_{ij} \in \Re^{n\times m}, u_{ij} \neq 0\ \forall i,j$,because the metric parameters in $dS^2 = u_{ij}dx^i dx^j$ vanish. The extension of the analysis results depends on the type of state input ψ_t, for example, the state of the qubit is $\{\alpha{\psi_0}^n + \beta{\psi_1}^n\}$, and with only one unitary transformation, there will be two qubit states $|q = 0\rangle_0$ and $|q = 1\rangle_1$ together, but with a probability% of $\alpha = \beta = 1/\sqrt{2}$ for each qubit $|q\rangle_{0,1}$. We notice that the effect of changing the qubit state in the general formula for a global gate will be at the limit $d\theta^2$ is $(\theta = \pi + 4k\pi)$, and the limit $d\emptyset^2$ is $\frac{1}{4}sin^2(\theta)/1 - sin^2(\theta/2)$. This is at certain angles, the difference may be small, but at other angles the difference is very large. Therefore, Eq.(9) and Eq.(10) differ in the number of terms dx.

Conflicts Of Interest

The authors declare no conflicts of interest.

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